Sp $\quad \infty$-category of spectra

HoSp $\quad$ stable homotopy category $\begin{cases} \text{cohomology theories representable} \\ \text{finite spectra (e.g. compact manifolds) fully dualizable} \end{cases}$

objects Sp: Spectra

A spectrum X can be viewed as a sequence of pointed spaces $\in sSet_*$

$$X = \{X_n\}_{n \geq 0}$$

$$\Sigma X_n \to X_{n+1}$$

since $Map(\Sigma X_n, X_{n+1}) = Map(X_n, \Omega X_{n+1})$ (based loop space)

we equivalently have $X_n \to \Omega X_{n+1}$

we may assume this is an equivalence "omega spectrum"

morphisms: reference: Adams "Stable Homotopy and generalized Homology" Part III

2-simplices

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

and a homotopy $g \circ f \simeq h$

$\vdots$

Universal property: $Sp = sSet_*[\Sigma^{-1}]$

Let $Fun^{\otimes}(-,-)$ denote functors preserving colimits and tensor product of symmetric monoidal ∞-cat

There is a fully faithful embedding

$$\mathrm{Fun}^{\otimes}(Sp, \mathcal{D}) \hookrightarrow \mathrm{Fun}^{\otimes}(sSet_*, \mathcal{D})$$

with image $\{ \mathcal{F} : sSet_* \to \mathcal{D} \text{ s.t. } \mathcal{F}(S^1) \otimes \text{invertible} \}$

$X \in Sp$ $\quad \Sigma X \xrightarrow{\sim} X[-1]$ shifted by -1

$\quad\quad\quad\quad \Omega X \xrightarrow{\sim} X[1]$

$$\Sigma^\infty : sSet_* \rightleftarrows Sp : \Omega^\infty$$

$$\Omega^\infty X = \operatorname*{colim}_{n \to \infty} \Omega^n X_n$$

where $\Omega^n = \mathrm{Map}_*(S^n, X) = \underbrace{\Omega \circ \cdots \circ \Omega}_{n\text{-times}}$

<u>Ex</u> R ring

$\mathcal{D}^{\mathrm{Perf}}(R)$ is a stable ∞-cat

objects: bounded chain complexes of finitely generated projective R-modules

$$\cdots \to 0 \to 0 \to P_N \to P_{N-1} \to \cdots \to P_M \to 0 \to \cdots$$

morphisms: maps of chain complexes $f : P_\bullet \to Q_\bullet$

2-simplices: diagrams

$$\begin{tikzcd} & Q_\bullet \arrow[dr, "g"] & \\ P_\bullet \arrow[ur, "f"] \arrow[rr, "h"] & & R_\bullet \end{tikzcd}$$

with a chain homotopy from h to $g \circ f$

X Scheme, can define

$$\mathcal{D}^{Perf}(X) = \lim_{\operatorname{spec} R \subseteq X} \mathcal{D}^{Perf}(R)$$

<u>Spaces and spectra of maps</u>

- $\mathcal{C}$ ∞-cat, $X, Y \in \mathcal{C}$ $\quad Map_{\mathcal{C}}(X,Y) \in sSet$
- $\mathcal{C}$ stable ∞-cat, $X, Y \in \mathcal{C}$ $\quad Mor_{\mathcal{C}}(X,Y) \in Sp$

$$Mor_{\mathcal{C}}(X,Y) = \{Map(X, \Sigma^n Y)\}_{n \geq 0}$$

<u>Ex</u>: $\pi_0 \Omega^\infty Mor_{\mathcal{D}^{Perf}(R)}(P_\bullet, Q_\bullet) \cong$

$$Hom_{ho\mathcal{D}^{Perf}(R)}(P_\bullet, Q_\bullet) \cong Hom_{\substack{\text{chain}\\\text{complexes}}}(P_\bullet, Q_\bullet)$$

usual derived category

since $P_\bullet$ is a complex of proj

<u>Ex</u>: M rank 1 projective R-module

$$\pi_0 \Omega^\infty \mathrm{Mor}_{\mathcal{D}^{\mathrm{Perf}}(R)}(P_\bullet \otimes Q_\bullet, M[-n]) \cong$$

$$\mathrm{Hom}_{h\mathcal{D}^{\mathrm{Perf}}(R)}(P_\bullet \otimes Q_\bullet, M[-n]) \cong$$

Since $P_\bullet, Q_\bullet$ complexes of proj

$$\mathrm{Hom}_{\substack{\text{chain}\\ \text{complexes}}}(P_\bullet \otimes Q_\bullet, M[-n])$$

Rmk: $M[-n] \cong \Sigma^{-n} M$ is a $\otimes$-invertible object

Examples of quadratic functors on symmetric monoidal stable ∞-categories:

