Surgery following Raniki Ch 1

M m-manifold

with embedding $S^n \times D^{m-n} \hookrightarrow M$

Def: The *surgery* is the new m-manifold

$$M' = \overline{M - (S^n \times D^{m-n})} \cup_{S^n \times S^{m-n-1}} (D^{n+1} \times S^{m-n-1})$$

Ex 1: $S^m = \partial(D^{n+1} \times D^{m-n}) =$

$$S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1}$$

$$S^n \times D^{m-n} \hookrightarrow S^m$$

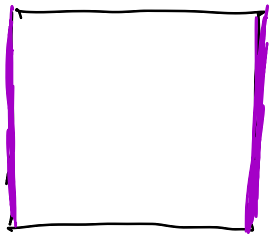
$$M' = D^{n+1} \times S^{m-n-1} \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}$$

$$= S^{n+1} \times S^{m-n-1}$$

Rmk: For $m=n$, $M' = \emptyset$

Pictures: (a) $S^1 = \partial (D^1 \times D^1)$

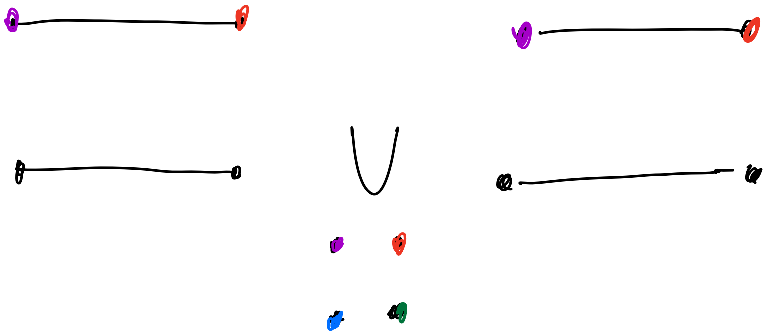
$$\simeq S^0 \times D^1 \cup D^1 \times S^0$$



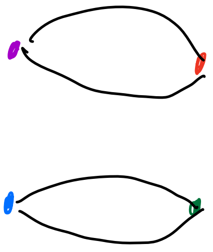
$\simeq S^1$

$$M^1 = D^1 \times S^0 \underset{S^0 \times S^0}{\cup} D^1 \times S^0$$

$\simeq$



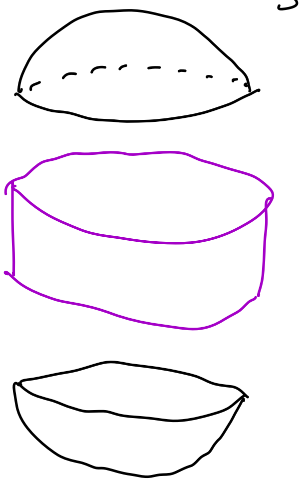
$\simeq$



$\simeq S^1 \amalg S^1$

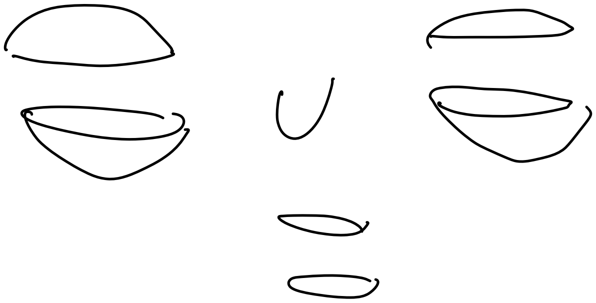
(b) $S^2 \simeq \partial(D^2 \times D^1)$

