Witt groups

R commutative ring, $\mathcal{M}$ line bundle on Spec R
i.e. Projective R module rank 1
$\sigma : \mathcal{M} \to \mathcal{M}$ $\sigma^2 = 1_{\mathcal{M}}$ ex: $\sigma = \pm 1_{\mathcal{M}}$
R linear

$\mathrm{Proj}(R)$ = finitely generated projective R-modules
aka vector bundles on the space whose sheaf of functions is the sheaf associated to R

Def: • $D_{\mathcal{M}} : \mathrm{Proj}(R)^{op} \xrightarrow{\simeq} \mathrm{Proj}(R)$
$P \longmapsto \mathrm{Hom}_R(P, \mathcal{M})$

is the duality associated to $\mathcal{M}$

• $\mathrm{Hom}_R(P \otimes P, \mathcal{M})$ = $\mathcal{M}$-valued bilinear forms on P
$\circlearrowleft C_2 = \{1, \tau\}$ τ acts by $\sigma \circ$ flip

- $$\mathrm{Hom}_R(P\otimes P, M)_{C_2} = M\text{-valued quadratic forms}$$

$$\big\downarrow \mathrm{Sym} \qquad x \mapsto x + \tau x$$

$$\mathrm{Hom}_R(P\otimes P, M)^{C_2} = M\text{-valued symmetric bilinear forms}$$

Land LI Exercise 1.1 Theorem 1 C.T.C. Wall "On the axiomatic foundations of the theory of Hermitian forms"

Thm: (Wall): Show that M-valued quadratic forms are equivalently described by

a) $b: P\otimes P \longrightarrow M$ symmetric bilinear

b) $q: P \longrightarrow M_{C_2}$ s.t.

$$q(rx) = r^2 q(x)$$

$$q(x+y) - q(x) - q(y) = [b(x,y)]$$

$$b(x,x) = q(x) + \tau q(x)$$

Pf: Take $\tilde{b} \in \mathrm{Hom}_R(P \otimes P, M)_{C_2}$

Define $q : P \to M_{C_2}$ by $q(x) = \tilde{b}(x,x)$ in $M/(1-\sigma)M$

well-defined because $\sigma\tilde{b}(\text{flip})(x,x) = \sigma\tilde{b}(x,x)$ and $\tilde{b}(x,x) = \sigma\tilde{b}(x,x)$ in $M/(1-\sigma)M$

$$q(rx) = \tilde{b}(rx,rx) = r^2\tilde{b}(x) \quad \text{in } M_{C_2}$$

$$q(x+y) - q(x) - q(y) = \tilde{b}(x,y) + \tilde{b}(y,x)$$

Define: $b = \mathrm{Sym}\,\tilde{b}$

$$b(x,y) = \tilde{b}(x,y) + \sigma\tilde{b}(y,x)$$

In M_{C_2}, $\sigma\tilde{b}(y,x) = \tilde{b}(y,x) \Rightarrow$

$$q(x+y) - q(x) - q(y) = b(x,y) \quad \text{in } M_{C_2}$$

- $b(x,x) = \tilde{b}(x,x) + \sigma \tilde{b}(x,x)$

$$= q(x) + \sigma q(x)$$

in M

<u>other direction</u>

Take b, q as above.

$$\tilde{b}(x,x) = q(x) \quad \text{in } M_{C_2}$$

$$\tilde{b}(x,y) + \sigma \tilde{b}(y,\cancel{x}) = b(x,y) = \sigma b(y,x) \text{ in } M \quad (\cancel{\star})$$

Suppose P is free. Let $\{e_1, \dots, e_n\}$ be a basis

Define $\tilde{b}(e_i, e_j) = \begin{cases} b(e_i, e_j) & i > j \\ 0 & i < j \\ \tilde{q}(e_i) & i = j \end{cases}$

$\tilde{q}$ any lift to M

$$(\mathrm{Sym}\,\tilde{b})(e_i, e_j) = \tilde{b}(e_i, e_j) + \tau \tilde{b}(e_j, e_i)$$

$$= \begin{cases} b(e_i, e_j) & i > j \\ \tau b(e_j, e_i) & i < j \\ \tilde{q}(e_i) + \tau \tilde{q}(e_i) & i = j \end{cases}$$

$= b(e_i, e_j)$

Question: Is this really well-defined in $\mathrm{Hom}_R(P \otimes P, M)_{C_2}$?

