$\mathcal{C}$ stable $\infty$-category $\qquad Q : \mathcal{C}^{op} \longrightarrow Sp$ quadratic functor

$B : \mathcal{C}^{op} \times \mathcal{C}^{op} \longrightarrow Sp \qquad B(P, Q) = Mor(P \otimes Q, \Omega^n M)$

$M$ $\otimes$-invertible object

$\mathbb{D}_Q : \mathcal{C}^{op} \to \mathcal{C} \qquad \mathbb{D}_Q(P) = Mor(P, \Omega^n M)$

$Q : \mathcal{C}^{op} \longrightarrow Sp \qquad Q =$

$$\begin{tikzcd}
Q^q \arrow[r] \arrow[d, equal] & \overbrace{Q^{gq} \arrow[r] & Q^{ge} \arrow[r] & Q^{gs}}^{\mathcal{C} = \mathcal{D}^{perf}(R)} \arrow[r] & Q^s \arrow[d, equal] \\
P \mapsto B(P,P)_{hC_2} & & & & P \mapsto B(P,P)^{hC_2}
\end{tikzcd}$$

Claim:

Given a (co)fiber sequence

$$Y \longrightarrow X \longrightarrow C \simeq X/Y$$

There is a (co)fiber sequence

$$Q(C) \to Q(X) \to Q(Y) \times_{B(Y,Y)} B(Y, X)$$

Pf: Say $Q(-) = B(-,-)^{hC_2}$

$B(X',-)$ and $B(-,X')$ take fiber sequences to fiber sequences. Thus $B(C,C)$ is the total homotopy fiber of

$$\begin{tikzcd}
B(X,X) \arrow[r] \arrow[d] & B(X,Y) \arrow[d] \\
B(Y,X) \arrow[r] & B(Y,Y)
\end{tikzcd}$$

in the sense that we have a fiber sequence

$$B(C,C) \to B(X,X) \longrightarrow B(Y,X) \underset{B(Y,Y)}{\times} B(X,Y)$$

$$\simeq \left(B(Y,X) \times B(X,Y)\right) \underset{B(Y,Y)\times B(Y,Y)}{\times} B(Y,Y)$$

diagonal

Apply $(-)^{hC_2}$ to obtain the fiber sequence

$$Q(C) \longrightarrow Q(X) \to B(Y,X) \underset{B(Y,Y)}{\times} Q(Y)$$

$\square$

## Algebraic Surgery

Given $X \in \mathcal{C}$ and $q \in \Omega^\infty Q(X)$

and $Y \xrightarrow{\alpha} X$ together with a nullhomotopy of $q|_Y \in \Omega^\infty Q(Y)$

<u>Rmk</u>: By claim, this is not enough data to determine an element of $Q(C)$

Obtain a form on modified object:

We have the composition

$$Y \xrightarrow{\alpha} X \xrightarrow{q_\#} \mathbb{D}_Q X \xrightarrow{\mathbb{D}_Q(\alpha)} \mathbb{D}_Q Y$$

is $(q|_Y)_\# \simeq *$

Let $B = \mathbb{D}_Q(\alpha) \circ q_\#$. Consider

$$Y \xrightarrow{\alpha} X \xrightarrow{B} \mathbb{D}_Q(Y) \quad (\star)$$

Since $B\alpha \simeq *$, $(\star)$ is called a triangle

We may take the <u>homology</u> of a triangle

$$Y \to \mathrm{Fib}(B)$$

$$\mathrm{cofib}(\alpha) \to \mathbb{D}_Q(Y)$$

and an equivalence

$$\mathrm{cofib}(Y \to \mathrm{Fib}(B)) \simeq \mathrm{Fib}(\mathrm{cofib}(\alpha) \to \mathbb{D}_Q(Y))$$

<u>Note</u>: A triangle a (co)fiber sequence $\Leftrightarrow$ homology $\simeq 0$

Let $X_\alpha := \mathrm{cofib}(Y \to \mathrm{Fib}(B))$

$$\simeq \mathrm{Fib}(\mathrm{cofib}(\alpha) \to \mathbb{D}_Q(Y))$$

By claim, there is a fiber sequence

$$Q(X_\alpha) \longrightarrow Q(\mathrm{Fib}(B)) \longrightarrow Q(Y) \underset{B(Y,Y)}{\times} B(\mathrm{Fib}(B), Y)$$

We claim that $q|_{\mathrm{Fib}(B)}$ determines a point of $Q(X_\alpha)$, or in other words, the image of $q|_{\mathrm{Fib}(B)}$ in $\Omega^\infty\left(Q(Y) \underset{B(Y,Y)}{\times} B(\mathrm{Fib}(B), Y)\right)$ has a canonical associated null-homotopy:

We have a nullhomotopy $q|_B \simeq 0 \in \Omega^\infty Q(Y)$

The image of $q|_{\mathrm{Fib}(B)}$ in $\Omega^\infty B(\mathrm{Fib}(B), Y)$ $\simeq \Omega^\infty \mathrm{Mor}(\mathrm{Fib}(B), \mathbb{D}_Q Y)$ admits the description:

$$\mathrm{Fib}(B) \longrightarrow X \xrightarrow{q_\#} \mathbb{D}_Q X \longrightarrow \mathbb{D}_Q Y$$

which has an associated canonical null homotopy by construction.

Thus we may let $q_\alpha \in \Omega^\infty Q(X_\alpha)$ be the associated point.

<u>Def:</u> $(X_\alpha, q_\alpha)$ is obtained from surgery on $(X, q)$ and $Y \xrightarrow{\alpha} X$, $q|_Y \simeq 0$

null homotopy

<u>Claim</u>: If $(X, q)$ is a Poincaré object, so is $(X_\alpha, q_\alpha)$.

<u>pf</u>: The triangle $Y \xrightarrow{\alpha} X \xrightarrow{\beta} \mathbb{D} Y$ has a dual triangle

$$\mathbb{D}^2 Y \xrightarrow[\mathbb{D}\beta]{} \mathbb{D} X \xrightarrow[\mathbb{D}\alpha]{} \mathbb{D} Y$$

$\beta = \mathbb{D}\alpha \circ q_\#$

$\Rightarrow \mathbb{D}\beta = \mathbb{D}q_\# \circ \alpha$

The map $X_\alpha \xrightarrow{q_{\alpha\#}} \mathbb{D}_Q X_\alpha$ is the map on homology induced by the map of triangles

$$\begin{tikzcd}
Y \arrow[r, "\alpha"] \arrow[d, "\cong"'] & X \arrow[r, "\beta"] \arrow[d, "q_\# \simeq \mathbb{D}_Q q_\#"] & \mathbb{D} Y \arrow[d, "1"] \\
\mathbb{D}^2 Y \arrow[r, "\mathbb{D}\beta"'] & \mathbb{D} X \arrow[r, "\mathbb{D}\alpha"'] & \mathbb{D} Y
\end{tikzcd} \qquad (\star\star)$$

induced from $q \in \Omega^\infty Q(X)$

(The fib and cofib descriptions of the homology $X_\alpha$ are dual to each other.)

Taking cofibers of $(\star\star)$ we have

$$\begin{tikzcd}
Y \arrow[r, "\alpha"] \arrow[d, "\cong"'] & X \arrow[r, "\beta"] \arrow[d, "q_\# \simeq \mathbb{D}_Q q_\#"] & \mathbb{D} Y \arrow[d, "1"] \\
\mathbb{D}^2 Y \arrow[r, "\mathbb{D}\beta"'] \arrow[d] & \mathbb{D} X \arrow[r, "\mathbb{D}\alpha"'] \arrow[d] & \mathbb{D} Y \arrow[d] \\
0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

The associated (co)fiber sequence on homology is

$$X_2 \xrightarrow{q_a\#} \mathbb{D}X_a \longrightarrow C$$

Since $(X, q)$ is a Poincaré object, $C \simeq 0$, showing that $q_a\#$ is an equivalence as desired. $\square$

Warning: To make this line up with the discussion of geometric surgery, we must take duals

<u>Ex</u>: Given: $M$ an $m$-manifold

$$S^n \longrightarrow M \quad \text{with} \quad N_{S^n}M \simeq 1^{m-n}_{S^n}$$

$$\Rightarrow [S^n] \cdot [S^n] = 0$$

intersection pairing on homology

$$\mathcal{C} = \mathcal{D}^{\mathrm{Perf}}(\mathbb{Z}) \qquad B(P_\cdot, Q_\cdot) = \mathrm{Mor}(P_\cdot \otimes Q_\cdot, \mathbb{Z})$$

$$Q = Q^s = P_\cdot \longmapsto B(P_\cdot, P_\cdot)^{hC_2}$$

$$\mathbb{D} = \mathbb{D}_Q = \mathrm{Hom}(-, \mathbb{Z})$$

$(C^*(M;\mathbb{Z}), q_M)$ Poincaré object $(\mathcal{C}, \Omega^m Q)$
intersection pairing on cohomology

