

K field
 more intuitive description Art invariant from:

Milnor conjecture filtration on $W(K)$

Metabolic forms have even dimension \Rightarrow

$$d: W(K) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$(P, b) \longmapsto \dim P$$

$$(P, q) \longmapsto \text{let's use this}$$

Fundamental ideal: $I := \ker d$

Let $(V, q) \in I$ $\dim V$ even

\leadsto Clifford algebra $Cl(V, q) = \frac{\text{Tens}(V)}{x^2 = q(x)}$

elts of V in deg 1
 $x \in V$

$\mathbb{Z}/2$ -graded

$$\cong Cl_0(V, q) \oplus Cl_1(V, q)$$

Fact: Center $Cl_0(V, q)$

is a deg 2 étale extension of K

$\rightarrow K \times K$
 or
 $\rightarrow K \subseteq K'$
 separable
 degree 2
 field extension

Galois theory then defines

$$\exists \text{ Gal}(\bar{K}/K) \rightarrow \mathbb{Z}/2 \text{ homomorphism}$$

{ homomorphisms $\text{Gal}(\bar{k}/k) \rightarrow \mathbb{Z}/2$ }

discriminant:
$$\begin{array}{ccc} \mathbb{I} & \longrightarrow & H^1(\text{Gal}(\bar{k}/k), \mathbb{Z}/2) \\ (V, q) & \longmapsto & \mathbb{Z} \end{array}$$

char $k \neq 2$
$$H^1(\text{Gal}(\bar{k}/k), \mathbb{Z}/2) = \text{coker} \left(k^* \longrightarrow k^* \right) \cong k^*/(k^*)^2$$

 $x \mapsto x^2$

When char $k=2$
$$H^1(\text{Gal}(\bar{k}/k), \mathbb{Z}/2) =$$

$$\text{coker} \left(k \longrightarrow k \right)$$

 $x \mapsto x^2 - x$

when $k = \mathbb{F}^2$
$$H^1(\text{Gal}(\bar{k}/k), \mathbb{Z}/2) = \mathbb{Z}/2$$

and Art invariant = discriminant

Exercise: prove this

Let $\mathcal{J} = \text{Ker discriminant}$

$(V, q) \in \mathcal{J} \Rightarrow \text{Cl}_0(V, q) = A_0 \times A_1$

A_i central simple algebras of order 2 in Brauer group

char $k \neq 2$
$$\mathcal{J} \longrightarrow H^2(\text{Gal}(\bar{k}/k), \mathbb{Z}/2)$$

Thm "Milnor Conjecture" = (Voevodsky, Orlov-Vishik-Voevodsky)
-Rost

$$\text{char } k \neq 2 \quad \mathbb{I}^m / \mathbb{I}^{m+1} \cong H^m(\text{Gal}(k/k), \mathbb{Z}/2)$$

Def: Δ is the category with objects

$$\underline{i} = \{0, 1, \dots, i\} \quad i = 0, 1, 2, \dots$$

$\text{Mor}(\underline{i}, \underline{j}) =$ maps of sets
preserving \leq

Quadratic
functors
on stable
 ∞ -cat

• a simplicial set is a functor

$$X: \Delta^{op} \rightarrow \text{Set}$$

Simplicial sets are a good substitute for
topological spaces in many contexts

$X_i = X(\underline{i}) =$ the set of i simplices

Ex: X top space \rightsquigarrow

$$\text{Sing } X: \Delta^{op} \rightarrow \text{Set}$$

$$\text{Sing}(\underline{i}) = \text{Map}(i\text{-simplex}, X)$$

is a corresponding simplicial set

$$\begin{aligned} \text{Top} &\rightarrow \text{Set} \\ X &\mapsto \text{Sing } X \end{aligned}$$

$$|-|: \mathcal{S}\text{Set} \rightarrow \text{Top}$$

$$X \rightarrow \frac{\coprod_i \Delta^i \times X_i}{\substack{\vee f: i \rightarrow j \\ \text{and } \sigma \in X_j, \rho \in \Delta^i}}$$