Rmk: When $\frac{1}{2} \in R$, Sym is an iso

Given b let $q(x) = \frac{b(x,x)}{2}$ in M_{C_2}

Rmk: When $R = \mathbb{F}_2$, Sym is not an iso

Fact: M $(4n+2)$-manifold with $TM \cong \mathbb{1}_M^{4n+2}$ stably

in the sense $TM = \mathbb{1}_M^{4n+2}$ in $KO^0(M)$ or

$TM \oplus V \cong \mathbb{1}_M^{4n+2} \oplus V$ "framed manifold"

Then intersection pairing

$$H^{2n+1}(M;\mathbb{F}_2) \times H^{2n+1}(M;\mathbb{F}_2) \longrightarrow \mathbb{F}_2$$

admits a quadratic refinement

$\mathrm{Im}(\mathrm{Sym})$ = M-valued even forms

$$\mathrm{Hom}_R(P \otimes P, M) \xrightarrow{\cong} \mathrm{Hom}_R(P, \mathrm{Hom}_R(P, M))$$

$$b \longmapsto \tilde{b} : P \to D_M P$$

- An M-valued form b is <u>perfect</u> or <u>unimodular</u> if the associated map $P \xrightarrow{\cong} D_M P$ is an isomorphism

- For $P \in \mathrm{Proj}(R)$ define

$$\mathrm{hyp}(P) = (P \oplus D_M P, ev)$$

$$(P \oplus D_M P) \otimes (P \oplus D_M P) \xrightarrow{(0, ev, ev, 0)} M$$

is a symmetric, unimodular M-valued form

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Exercise: Show $\mathrm{hyp}(P)$ is canonically a quadratic form

- A symmetric unimodular form b on P is called metabolic if $\exists$ a Lagrangian

i.e. $L \subseteq P$ $L \in \mathrm{Proj}(R)$ s.t.

$$0 \to L \to P \underset{\cong}{\overset{\tilde{b}}{\to}} D_M P \to D_M L \to 0$$

is exact $(\Rightarrow b|_L = 0)$

- if (b, q) is quadr and L a Lagrangian for b, we say that L is a quad Lagrangian if in addition $q|_L = 0$

Land LI

Exercise: Let (b, q) be a quadr form on P

show: if $\exists$ a quadr Lagrangian L, then $(b, q) \cong hyp(L)$

I owe you hint

- Show that not every sym. metabolic form is hyperbolic

Def: $GW^s_\sigma(R; M)$ = Group completion $\{$ isom classes of unimod M-valued sym forms, $\oplus \}$

- $GW^q_\sigma(R; M)$ = Group Completion $\{$ isom classes of unimod M-valued quadr forms, $\oplus \}$

- $W^s_\sigma(R; M) = \{$ isom classes of unimod M-valued sym forms, $\oplus \} / \langle$ metabolic forms $\rangle$

- $W^q_\sigma(R; M) = \{$ isom classes of unimod M-valued quadr forms, $\oplus \} / \langle$ quad metabolic forms $\rangle$

__Terminology:__

$GW(R)$ or $GW^s(R) := GW_1^s(R, R)$ "Grothendieck-Witt group"

$W(R)$ or $W^s(R) := W_1^s(R, R)$ "Witt group"

For $u \in R^*$, $\langle u \rangle$ denotes the symmetric, bilinear unimodular form

$$\langle u \rangle : R \times R \to R \qquad \langle u \rangle (x \otimes y) = uxy$$

__Lemma:__ Let K be a field. Let $V = Kx \oplus Ky$ and $b : V \times V \to K$ the symmetric unimodular bilinear form

$$b(x,x) = b(y,y) = 0 \qquad \text{Gram matrix} \quad \begin{array}{c|cc} & x & y \\ \hline x & 0 & 1 \\ y & 1 & 0 \end{array}$$
$$b(x,y) = 1$$

Then (i) $\langle x \rangle$ is a Lagrangian for (V, b)

and (ii) $[b] = \langle 1 \rangle \oplus \langle -1 \rangle$ in $GW(K)$

__Pf:__ (i) $0 \to \langle x \rangle \to \langle x, y \rangle \cong \langle x, y \rangle^* \to X^* \to 0$

$$x \mapsto y^*$$
$$y \mapsto x^*$$

$\Rightarrow$ $\langle x \rangle$ is a Lagrangian

(ii) We claim $[b] \oplus \langle -1\rangle \cong \langle 1\rangle \oplus \langle -1\rangle \oplus \langle 1\rangle$.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array} \cong \begin{array}{c} \begin{array}{ccc} z+x & z+y & x+y+z \end{array} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

ref: Milner-Husemüller p. 9

$\Rightarrow [b] = \langle 1\rangle + \langle -1\rangle$ in $GW(k)$

Rmk: when char $k \neq 2$, $[b] \cong \langle 1\rangle \oplus \langle -1\rangle$

using basis $\left\{ \frac{x+y}{2}, \frac{x-y}{2} \right\}$

Prop: Let K be a field. Then $GW^s(K)$ is generated by the forms $\langle u\rangle$ for $u \in K^*$