$\mathcal{C} = \mathrm{Sp}, \mathcal{D}^{\mathrm{Perf}}(R), \mathcal{D}^{\mathrm{Perf}}(X)$

$B : \mathcal{C}^{op} \times \mathcal{C}^{op} \longrightarrow \mathrm{Sp} \qquad B(X, Y) \cong \mathrm{Mor}(X \otimes Y, L)$

$L = \Sigma^{-n} \mathbb{S}, \quad M[-n], \quad M[-n]$ respectively $\forall n \in \mathbb{Z}$

line bundle on X

- Associated to B is the duality functor

$$\mathbb{D}_Q = \mathrm{Mor}_{\mathrm{Sp}}(-, \mathbb{S}), \ \mathrm{Mor}_{\mathcal{D}^{\mathrm{Perf}}(R)}(-, M[-n]), \ \mathrm{Mor}_{\mathcal{D}^{\mathrm{Perf}}(X)}(-, M[-n])$$

- Associated to B are the quadratic functors

$$Q^q : \mathcal{C}^{op} \to Sp \qquad Q^q(X) = B(X,X)_{hC_2}$$

$$Q^s : \mathcal{C}^{op} \to Sp \qquad Q^s(X) = B(X,X)^{hC_2}$$

Theorem (9 authors = Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nikolaus, Steimle)

R ring, M rank 1 projective R-module $\quad \sigma : M \to M$ involution

There exist essentially unique quadratic functors

$$Q^{ge}_M,\ Q^{ge}_M,\ Q^{gs}_M : \mathcal{D}^{perf}(R)^{op} \to Sp$$

whose restriction to $\mathrm{Proj}(R) \subseteq \mathcal{D}^{perf}(R)^{op}$ are

"genuine quadratic" $\quad Q^{gq}_M = \mathrm{Hom}(P \otimes P, M)_{C_2}$

"genuine symmetric" $\quad Q^{gs}_M = \mathrm{Hom}(P \otimes P, M)^{C_2}$

"genuine even" $\quad Q^{ge}_M = \mathrm{im}\left[Q^{gq}_M(P) \to Q^{gs}_M(P) \right]$

There is a sequence of natural transformations $\quad Q^q_M \Rightarrow Q^{gq}_M \Rightarrow Q^{ge}_M \Rightarrow Q^{gs}_M \Rightarrow Q^s_M$

which are equivalences when $\frac{1}{2} \in R$

<u>Def</u>: A <u>quadratic object</u> of $(\mathcal{C}, Q)$ is a pair (X, q) with $X \in \mathcal{C}$ and $q \in \Omega^\infty Q(X)$

- The map $Q(X) \longrightarrow B(X,X)^{hC_2} \longrightarrow B(X,X)$ induces

$$\Omega^\infty Q(X) \to \Omega^\infty B(X,X)^{hC_2} \longrightarrow \Omega^\infty B(X,X) \cong \Omega^\infty \mathrm{Mor}(X, D_Q X)$$

Let $q_\# : X \to D_Q X$ denote the image of q

<u>Def</u>: A quadratic object (X, q) of $(\mathcal{C}, Q)$ is a <u>Poincaré object</u> if $q_\#$ is invertible

<u>Ex</u>: M compact oriented manifold dimension n

Let $C^*(M;\mathbb{Z})$ be the singular cochain complex

$$q_M : C^*(M;\mathbb{Z}) \otimes C^*(M;\mathbb{Z}) \to C^*(M;\mathbb{Z}) \xrightarrow{[M]} \mathbb{Z}[-n]$$

determines a point $q_M \in \Omega^\infty Q(C^*(M;\mathbb{Z}))$

Poincaré duality $\Rightarrow$ $(C^*(M;\mathbb{Z}), q_M)$ is a Poincaré object of $(\mathcal{D}^{\mathrm{Perf}}(\mathbb{Z}), Q)$

Ex: 0 object of $\mathcal{C}$ stable ∞-category
Q quadratic functor from our examples
$\Omega^\infty Q(0) \simeq *$ contractible

$q \in \Omega^\infty Q(0)$ any point

Then $(0, q)$ is a Poincaré object of $(\mathcal{C}, Q)$

Ex: $X \in \mathcal{C}$

hyperbolic Poincaré objects

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(X,X) \simeq B(X, \mathbb{D}X) \arrow[r, hook, "\text{Summand}"] & B(X\oplus \mathbb{D}X, X \oplus \mathbb{D}X) \arrow[d] \\
1_X \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & h \in Q(X \oplus \mathbb{D}(X))
\end{tikzcd}$$

$\mathrm{hyp}(X) := (X \oplus \mathbb{D}X, h)$ is a Poincaré object

Monoid structure: Let (X, q) and (X', q') be quadratic objects of $(\mathcal{C}, Q)$

$$X \oplus X' \to X$$

$\rightsquigarrow$ $Q(X) \to Q(X \oplus X')$, same for X'

$$\rightsquigarrow \Omega^\infty Q(X) \oplus \Omega^\infty Q(X') \longrightarrow \Omega^\infty Q(X \oplus X')$$
$$(q, q') \longmapsto q \oplus q'$$

