- We have defined L-groups of $(\mathcal{C}, Q)$ with $\mathcal{C}$ a stable ∞-category
  Q a quadratic functor given by one of our examples

$$L_i(\mathcal{C}, Q) := L_0(\mathcal{C}, \Omega^i Q) \cong \frac{\left\{\begin{matrix}\text{homotopy classes} \\ \text{Poincaré objects} \\ \text{of } (\mathcal{C}, \Omega^i Q)\end{matrix}\right\}}{\begin{matrix}\text{metabolic} \\ \text{objects}\end{matrix}}$$

$$\cong \left\{\begin{matrix}\text{cobordism classes} \\ \text{Poincaré objects} \\ \text{of } (\mathcal{C}, \Omega^i Q)\end{matrix}\right\}$$

- In Lurie L6, 7, 8, 9
  $L_i(\mathcal{C}, Q)$ is expressed as

$$L_i(\mathcal{C}, Q) \cong \pi_i(\mathbb{L}(\mathcal{C}, Q))$$

for a spectrum $\mathbb{L}(\mathcal{C}, Q) \in Sp$

- For a cohomology theory $H \in Sp$, and a rank r vector bundle $V \to X$, V is <u>oriented</u> with respect to H if

$$Th(V) \wedge H \simeq \Sigma^r X \wedge H$$

- A manifold M is oriented in the classical sense iff

  the tangent bundle $TM \to M$ is oriented with respect to $H\mathbb{Z}$ = ordinary singular (co)homology with $\mathbb{Z}$-coefficients

- Our current goal is to show:

<u>Thm</u>: Let M be a compact, smooth, oriented manifold. Then $TM \to M$ has a canonical orientation with respect to $\mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), \Omega^S)$ and $\mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), \Omega^S)$

reference: L24 Lurie prop 2, prop 1 ...

$$\mathrm{Shv}_{const}(M, \mathcal{C}) = \infty\text{-category of constructible } \mathcal{C}\text{-valued sheaves on } M$$

$$= \varprojlim_{\text{triangulations } T} \mathrm{Shv}_T(X; \mathcal{C})$$

$$\mathrm{Shv}_T(X; \mathcal{C}) := \mathrm{Functors}(T, \mathcal{C})$$

view as partially ordered set $\Rightarrow$ category

$\Rightarrow \infty$-cat

Compare: $\mathcal{D}^{\mathrm{Perf}}(X) = \varprojlim_{\mathrm{Spec}\, R \subseteq X} \mathcal{D}^{\mathrm{Perf}}(R)$

$Q_{\mathrm{const}} : \mathrm{Shv}_{\mathrm{const}}(M, \mathcal{C})^{op} \longrightarrow \mathrm{Sp}$

is the amalgamation of the quadratic functors

$$Q_T : \mathrm{Shv}_T(X, \mathcal{C})^{op} \longrightarrow \mathrm{Sp}$$

$$Q_T(\mathcal{F}) = \varinjlim_{\tau \in T} \left(Q(\mathcal{F}(\tau))\right)$$

Let V be a vector bundle.

$Q_{\mathrm{const}, V} : \mathrm{Shv}_{\mathrm{const}}(M, \mathcal{C})^{op} \longrightarrow \mathrm{Sp}$

is the amalgamation of the quadratic functors

$$Q_{T,V} : \mathrm{Shv}_T(X, \mathcal{C})^{op} \longrightarrow \mathrm{Sp}$$

$$Q_{T,V}(\mathcal{F}) = \varinjlim_{\tau \in T} \left(\frac{V(\tau)}{V(\tau)-0} \wedge Q(\mathcal{F}(\tau))\right)$$

$$\simeq \mathrm{Th}(V) \wedge \varinjlim_{\tau \in T} Q(\mathcal{F}(\tau))$$

because $\varinjlim_{\tau \in T} \frac{V(\tau)}{V(\tau)-0} \simeq Th(V)$

<u>Rmk</u>: Lurie's notation replaces $V \to M$ by its associated spherical fibration ζ

$$\zeta_m = V_m / V_m \setminus \{0\} \quad \forall\, m \in M$$

I think of $Q_{const, V}$ perhaps erroneously, as follows:

$$\mathcal{C} = \mathcal{D}^{Perf}(R) \quad B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to Sp$$

$$B(P_\cdot, Q_\cdot) = Mor(P_\cdot \otimes Q_\cdot, \mathbb{Z})$$

$$Q^q(P_\cdot) = B(P_\cdot, P_\cdot)_{hC_2} \to B(P_\cdot, P_\cdot)^{hC_2} = Q^s(P_\cdot)$$

