Comment on geometric surgery and surgery of Poincaré objects:

$$\begin{tikzcd} S^n \times D^{m-n} \times \{1\} \arrow[r, hook] \arrow[d] & M \times I \arrow[d] \\ D^{n+1} \times D^{m-n} \arrow[r] & W \simeq M \times I \cup_{S^n \times D^{m-n} \times \{1\}} D^{n+1} \times D^{m-n} \end{tikzcd}$$

homotopy pushout or colimit

$$\Rightarrow \quad \begin{tikzcd} S^n \arrow[r] \arrow[d] & M \arrow[d] \\ * \arrow[r] & W \end{tikzcd}$$

homotopy colimit

$(S^n \times D^{m-n} \times \{1\} \simeq S^n$

$M \times I \simeq M$

$D^{n+1} \times D^{m-n} \simeq *$

$$\Rightarrow C_*(S^n; \mathbb{Z}) \to C_*(M; \mathbb{Z}) \to C_*(W; \mathbb{Z})$$

is a cofiber sequence

$$B(P_\cdot, Q) = \mathrm{Mor}_{\mathcal{D}^{\mathrm{Perf}}(\mathbb{Z})}(P_\cdot \otimes Q_\cdot, \mathbb{Z})$$

$$\mathbb{D} = \mathrm{Hom}_{\mathcal{D}^{\mathrm{Perf}}(\mathbb{Z})}(-, \mathbb{Z})$$

$Q = P_\bullet \mapsto B(P_\bullet, P_\bullet)^{hC_2}$

Apply $\mathbb{D}$. Obtain fiber sequence

$$C^*(S^n;\mathbb{Z}) \leftarrow C^*(M;\mathbb{Z}) \leftarrow C^*(W;\mathbb{Z})$$

Geometric surgery data: $S^n \times D^{m-n} \hookrightarrow M$

Algebraic surgery data:

$$C_*(S^n;\mathbb{Z}) \longrightarrow C_*(M,\mathbb{Z})$$

And nullhomotopy of pullback of intersection form from trivialization of the normal bundle

Surgery:

$$C_*(S^n;\mathbb{Z}) \longrightarrow C_*(M;\mathbb{Z}) \xrightarrow[\simeq]{i^\#} \mathbb{D}_{\Omega^m Q} C_*(M;\mathbb{Z}) \xrightarrow{\mathbb{D}_{\Omega^m Q}\alpha} \mathbb{D}_{\Omega^m Q} C_*(S^n;\mathbb{Z})$$

$$(\star)\quad C_*(S^n;\mathbb{Z}) \xrightarrow{\alpha} C_*(M;\mathbb{Z}) \xrightarrow{\beta = D_{\Omega^m Q}\alpha\, i^\#} \mathbb{D}_{\Omega^m Q} C_*(S^n;\mathbb{Z})$$

Last time we saw:

Have cobordism b/w $(C_*(M;\mathbb{Z}), i)$ and $(C_*(M;\mathbb{Z})', i')$:
of Poincaré objects

$$C_*(M;\mathbb{Z}) \longleftarrow \mathrm{fib}(\beta) \longrightarrow C_*(M;\mathbb{Z})'$$

We have a dual triangle $\mathbb{D}(\star)$

$$\operatorname{cofib} \alpha \simeq \mathbb{D} \operatorname{fib} \mathbb{D}\alpha$$

$$\Rightarrow \quad \operatorname{fib} \mathbb{D}\alpha \simeq C^*(W;\mathbb{Z})$$

$\operatorname{fib} \mathbb{D}\alpha$ gives cobordism of the Poincaré objects dual to $C_*(M;\mathbb{Z})$ and $C_*(M;\mathbb{Z})'$

Thus $C^*(W;\mathbb{Z})$ gives cobordism b/w the Poincaré objects $C^*(M;\mathbb{Z})$ and $\mathbb{D} C_*(M;\mathbb{Z})'$

This is consistent with the notion that $C_*(M;\mathbb{Z})' \simeq C_*(M';\mathbb{Z})$ where M' is the geometric surgery.

Hirzebruch Signature theorem

Orientations

Let $H \in Sp$. So H represents a cohomology theory

$H^n(X) = [X, \Sigma^n H]$ $\quad H_n(X) = \pi_n(X \wedge H)$

Ex: $H = H\mathbb{Z}$ ordinary singular cohomology with $\mathbb{Z}$ coefficients

$H\mathbb{Q}$ ordinary singular cohomology with $\mathbb{Q}$ coefficients

MU cobordism

$\mathbb{L} = \mathcal{L}(\mathcal{D}^{perf}(\mathbb{Z}), Q^s_{\mathbb{Z}})$ a spectrum

with $\pi_i \mathbb{L} = \mathbb{L}_n(\mathcal{D}^{perf}(\mathbb{Z}), Q^s_{\mathbb{Z}})$

When H has a ring structure, have $\mathbb{S} = S^0 \to H$ which we will assume.