$$\simeq S^1 \times D^1 \underset{S^1 \times S^0}{\cup} D^2 \times S^0$$



$$M' = D^2 \times S^0 \underset{S^1 \times S^0}{\cup} D^2 \times S^0$$

$\simeq$



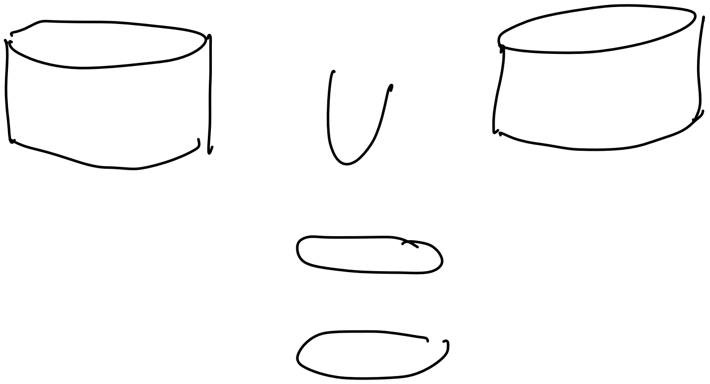
$\simeq S^2 \cup S^2$

(c) $S^2 \simeq \partial(D^2 \times D^1)$

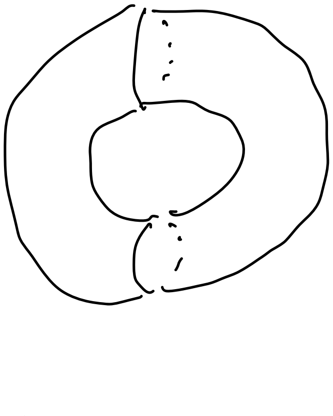
$$\simeq S^1 \times D^1 \underset{S^1 \times S^0}{\cup} D^2 \times S^0$$

$$M' \simeq S^1 \times D^1 \underset{S^1 \times S^0}{\cup} S^1 \times D^1$$

$\simeq$



$\simeq$



$\simeq S^1 \times S^1$

Ex: Mumford's plumbing game

$V \hookrightarrow \mathbb{P}^m$ complex variety

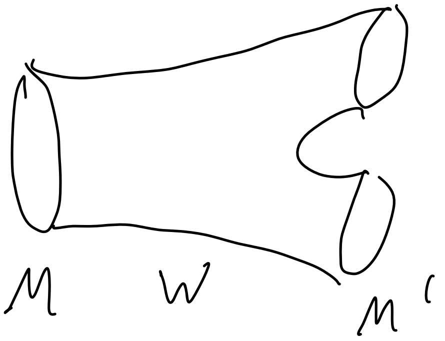
$p \in V$ isolated singularity

$$\begin{tikzcd} M = V \cap S^{m-1}_{\varepsilon, p} \arrow[d] \\ S^{m-1} \end{tikzcd}$$

← sphere of radius ε around p

Dubouloz-Déglise-Østvaer, '21, version enriched in bilinear forms

<u>Def</u>: An $(m+1)$-dim'l geometric cobordism $(W; M, M')$ is an $(m+1)$-dim'l manifold W with $\partial W = M \amalg M'$



- A surgery on M determines a cobordism between M and M' called the trace W of the surgery

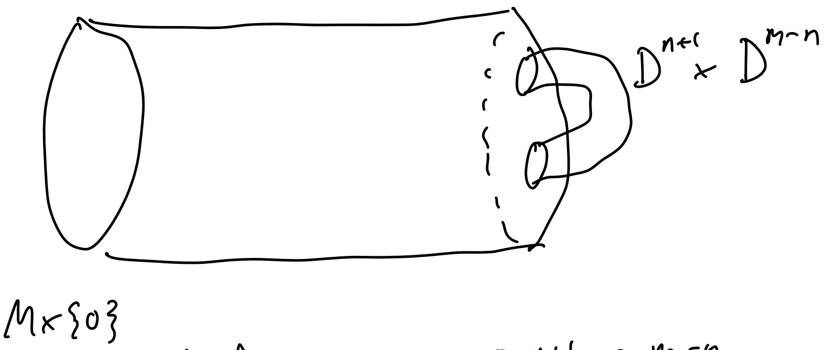
$$W = D^{n+1} \times D^{m-n} \cup M \times I$$

$$S^n \times D^{m-n} \subseteq M \times \{1\}$$

equivalently

$$\begin{tikzcd} S^n \times D^{m-n} \arrow[r, "m \mapsto m\times 1"] \arrow[d] & M \times I \arrow[d] \\ D^{n+1} \times D^{m-n} \arrow[r] & W \end{tikzcd}$$

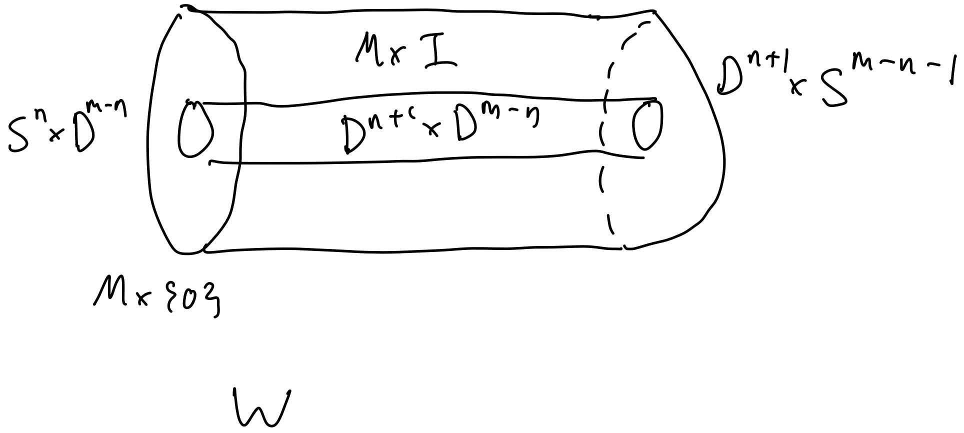
Picture:



$$W = M \times I \cup D^{n+1} \times D^{m-n}$$

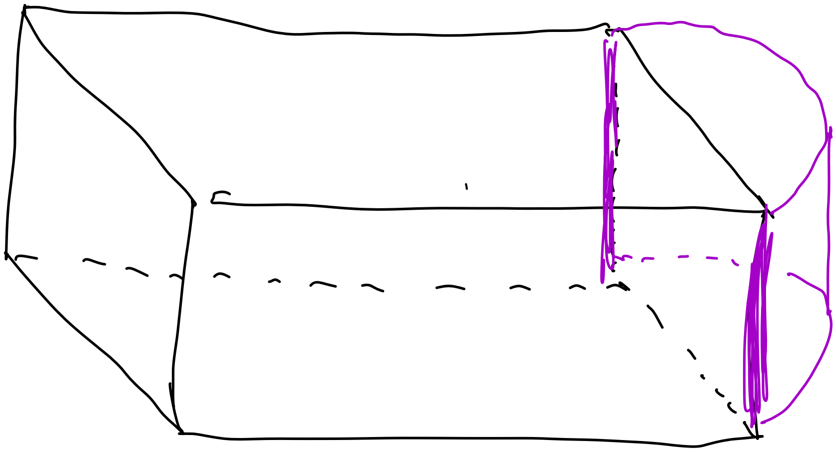
$$\partial W = \underbrace{D^{n+1} \times S^{m-n-1} \cup M \times \{1\}}_{M'} \amalg M \times \{0\}$$

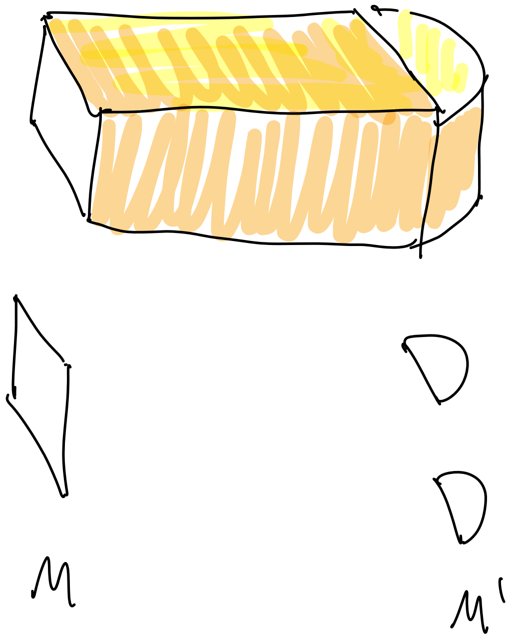
more symmetric picture



Ex 1a: $S^1 = \partial(D^1\times D^1)$

$= S^0\times D^1 \cup D^1\times S^0$





Algebraic cobordism: Sp stable ∞-category of spectra

$Q: \mathcal{C}^{op} \longrightarrow Sp$ quadratic functor

with $\mathcal{C}$ a stable ∞-category

Ex: R commutative ring, M rank 1 projective R-module

$\sigma: M \to M$ involution

$$\mathcal{C} = \mathcal{D}^{Perf}(R)$$

$$B: \mathcal{C}^{op} \times \mathcal{C}^{op} \longrightarrow Sp$$

$$B(P_\bullet, Q_\bullet) = Mor_{\mathcal{C}}(P_\bullet \otimes Q_\bullet, M[-n])$$

$$Q^{q}(P_\cdot) = B(P_\cdot, P_\cdot)_{hC_2}$$
$$Q^{s}(P_\cdot) = B(P_\cdot, P_\cdot)^{hC_2}$$

Let (X, q) and (X', q') be Poincaré objects

Def: A *cobordism* from (X, q) to (X', q') is the data:

(i) $L \in \mathcal{C}$ and maps $\alpha : L \to X$
$\alpha' : L \to X'$

(ii) a path joining the images of q and q' in $\Omega^\infty Q(\mathbf{L})$

This path gives a homotopy between the two maps $L \to \mathbb{D}_Q(L)$ in the diagram:

$$\begin{tikzcd}
X \arrow[d, "q_\#"'] & L \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & X' \arrow[d, "q'_\#"] \\
\mathbb{D}_Q X \arrow[r, "\mathbb{D}_Q \alpha"'] & \mathbb{D}_Q L & \mathbb{D}_Q X' \arrow[l, "\mathbb{D}_Q(\alpha')"]
\end{tikzcd}$$

This

$$\begin{tikzcd}
\mathrm{Fib}(\alpha) \arrow[r] & L \arrow[r, "\alpha'"] & X' \arrow[d, "q'_{\#}"] \\
& \mathbb{D}_Q L & \mathbb{D}_Q X' \arrow[l, "\mathbb{D}_Q(\alpha')"]
\end{tikzcd}$$

is null homotopic.