Natural weak equivalence $(p, f^* \sigma) \sim (f_* p, \sigma)$

$$|-| \text{Sing} X \rightarrow X$$

Def: The nerve of a category \mathcal{C} is the simplicial set $N(\mathcal{C})$ whose n -simplices are composable sequences of morphisms

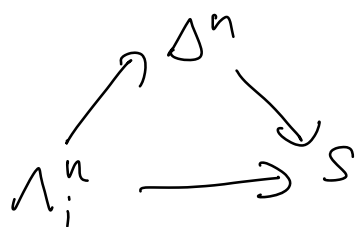
$$C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n$$

- Let Λ_i^n denote the i th horn, the simplicial subset of Δ^n obtained by removing the interior and the face opposite the i th vertex

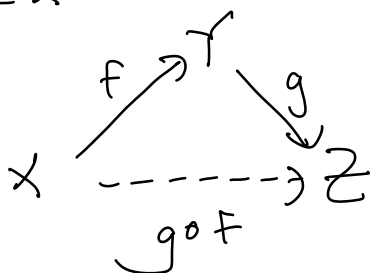
Fact: A simplicial set S is isomorphic to the nerve of a category \Leftrightarrow for all $0 < i < n$ and

$$\Lambda_i^n \rightarrow S \quad \text{there is a unique } \Delta^n \rightarrow S$$

s.t.



Ex: $i=1, n=2$



Def: An ∞ -category (also called $(\infty, 1)$ -cat and quasicategory) is a simplicial set \mathcal{C}

satisfying the condition: for all $0 < i < n$ and

$\Lambda_i^n \rightarrow \mathcal{C}$ there is $\Delta^n \rightarrow \mathcal{C}$

s.t.



$\mathcal{C}_0 =$ "objects"

$\mathcal{C}_1 =$ "morphisms"

multiple choices of compositions
using previous example
unique up to (a lot of) homotopy

- can do homotopy theory in \mathcal{C} , (co)fiber sequences
(co)limits

↑ one of the major ways to have homotopy theory. Also simplicial model cat

- For objects $A, B \in \mathcal{C}_0$, can associate a simplicial set $\text{Map}(A, B)$, $\pi_0 \text{Map}(A, B) = \text{htpy classes of maps } A \rightarrow B$

homotopy category $ho \mathcal{C}$: objects \mathcal{C}_0 morphisms $\pi_0 \text{Map}(A, B)$

Lurie L2 Ex 11 L3 Ex 6

Example (sketch)

R ring

There is an ∞ -category $D^{perf}(R)$

0-simplices: bounded chain complexes of finitely generated projective R -modules

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow P_N \rightarrow P_{N-1} \rightarrow \dots \rightarrow P_M \rightarrow 0 \rightarrow \dots$$

1-simplices: pairs (P, Q) of 0-simplices together with a map of chain complexes

$$f: P_\bullet \rightarrow Q_\bullet$$

2-simplices: diagrams

$$\begin{array}{ccc} & Q_\bullet & \\ f \nearrow & & \searrow g \\ P_\bullet & \xrightarrow{h} & R_\bullet \end{array}$$

with a chain homotopy from h to $g \circ f$

- $\mathcal{D}^{\text{perf}}(R)$ is the derived category of R from homological algebra

see Lurie L3 Def 9

- $\mathcal{D}^{\text{perf}}(R)$ is a stable ∞ -category

0 object = $\dots \rightarrow 0 \rightarrow 0 \rightarrow \dots$

maps have fibers and cofibers

fiber sequences = cofiber sequences = mapping cones

objects can be desuspended by shift

($\Sigma X = \text{cofib}(X \rightarrow 0)$)

- can do stable homotopy theory in a stable ∞ -category

- X, Y objects of a stable ∞ -cat

The simplicial sets $\{ \text{Map}_{\mathcal{C}}(Y, \Sigma^n X) \}_{n \geq 0}$

define a spectrum, i.e. object in classical stable htpy

Let $\text{Mor}_{\mathcal{C}}(X, Y)$ denote this spectrum.