Note: $(X \oplus X', q \oplus q')$ is a quadratic object of $(\mathcal{C}, Q)$

__Claim:__ If (X, q) and (X', q') are Poincaré objects, so is $(X \oplus X', q \oplus q')$

__Pf:__

$$B(X\oplus X', X\oplus X') \cong B(X,X) \oplus B(X',X') \oplus B(X,X') \oplus B(X',X)$$

$$\begin{tikzcd}
B(X\oplus X', X\oplus X')_{hC_2} \arrow[r, "\mathrm{Sym}"] & B(X\oplus X', X\oplus X')^{hC_2} \arrow[r] & B(X\oplus X', X\oplus X') \\
B(X,X)_{hC_2} \oplus B(X',X')_{hC_2} \arrow[u] \arrow[r, "(\mathrm{Sym},\mathrm{Sym})"] & B(X,X)^{hC_2} \oplus B(X',X')^{hC_2} \arrow[u] \arrow[r] & B(X,X) \oplus B(X',X') \arrow[u, hook]
\end{tikzcd}$$

$$(q \oplus q')_\# = q_\# \oplus q'_\#$$

__Thus:__ $\{\text{Poincaré objects } (C, Q)\} / \text{homotopy}$

is a commutative monoid with $\oplus$ and identity $(0, q)$

Not a group: $(X, q) \oplus (X', q') \simeq 0$

$\Rightarrow (X, q)$ and $(X', q') \sim (0, q)$

<u>Def:</u> A Lagrangian of a Poincaré object (X, q) is a pair (L, η) where

- $L \xrightarrow{f} X$ is a map and
- $\eta : f^*(q) \sim 0$ in $\Omega^\infty Q(L)$

such that

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \overset{q}{\simeq} D_Q X \arrow[d] \\ 0 \arrow[r] & D_Q L \end{tikzcd}$$

is a pullback

i.e. $L \longrightarrow X \simeq D_Q X \longrightarrow D_Q L$

is a fiber sequence

<u>terminology:</u> X is metabolic with Lagrangian L

<u>Def:</u>
$$GW_0(\mathcal{C}, Q) = \frac{\left\{ \substack{\text{homotopy classes} \\ \text{Poincaré objects } (X, q) \\ \text{of } (\mathcal{C}, Q)} \right\}}{[X, q] = [hyp(L)]}$$

for X metabolic with Lagrangian L

<u>Rmk</u>: This is not isomorphic to

group completion $\left\{ \begin{array}{l} \text{homotopy classes} \\ \text{Poincaré objects } (X, q) \\ \text{of } (\mathcal{C}, Q) \end{array} \right\}$

<u>Def</u>: $L_0(\mathcal{C}, Q) = \left\{ \begin{array}{c} \text{homotopy} \\ \text{classes of} \\ \text{Poincaré} \\ \text{objects in } (\mathcal{C}, Q) \end{array} \right\} \Big/ \langle \text{metabolic objects} \rangle$

- Given a quadratic functor $Q: \mathcal{C}^{op} \longrightarrow Sp$ on a stable ∞-category

$\Omega^n Q : \mathcal{C}^{op} \longrightarrow Sp$ is a quadratic functor

<u>Def</u>: $GW_n(\mathcal{C}, Q) := GW_0(\mathcal{C}, \Omega^n Q)$

$L_n(\mathcal{C}, Q) := L_0(\mathcal{C}, \Omega^n Q)$

<u>Rmk</u>: A better definition is to create spaces and spectra $\mathcal{GW}(\mathcal{C}, Q)$ and $\mathcal{L}(\mathcal{C}, Q)$ and to define

$$GW_n(\mathcal{C}, Q) = \pi_n \mathcal{GW}(\mathcal{C}, Q)$$
$$L_n(\mathcal{C}, Q) = \pi_n \mathcal{L}(\mathcal{C}, Q)$$

__Theorem__ (Hebestreit–Steimle)

R ring. M rank 1 projective R-module

$$GW_n(\mathcal{D}^{perf}(R), Q^{gq}_M) \cong \text{classical } GW^q_n(R, M)$$

$$GW_n(\mathcal{D}^{perf}(R), Q^{ge}_M) \cong \text{classical } GW^{ev}_n(R, M)$$

$$GW_n(\mathcal{D}^{perf}(R), Q^{gs}_M) \cong \text{classical } GW^{s}_n(R, M)$$

classical Grothendieck-Witt groups defined using unimodular forms and group completion

cf. L3 Witt groups

__references__: Lurie L5, L7

Harpaz L1

Land L1