Thus so is $W^s(K)$.

pf: Let (V, b) be unimod symmetric bilinear form

Assume $\exists\, v \in V$ s.t. $b(v,v) = a \neq 0$

Let $V' = \langle v\rangle^\perp = \{ w \in V \mid b(w,v) = 0 \}$

$V' \oplus \langle v\rangle \xrightarrow{\cong} V$

injective b/c $\langle v\rangle \cap V' = 0$

Since $b(rv, v) = ar \neq 0$ for $r \neq 0$

Surjective b/c $\forall\, x \in V$

$x - \frac{b(x,v)}{b(v,v)} v$ is in W

Let $b' = b|_{V'}$.

check (V', b') unimod symmetric bilinear form

$\Rightarrow (V, b) \cong (V', b') \oplus \langle a \rangle$

Inductively we get

$(V, b) \cong \langle a_1 \rangle + \langle a_2 \rangle + \dots + \langle a_i \rangle \oplus (W, b')$

Such that $\forall\ w \in W \quad b'(w,w) = 0$.

Choose $w \in W$ and $y \in W$ s.t. $b'(w,y) = 1$

As above $(W, b') \cong \left(\langle x, y \rangle, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \oplus \underbrace{(W', b')}_{W'}$

Apply above lemma and induction $\square$

Ex: $W^s(\mathbb{R}) \cong \mathbb{Z}$, $W(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/4 & q \equiv 3 \bmod 4 \\ \mathbb{Z}/2 [F^*/F^{*2}] & q \equiv 1 \bmod 4 \end{cases}$

Land L1 Exercise 4: K field s.t. $K^\times \xrightarrow{(-)^2} K^\times$ surjective $\Rightarrow W^s(K) \cong \mathbb{Z}/2$

Exercise 5: K field, $\mathrm{char}(K) \neq 2 \Rightarrow W^s_{-1}(K) = 0$

Example 11 Lurie L13: $W^q(\mathbb{F}_2) \cong \mathbb{Z}/2$

(V, q)

Let $(V, (b, q))$ be a unimodular quadric space over $\mathbb{F}_2$

S'pose $q(v) = 1 \quad \forall\ v \in V - 0$ "q anisotropic"

$\Rightarrow b(v,w) = q(v+w) - q(v) - q(w) = 1$

If $u, v, w \in W$ are linearly independent

$$1 = b(u, v+w) = b(u,v) + b(u,w) = 0$$

which is impossible.

$\Rightarrow$ any nontrivial anisotropic quadratic space is dimension 2

Let $V_0 = \mathbb{F}_2 \oplus \mathbb{F}_2 \qquad q_0(a,b) = a^2 + ab + b^2$

This element is non-trivial and gives an isomorphism

$$\mathbb{Z}/2 \xrightarrow{\cong} W^q(\mathbb{F}_2)$$

<u>Arf invariant</u>:

(V, q) non-degenerate quadratic space over $\mathbb{F}_2$

$$\mathrm{Arf}(V,q) = \begin{cases} 1 & \text{if } |q^{-1}(\{1\})| \geq |q^{-1}(\{0\})| \\ 0 & \text{if } |q^{-1}(\{0\})| > |q^{-1}(\{1\})| \end{cases}$$

$$W^q(\mathbb{F}_2) \xrightarrow[\text{homomorphism}]{} \mathbb{Z}/2$$

Ex: $\mathrm{Arf}(V_0, q_0) = 1$

Def: The Kervaire invariant of a framed compact, oriented, smooth $4n+2$ dim'l ~~dim'l~~ manifold is the Arf invariant of the quadratic form refining the intersection pairing

$$H^{2n+1}(M, \mathbb{F}_2) \times H^{2n+1}(M, \mathbb{F}_2) \to \mathbb{F}_2$$

Thm (Hopkins–Hill–Ravenel, Browder...)
Differentiable manifolds of Kervaire invariant 1 exist only in dimensions

2, 6, 14, 30, 62 and possibly 126

Ex: $W^q(\mathbb{R}) \cong \mathbb{Z}$ by Sylvester's signature theorem

- When we define L theory and thus L groups we will have:

K field char $K \neq 2$

$$L^q_m(K) \cong \begin{cases} 0 & m \cong 1,2,3 \mod 4 \\ W(K) & m \equiv 0 \mod 4 \end{cases}$$

we may need a "g" here see lecture 5

K field char 2

$$L^{gq}_m(K) \cong \begin{cases} 0 & m \equiv 1 \mod 2 \\ W(K) & m \cong 0 \mod 2 \end{cases}$$