k field

more intuitive description Arf invariant from:

Milnor conjecture filtration on $W(k)$

Metabolic forms have even dimension $\Rightarrow$

$$d: W(k) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$
$$(P, b) \longmapsto \dim P$$
$$(P, q)$$

$\leftarrow$ let's use this

Fundamental ideal: $I := \ker d$

Let $(V, q) \in I$ $\quad \dim V$ even

$\rightsquigarrow$ Clifford algebra $\quad Cl(V, q) = \dfrac{\mathrm{Tens}(V)}{x^2 = q(x)}$

elts of V in deg 1

$\uparrow x \in V$

$\mathbb{Z}/2$-graded

$\cong Cl_0(V, q) \oplus Cl_1(V, q)$

__Fact:__ Center $Cl_0(V, q)$ is a deg 2 étale extension of k

$\rightarrow k \times k$ or

$\rightarrow k \subseteq k'$ separable degree 2 field extension

Galois theory then defines

$z: \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}/2$ homomorphism

$$\{\text{homomorphisms } \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}/2\}$$
$$\|$$

discriminant: $I \longrightarrow H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2)$

$(V, q) \mapsto z$

char $k \neq 2$ $\quad H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) =$

$$\mathrm{coker}\,(k^* \xrightarrow[x \mapsto x^2]{} k^*) \cong k^*/(k^*)^2$$

when char $k = 2$ $\quad H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) =$

$$\mathrm{coker}\,(k \xrightarrow[x \mapsto x^2 - x]{} k)$$

when $k = \mathbb{F}_2$ $\quad H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) = \mathbb{Z}/2$

and Arf invariant = discriminant

Exercise: prove this

Let $J = \mathrm{Ker}$ discriminant

$(V, q) \in J \Rightarrow Cl_0(V, q) = A_0 \times A_1$ $\quad$ A_i central simple algebras of order 2 in Brauer group

Char $k \neq 2$ $\quad J \longrightarrow H^2(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2)$

Thm "Milnor Conjecture" (Voevodsky, Orlov-Vishik-Voevodsky -Rost)

Char $k \neq 2$ $\quad I^m / I^{m+1} \cong H^m(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2)$

Quadratic functors on stable ∞-cat

Def: • Δ is the category with objects

$\underline{i} = \{0, 1, \dots, i\}$ $\quad i = 0, 1, 2, \dots$

$\mathrm{Mor}(\underline{i}, \underline{j})$ = maps of sets preserving $\leq$

• a simplicial set is a functor

$$X : \Delta^{op} \longrightarrow \mathrm{Set}$$

Simplicial sets are a good substitute for topological spaces in many contexts

$X_i = X(\underline{i})$ = the set of i simplices

Ex: X top space $\rightsquigarrow$

$$\mathrm{Sing}\, X : \Delta^{op} \longrightarrow \mathrm{Set}$$

$$\mathrm{Sing}(\underline{i}) = \mathrm{Map}(i\text{-simplex}, X)$$

is a corresponding simplicial set

$$\mathrm{Top} \longrightarrow s\mathrm{Set}$$
$$X \longmapsto \mathrm{Sing}\, X$$

$$|-|: sSet \to Top$$

$$X \to \frac{\coprod_i \Delta^i \times X_i}{\begin{gathered}\forall\ f: \underline{i} \to \underline{j} \\ \text{and } \sigma \in X_j,\ p \in \Delta^i \\ (p, f^*\sigma) \sim (f_* p, \sigma)\end{gathered}}$$

Natural weak equivalence

$$|Sing X| \xrightarrow{\sim} X$$

<u>Def</u>: The nerve of a category $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are composable sequences of morphisms

$$C_0 \to C_1 \to \dots \to C_n$$

- Let Λ^n_i denote the ith horn, the simplicial subset of Δ^n obtained by removing the interior and the face opposite the ith vertex

<u>Fact</u>: A simplicial set S is isomorphic to the nerve of a category $\Leftrightarrow$ for all $0<i<n$ and

$\Lambda^n_i \to S$ there is <u>a unique</u> $\Delta^n \to S$

s.t.

$$\begin{tikzcd} & \Delta^n \arrow[dr] & \\ \Lambda^n_i \arrow[ur] \arrow[rr] & & S \end{tikzcd}$$

Ex: $i=1$, $n=2$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & Z \end{tikzcd}$$

Def: An ∞-category (also called $(\infty,1)$-cat and quasicategory) is a simplicial set $\mathcal{C}$ satisfying the condition: for all $0<i<n$ and $\Lambda^n_i \longrightarrow \mathcal{C}$ there is $\Delta^n \longrightarrow \mathcal{C}$

s.t.

$$\begin{tikzcd} & \Delta^n \arrow[dr] & \\ \Lambda^n_i \arrow[ur] \arrow[rr] & & \mathcal{C} \end{tikzcd}$$

$\mathcal{C}_0$ = "objects"

$\mathcal{C}_1$ = "morphisms"

multiple choices of compositions using previous example unique up to (a lot of) homotopy

- Can do homotopy theory in $\mathcal{C}$, (co)fiber sequences (co)limits

One of the major ways to have homotopy theory. Also simplicial model cats

- For objects $A, B \in \mathcal{C}_0$, can associate a simplicial set $\mathrm{Map}(A,B)$, $\pi_0 \mathrm{Map}(A,B)$ = htpy classes of maps $A \to B$

homotopy category $\mathrm{ho}\,\mathcal{C}$: objects $\mathcal{C}_0$ morphisms $\pi_0 \mathrm{Map}(A,B)$