Thus, there is an induced map

$$\mathrm{Fib}(\alpha) \xrightarrow{u} \mathrm{Fib}(\mathbb{D}_Q(\alpha'))$$

We require that u is an equivalence

<u>Rmk</u>: Informally $\mathrm{Fib}\,\alpha = \Omega\,\mathrm{cofib}(\alpha)$

$$= \Omega(X/L)$$

and $\mathrm{Fib}(\mathbb{D}_Q(\alpha')) = \mathbb{D}_Q(\mathrm{cofib}\,\alpha')$

$$= \mathbb{D}_Q(X'/L)$$

So u is an equivalence

$$\Omega(X/L) \xrightarrow{u} \mathbb{D}_Q(X'/L)$$

Ex: Let $(W; M, M')$ be a $(m+1)$-dim'l geometric cobordism as above, with compatible orientations on W, M and M'

Let $C^*(-;\mathbb{Z}) : sSet \longrightarrow \mathcal{D}^{Perf}(\mathbb{Z})$ denote the functor taking M to its singular cochain complex

and let

$$q_M : C^*(M;\mathbb{Z}) \otimes C^*(M;\mathbb{Z}) \longrightarrow C^*(M;\mathbb{Z}) \xrightarrow{[M]} \mathbb{Z}[-n]$$

$q_M \in \Omega^\infty Q^S_{\mathbb{Z}[-m]}(C^*(M;\mathbb{Z}))$ be the point of the 0-space

of $Q^S_{\mathbb{Z}[-m]}(C^*(M;\mathbb{Z}))$ corresponding to the intersection pairing

Let $L = C^*(W;\mathbb{Z})$ giving

$$\xrightarrow{\alpha'} C^*(M';\mathbb{Z})$$

$$d[W] = [M] - [M']$$

giving a path between the images of $e_{M'}$ and e_M in $\Omega^\infty Q^s(L)$

$$\Omega \operatorname{cofib}(C^*(M;\mathbb{Z}) \xleftarrow{\alpha} C^*(W;\mathbb{Z})) \simeq$$

$$C^*(W, M; \mathbb{Z})$$

$$\operatorname{fib}\left(\operatorname{Mor}_{\mathcal{D}^{\mathrm{perf}}(\mathbb{Z})}(C^*(M';\mathbb{Z}), \mathbb{Z}[-m]) \xrightarrow{\mathbb{D}_Q \alpha'} \operatorname{Mor}_{\mathcal{D}^{\mathrm{perf}}(\mathbb{Z})}(L, \mathbb{Z}[-m]) \right) :\simeq$$

$$\mathbb{D}_Q(\operatorname{cofib} \alpha') = \mathbb{D}_Q\left(\Sigma C^*(W, M')\right)$$

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$$\operatorname{Mor}_{\mathcal{D}^{\mathrm{perf}}(\mathbb{Z})} C_*(W, M')[-(m+1)]$$

$$u: C^*(W, M; \mathbb{Z}) \xrightarrow{\cap [W]} C_*(W, M'; \mathbb{Z})[-(m+1)]$$

in negative degrees now

Poincaré duality implies the induced map

$$H^i(W, M; \mathbb{Z}) \xrightarrow{\cap [W]} H_{m+1-i}(W, M'; \mathbb{Z})$$

is an iso

$\Rightarrow$ u is an equivalence

Thus an oriented (m+1)-dim'l geometric cobordism gives an algebraic cobordism b/w Poincaré objects of $(\mathcal{D}^{perf}(\mathbb{Z}), Q^s_{\mathbb{Z}[-m]})$

Ex When $(X, q) = (0, q)$ is the 0 Poincaré object a cobordism is

(i) $L \xrightarrow{\alpha'} X'$

(ii) a null homotopy of the image of q' in $\Omega^\infty Q(L)$

i.e. the Lagrangians we defined last time

Lurie LS Prop 5: Cobordism of Poincaré objects of $(\mathcal{C}, Q)$ is an equivalence relation

We have

$$L_n(\mathcal{C}, Q) = \left\{ \begin{array}{c} \text{cobordism classes of} \\ \text{Poincaré objects of} \\ (\mathcal{C}, \Omega^n Q) \end{array} \right\}$$

<u>references</u>

Raniki Ch1 "Algebraic and Geometric Surgery"

Lurie L5