$(\underbrace{\mathbb{D}}_{\Omega^m Q} \underbrace{C^*(M;\mathbb{Z})}_{\substack{\text{12} \\ C_*(M;\mathbb{Z})[-m]}}, \underbrace{\mathbb{D}}_{\Omega^m Q} q_M)$ Poincaré object $(\mathcal{C}, \Omega^m Q)$
intersection pairing on homology

$$\begin{tikzcd} C_*(S^n;\mathbb{Z})[-m] \arrow[r] \arrow[d, equal] & C_*(M;\mathbb{Z})[-m] \arrow[d, equal] \\ Y & X \end{tikzcd}$$

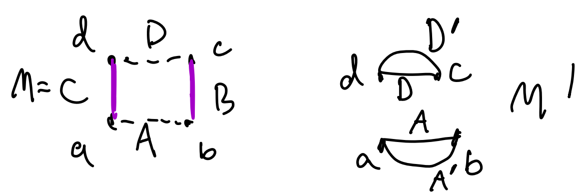
The trivialization of the normal bundle gives a nullhomotopy $\mathbb{D}\varphi_M\big|_Y \simeq *$

We may therefore perform an algebraic surgery

I believe the result is the intersection pairing on the homology of the associated geometric surgery (Raniki ?)

<u>Ex:</u> $\quad S^1 = \partial(D^1\times D^1) = S^0\times D^1 \cup D^1\times S^0$

$S^0\times D^1$



$$C_*(S^0 \times D^1)[-1] \to C_*(S^1)[-1] \to C^*(S^1) \longrightarrow C^*(S^0 \times D^1)$$

$$\begin{array}{llllll}
\deg 0 & \mathbb{Z}\{C, B\} & \mathbb{Z}\{A, B, C, D\} & & \mathbb{Z}\{a^*, b^*, c^*, d^*\} & \mathbb{Z}\{a^*, b^*, c^*, d^*\} \\
& \downarrow & \downarrow & \xrightarrow{\;\;} & \downarrow & \downarrow \\
\deg -1 & \mathbb{Z}\{a, b, c, d\} & \mathbb{Z}\{a, b, c, d\} & PD & \mathbb{Z}\{A^*, B^*, C^*, D^*\} & \mathbb{Z}\{C^*, B^*\}
\end{array}$$

$$\mathrm{cofib}(C_*(S^0 \times D^1)[-1] \to C_*(S^1)[-1]) \simeq \mathbb{Z}\{A, D\} \ \deg 0$$

$$X_\alpha = \mathrm{fib}(\mathrm{cofib} \longrightarrow C^*(S^0 \times D^1))$$

$$\begin{array}{cc}
\mathbb{Z}\{A, D\} & \mathbb{Z}\{a^*, b^*, c^*, d^*\} \ \deg 0 \\
& \downarrow \\
& \mathbb{Z}\{C^*, B^*\}
\end{array}$$

$$= \Omega \, \mathrm{cofib}(\mathrm{cofib} \longrightarrow C^*(S^0 \times D^1))$$

$$= \begin{array}{cl}
\mathbb{Z}\{A, D\} & \deg 0 \\
\downarrow & \\
\mathbb{Z}\{a^*, b^*, c^*, d^*\} & \deg -1 \\
\downarrow & \\
\mathbb{Z}\{C^*, B^*\} & \deg -2
\end{array}$$

if the map $\mathrm{cofib} \to C^*(S^0 \times D^1)$ is really represented by a map of complexes

$$\text{homology} = H_*\left(\begin{array}{ll} \mathbb{Z}\{A, D\} & \deg 0 \\ \downarrow & \\ \operatorname{Ker} \mathbb{Z}\{a-d, b-c\} & \deg -1 \end{array}\right)$$

If $\downarrow$ is 0 we will get $H_1(S^1 \amalg S^1)$ in the right degrees, but this seems very shaky to me...

<u>Claim</u>: $(X, q)$ and $(X_\alpha, q_\alpha)$ are Cobordant

<u>pf</u>: Let $L = \mathrm{fib}(B)$,

$$L \to X \xrightarrow{B} \mathbb{D}Y$$

Thus $Y \to L \to X_\alpha$ is a fiber sequence by the construction of $X_\alpha$.

$$X_\alpha \hookleftarrow L \longrightarrow X$$

By construction, $q$ and $q_\alpha$ have the same restriction to $L$.

$$\begin{tikzcd}
Y \simeq \mathrm{fib}(L \to X_\alpha) \arrow[r] & \mathbb{D}\,\mathrm{cofib}(L \to X) \cong \mathbb{D}\mathbb{D}(Y) \arrow[d, phantom, sloped, "\simeq"] \\
 & Y
\end{tikzcd}$$

is invertible. $\square$

References

Lurie L9, L11

9 authors II Section 2.3