Lurie L2 Ex 11 L3 Ex 6

<u>Example</u> (sketch) R ring

There is an ∞-category $D^{\mathrm{Perf}}(R)$

0-simplices: bounded chain complexes of finitely generated projective R-modules

$$\cdots \to 0 \to 0 \to P_N \to P_{N-1} \to \cdots \to P_M \to 0 \to \cdots$$

1-simplices: pairs $(P_\bullet, Q_\bullet)$ of 0-simplices together with a map of chain complexes

$$f : P_\bullet \to Q_\bullet$$

2-simplices: diagrams

$$\begin{tikzcd} & Q_\bullet \arrow[dr, "g"] & \\ P_\bullet \arrow[ur, "f"] \arrow[rr, "h"] & & R_\bullet \end{tikzcd}$$

with a chain homotopy from h to $g \circ f$

- $\mathrm{ho}\,\mathcal{D}^{perf}(R)$ is the derived category of R from homological algebra

See Lurie L3 Def 9

- $\mathcal{D}^{perf}(R)$ is a stable ∞-category

  0 object $= \cdots \to 0 \to 0 \to \cdots$

  maps have fibers and cofibers

  fiber sequences = cofiber sequences = mapping cones

  objects can be desuspended by shift

  $(\Sigma X = \mathrm{cofib}(X \to 0))$

- can do stable homotopy theory in a stable ∞-category
- X, Y objects of a stable ∞-cat

  The simplicial sets $\{\mathrm{Map}_{\mathcal{C}}(Y, \Sigma^n X)\}_{n \geq 0}$ define a spectrum, i.e. object in classical stable htpy

  Let $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ denote this spectrum.

  True Story: You can take the derivative of a functor, and the second, third, ... etc. derivatives

  "GoodWillie Calculus"

<u>ex</u>

M, N mfld $\quad \mathrm{Emb}(M,N)$ = embeddings $M \hookrightarrow N$

$\mathrm{Imm}(M,N)$ = immersions $M \to N$

First derivative $\mathrm{Emb}(-,N) \Rightarrow \mathrm{Imm}(-,N)$

- The first derivative is <u>linear</u> in the sense that it takes takes fiber sequences to fiber sequences

- The functor taking an object of, say $\mathcal{D}^{\mathrm{Perf}}(R)$, to all the quadratic forms or bilinear forms valued in some line bundle is itself a <u>quadratic</u> functor, meaning that it can be recovered from its first two derivatives

- For the definition of a quadratic functor see <u>Lurie 24 Def 6</u> We will use the following examples:

Ex1: $Sp =$ stable ∞-category corresponding to classical stable htpy theory

"Sp" for "spectra" (not related to $\mathrm{Spec}\, R$)

$\mathbb{S} \in Sp$ sphere spectrum $\mathbb{S} = \Sigma^\infty_+ pt$

$$B : Sp^{op} \times Sp^{op} \longrightarrow Sp$$

$$B(X, Y) = Mor_{Sp}(X \wedge Y, \mathbb{S})$$

Rmk $B(X, Y) = Mor_{Sp}(Y, \mathbb{D}X)$

$$Q^s : Sp_f^{op} \longrightarrow Sp \qquad Q^s(X) = B(X,X)^{hC_2}$$

finite

$$Q^q : Sp_f^{op} \longrightarrow Sp \qquad Q^q(X) = B(X,X)_{hC_2}$$

Q^s and Q^q are quadratic functors

Ex 2: R ring, M projective R-module rank 1

$n \in \mathbb{Z}$, $\sigma : M \to M$ $\sigma^2 = 1$

$$B : D^{Perf}(R)^{op} \times D^{Perf}(R)^{op} \longrightarrow Sp$$

$$B(P_\bullet, Q_\bullet) = Mor_{D^{Perf}(R)}(P_\bullet \otimes Q_\bullet, M[-n])$$

$\circlearrowright C_2$

chain complex with the single R-module M in degree $-n$

$Q^s_\sigma : D^{Perf}(R)^{op} \longrightarrow Sp$ $\qquad Q^s_\sigma(P_\bullet) = B(P_\bullet, P_\bullet)^{hC_2}$

"spectrum of symmetric $M[-n]$ valued forms on $P_\bullet$"

$Q^q_\sigma : D^{Perf}(R)^{op} \longrightarrow Sp$ $\qquad Q^q_\sigma(P_\bullet) = B(P_\bullet, P_\bullet)_{hC_2}$

"spectrum of $M[-n]$ valued quadratic forms on $P_\bullet$"

Ex: M compact oriented manifold dim n

Singular cochain complex $C^*(M;\mathbb{Z})$ determines object $D^{perf}(\mathbb{Z})$

Intersection pairing

$$C^*(M;\mathbb{Z}) \otimes C^*(M;\mathbb{Z}) \to C^*(M;\mathbb{Z}) \xrightarrow{[M]} \mathbb{Z}[-n]$$

determines a point $b_M \in \Omega^\infty Q^s_1(C^*(M;\mathbb{Z}))$

L-theory and Hermitian K-theory

input: $(\mathcal{C}, Q)$

$\mathcal{C}$ stable ∞-category

Q quadratic functor (nondegenerate)

output: Spectum thus htpy groups

"L-groups" "Hermitian K-theory"

references

Lurie L2, 3, 4, 13

Land L1