Let $P_\bullet$ be a bounded complex of locally free sheaves of abelian groups on X

Let $\underline{\mathbb{Z}} \to M$ be the constant sheaf with value $\mathbb{Z}$ (in deg 0)

$$Q^q_{const}(P_\cdot) = Mor_{Shv_{cons}(M, \mathcal{D}^{Perf}(\mathbb{Z}))}(P_\bullet \otimes P_\bullet, \underline{\mathbb{Z}})_{hC_2}$$

$$Q^s_{const}(P_\cdot) = Mor_{Shv_{cons}(M, \mathcal{D}^{Perf}(\mathbb{Z}))}(P_\bullet \otimes P_\bullet, \underline{\mathbb{Z}})^{hC_2}$$

For the versions twisted by $V \to M$:

Dold-Kan correspondence:

connective chain complexes $\xleftrightarrow[\text{equiv}]{}$ simplicial abelian groups

R commutative ring.

$$\mathcal{D}(R) \longrightarrow HR\text{-modules} \subseteq Sp$$

$$P_\bullet \longmapsto \operatorname{colim}_{\Delta} HP_\bullet$$

- $R = \mathbb{Q}$ or $\mathbb{R}$, this is an equivalence.

  So $Th(V) \wedge HR$ determines an element of $Shv_{const}(M, \mathcal{D}^{perf}(R))$

- $R = \mathbb{Z}$, $Th(V) \wedge HR$ determines an element of $Shv_{const}(M, HR\text{-modules})$ and I believe we can work here too

Ex: $R \subset \mathbb{R}$

If the transition functions of $\det V$ are in R, then $Th(V) \wedge HR$

Corresponds to the complex of sheaves concentrated in degree $r = \operatorname{rank} V$ with the sheaf given by the locally free R module with transition functions equal to those from $\det V$

So for $R = \mathbb{R}$, $Th(V) \wedge H\mathbb{R}$

$$Th(V) \wedge H\mathbb{R} \simeq \det V[r]$$

is the complex $\det V[r]$ of $\mathbb{R}$-modules

and

$$Q^{q}_{const, V}(P_\cdot) = \underset{Shv_{cons}(M, \mathcal{D}^{Perf}(\mathbb{R}))}{Mor}(P_\cdot \otimes P_\cdot, \det V[r])_{hC_2}$$

$$Q^{s}_{const, V}(P_\cdot) = \underset{Shv_{cons}(M, \mathcal{D}^{Perf}(\mathbb{R}))}{Mor}(P_\cdot \otimes P_\cdot, \det V[r])^{hC_2}$$

<u>Rmk</u>: For a $\mathbb{R}$-line bundle such as $\det V$, we may reduce the structure group to

$O(1) = \{\pm 1\}$. Thus we may assume all transition functions are mult by 1 or -1 which are in $\mathbb{Z}^*$

This gives meaning to $\det V[r]$ in $Shv_{Cons}(M, \mathcal{D}^{Perf}(\mathbb{Z}))$. I believe we have

$$Q^q_{const,V}(P_\cdot) = Mor_{Shv_{cons}(M, \mathcal{D}^{Perf}(\mathbb{Z}))}(P_\cdot \otimes P_\cdot, \det V[r])_{nC_2}$$

$$Q^s_{const,V}(P_\cdot) = Mor_{Shv_{cons}(M, \mathcal{D}^{Perf}(\mathbb{Z}))}(P_\cdot \otimes P_\cdot, \det V[r])^{nC_2}$$

- Back to orienting TM with respect to

$\underline{\mathbb{L}} = \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), Q)$ with $Q = Q^q$ or Q^s:

$$TM \wedge \underline{\mathbb{L}} \simeq \underline{\mathbb{L}}(Shv_{const}(M, \mathcal{D}^{Perf}(\mathbb{Z})), Q_{Const,TM})$$

Since M is oriented, we have an equivalence

$$H\mathbb{Z} \wedge Th(TM) \simeq H\mathbb{Z} \wedge \Sigma^m M$$

This gives an equivalence of the corresponding ($\otimes$-invertible) elements of $\mathrm{Shv}_{\mathrm{cons}}(M, \mathcal{D}^{\mathrm{Perf}}(\mathbb{Z}))$ i.e, an equivalence

$$\det TM[r] \simeq \mathbb{Z}[r]$$

and thus an equivalence of functors

$$\begin{aligned} Q_{\mathrm{Const}, TM} &: \mathrm{Shv}_{\mathrm{const}}(M, \mathcal{D}^{\mathrm{Perf}}(\mathbb{Z}))^{op} \longrightarrow Sp \\ &\simeq \Sigma^m Q_{\mathrm{const}} \\ &\simeq Q_{\mathrm{const}, \mathbb{1}^m_M} \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Th}(TM) \wedge \mathbb{L} &\simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \mathcal{D}^{\mathrm{Perf}}(\mathbb{Z})), Q_{\mathrm{Const}, TM}) \\ &\simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \mathcal{D}^{\mathrm{Perf}}(\mathbb{Z})), Q_{\mathrm{Const}}, \mathbb{1}^m_M) \\ &\simeq \Sigma^m M \wedge \mathbb{L} \end{aligned}$$