Def: A rank r virtual vector bundle $V \to X$ is said to be oriented with respect to H if

- There is a Thom class $u \in H^r(Th(V))$ s.t. $\forall\ x \in X$ $u|_x \in H^r(Th(V_x)) \simeq H^0$ is a unit

$$\Leftrightarrow$$

- We are given an equivalence
$$Th(V) \wedge H \simeq \Sigma^r X \wedge H$$

<u>Def</u>: A smooth manifold M is <u>oriented</u> if TM is oriented with respect to $H\mathbb{Z}$

- M manifold dim m, compact
An orientation of TM with respect to H gives rise to a fundamental class $[M] \in H_m(M)$ s.t.
$$H^i(M) \xrightarrow{\cong} H_{m-i}(M)$$

<u>Construction</u>: cf L2 &1
$$M \hookrightarrow \mathbb{R}^k \quad \text{embedding}$$
Thom collapse
$$S^k = \mathbb{R}^k \cup \{\infty\} \xrightarrow{c} Th(N_M \mathbb{R}^k)$$

In Sp, $\mathbb{S} = S^0 \xrightarrow{\eta} \Sigma^{-k} Th(N_M \mathbb{R}^k) \simeq Th(-TM)$

(As before

$$TM + N_M \mathbb{R}^k = 1^k$$

$$\Rightarrow \quad \Sigma^{-k} Th\, N_M \mathbb{R}^k \simeq Th(-TM) \quad)$$

• apply $\mathbb{S} \to H$

$$\mathbb{S} \longrightarrow H \wedge Th(-TM)$$

<u>Rmk</u>: TM oriented with respect to H $\Rightarrow$
$-TM$ oriented with respect to H

<u>pf</u>: $N_M \mathbb{R}^k \to 1^k \to TM$

The orientation of 2 out of 3 vector bundles in a short exact sequence gives an orientation of the third

$\Rightarrow$ $N_M \mathbb{R}^k$ oriented

On the other hand $-TM = 1^{-k} + N_M \mathbb{R}^k$,
so $-TM$ is oriented

Thus we have

$$\mathbb{S} \longrightarrow \Sigma^{-m} H \wedge M$$

equivalently $S^m \longrightarrow H \wedge M$

equivalently $[M] \in H_m[M]$

- use $\mathbb{D} M \simeq M^{-TM}$ to obtain iso

$$H^i(M) \xrightarrow[\cong]{\cap [M]} H_{m-i}(M)$$

as in L2

- Two fundamental classes must differ by a unit in $H_0(M)$
- Thus two different orientations of TM with respect to a cohomology theory H give a "Signature formula"

$$[M]_1 \simeq F(TM)[M]_2$$

for some unit $f(TM) \in H_0(M)$

a characteristic class

Hirzebruch Signature formula:

We will have two canonical orientations of TM with respect to the cohomology theory

$$\mathbb{L} \wedge H\mathbb{Q}$$

for a smooth, compact oriented manifold M

Lurie has piecewise linear

First orientation: Since M is oriented, we have a given equivalence

$$H\mathbb{Z} \wedge Th(TM) \simeq \Sigma^n H\mathbb{Z} \wedge M$$

Smashing $\wedge_{H\mathbb{Z}} (H\mathbb{Q} \wedge \mathbb{L})$ we have our first orientation

$$(H\mathbb{Q} \wedge \mathbb{L}) \wedge Th(TM) \simeq \Sigma^m (H\mathbb{Q} \wedge \mathbb{L}) \wedge M$$

<u>Second orientation:</u> we will construct an orientation

$$\mathbb{L} \wedge Th(TM) \simeq \Sigma^m \mathbb{L} \wedge M$$

and then $\wedge_{\mathbb{L}} (\mathbb{L} \wedge H\mathbb{Q})$ produces the claimed second orientation.

<u>The canonical $\mathbb{L}$-theory orientation of M:</u>

Analogy: The K-groups of X are given by

$$K^i(X) = [X, \Sigma^i K]$$
$$K_i(X) = \pi_i(X \wedge K)$$

where $K \in Sp$

They are also given by forming a K-theory spectrum / space from a space of vector bundles

- Similarly the L-groups

$L^i(M) \simeq [M, \Sigma^i \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), Q)]$ $\quad L_i(M) = \pi_i(M \wedge \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), Q))$

of M are also L-groups of a stable ∞-category of sheaves

$$L_i(M) = L_i\left(Shv_{const}(M, \mathcal{D}^{Perf}(\mathbb{Z})), Q^s\right)$$

where $Shv_{const}(M, \mathcal{D}^{Perf}(\mathbb{Z}))$ is a stable ∞-category of sheaves of objects in $\mathcal{D}^{Perf}(\mathbb{Z})$ and

$$Q^s : Shv_{const}(M, \mathcal{D}^{Perf}(\mathbb{Z}))^{op} \longrightarrow Sp$$

is a quadratic functor

- For Thom spaces $V \to M$

$$L_i(Th(V)) \simeq L_i\left(Shv_{const}(M, \mathcal{D}^{Perf}(\mathbb{Z})), Q^s_V\right)$$

where

Q^s_V is a "twist" of Q^s by V

references:

Lurie L 11, L23