True Story: You can take the derivative of a functor, and the second, third, ... etc. derivatives

"GoodWillie Calculus"

ex

M, N mfd $\text{Emb}(M, N) = \text{embeddings } M \hookrightarrow N$

$\text{Imm}(M, N) = \text{immersions } M \rightarrow N$

First derivative $\text{Emb}(-, N) = \text{Imm}(-, N)$

- The first derivative is linear in the sense that it takes fiber sequences to fiber sequences
- The functor taking an object of, say $\mathcal{D}^{\text{perf}}(\mathbb{R})$, to all the quadratic forms or bilinear forms valued in some line bundle is itself a quadratic functor, meaning that it can be recovered from its first two derivatives
- For the definition of a quadratic functor see Lurie 24 Def 6 We will use the following examples:

Ex): $Sp =$ stable ∞ -category
corresponding to classical
stable htpy theory

"Sp" for "spectra" (not related to $\text{spec } R$)

$\mathbb{S} \in Sp$ sphere spectrum $\mathbb{S} = \Sigma_+^\infty pt$

$$B: Sp^{op} \times Sp^{op} \longrightarrow Sp$$

$$B(X, Y) = \text{Mor}_{Sp}(X \wedge Y, \mathbb{S})$$

Rmk $B(X, Y) = \text{Mor}_{Sp}(Y, \mathbb{D}X)$

$$Q^s: Sp_f^{op} \longrightarrow Sp \quad Q^s(X) = B(X, X)^{h\mathbb{C}_2}$$

finite \nearrow

$$Q^a: Sp_f^{op} \longrightarrow Sp \quad Q^a(X) = B(X, X)_{h\mathbb{C}_2}$$

Q^s and Q^a are quadratic functors

Ex 2: R ring, M projective R -module rank 1
 $n \in \mathbb{Z}$, $\sigma: M \rightarrow M$ $\sigma^2 = 1$

$$B: D^{\text{perf}}(R)^{\text{op}} \times D^{\text{perf}}(R)^{\text{op}} \longrightarrow Sp$$

$$B(P_*, Q_*) = \text{Mor}_{D^{\text{perf}}(R)}(P_* \otimes Q_*, M[-n])$$

\curvearrowright
 C_2

chain complex with the single R -module M in degree $-n$

$$Q_{\sigma}^s: D^{\text{perf}}(R)^{\text{op}} \longrightarrow Sp$$

$$Q_{\sigma}^s(P_*) = B(P_*, P_*)_{\mathbb{C}_2}^{\text{sym}}$$

"spectrum of symmetric $M[-n]$ valued forms on P_* "

$$Q_{\sigma}^q: D^{\text{perf}}(R)^{\text{op}} \longrightarrow Sp$$

$$Q_{\sigma}^q(P_*) = B(P_*, P_*)_{\mathbb{C}_2}^{\text{quad}}$$

"spectrum of $M[-n]$ valued quadratic forms on P_* "

Ex: M compact oriented manifold dim n

Singular cochain complex $C^*(M; \mathbb{Z})$
determines object $D^{\text{perf}}(\mathbb{Z})$

Intersection pairing

$C^*(M; \mathbb{Z}) \otimes C^*(M; \mathbb{Z}) \rightarrow C^*(M; \mathbb{Z}) \xrightarrow{[M]} \mathbb{Z}[-n]$
determines a point $b_M \in \Omega^\infty \mathbb{Q}^S(C^*(M; \mathbb{Z}))$

L-theory and Hermitian K-theory

input: $(\mathcal{C}, \mathcal{Q})$

\mathcal{C} stable ∞ -category

\mathcal{Q} quadratic functor (nondegenerate)

output: spectrum thus htpy groups

"L-groups"

"Hermitian K-theory"

references

Lurie [2, 3, 4, 13]

Land [1]