Giving a canonical orientation of TM with respect to $\mathbb{L}$.

Alternate description of the orientation for Q^s:

We give a Thom class for $-TM$

i.e. a canonical element of $\mathbb{L}^{-m}(\mathrm{Th}(-TM))$

$$L^{-m}(Th(-TM)) \cong L^0(\Sigma^m Th(-TM))$$

$$\cong \left\{ \begin{array}{l} \text{cobordism classes of} \\ \text{Poincaré objects} \\ \text{in } Shv_{const}(M, \mathcal{D}^{perf}(\mathbb{Z})) \end{array} \right\}$$

with $Q = Q^s$ or Q^q

$$B(P_\cdot, Q_\cdot) = \underset{Shv_{const}(M, \mathcal{D}^{perf}(\mathbb{Z}))}{Mor}(P_\cdot \wedge Q, \det^* TM)$$

Since M is oriented, $\det^* TM$ is trivial $\Rightarrow$

$$\cong \left\{ \begin{array}{l} \text{cobordism classes of} \\ \text{Poincaré objects} \\ \text{in } Shv_{const}(M, \mathcal{D}^{perf}(\mathbb{Z})) \end{array} \right\}$$

with $Q = Q^s$ or Q^q

$$B(P_\cdot, Q_\cdot) = \underset{Shv_{const}(M, \mathcal{D}^{perf}(\mathbb{Z}))}{Mor}(P_\cdot \wedge Q, \underline{\mathbb{Z}})$$

The constant sheaf $\underline{\mathbb{Z}}$ equipped with multiplication

$$\underline{\mathbb{Z}} \times \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}$$

gives a unimodular symmetric bilinear form on $\underline{\mathbb{Z}}$

giving the Thom class. $\square$

Combining with last time;

Pre-Hirzebruch Signature theorem

Let M be an oriented smooth compact manifold.

There is a unit $F(TM) \in H\mathbb{Q} \wedge \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), Q^{q \, ars})_0(M)$

such that

$$[M]_{\mathbb{L}} = F(TM)\,[M]_{H\mathbb{Q}} \quad \text{in } H\mathbb{Q} \wedge \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), Q^{q \, ars})_m(M)$$

where $[M]_{\mathbb{L}}$ is the fundamental class from the canonical $\mathbb{L}$-orientation of TM

and $[M]_{H\mathbb{Q}}$ is the fundamental class from the canonical $H\mathbb{Q}$ orientation of TM

L15 L16

Theorem: $L_i(\mathcal{D}^{Perf}(\mathbb{Z}), Q^q) \cong \begin{cases} 0 & i \equiv 1,3 \mod 4 \\ \mathbb{Z}/2 & i \equiv 2 \mod 4 \\ \mathbb{Z} & i \equiv 0 \mod 4 \end{cases}$

$$\Rightarrow \quad (\mathbb{L} \wedge H\mathbb{Q})^*(M) \simeq H^*(M; \mathbb{Q})[[\underset{\text{degree } 4}{t}]]$$

$$\Rightarrow \quad H\mathbb{Q} \wedge \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), \mathbb{Q}^q)_0 (M) \simeq \prod_{k \geq 0} H^{4k}(M; \mathbb{Q})$$

So $f(TM) =$

$$f_1(TM) + f_2(TM) + \dots$$

$$f_i(TM) \in H^{4k}(M; \mathbb{Q})$$

$p : M \longrightarrow *$ the point

$$\begin{array}{ccc} P_* : H\mathbb{Q} \wedge \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), \mathbb{Q}^{qars})_m (M) & \longrightarrow & H\mathbb{Q} \wedge \mathbb{L}(\mathcal{D}^{Perf}(\mathbb{Z}), \mathbb{Q}^{qars})_m (*) \\ \wr\wr & & \wr\wr \\ H^*(M; \mathbb{Q})[[t]] & & \mathbb{Q}[[t]] \end{array}$$

$P_* [M]_{\mathbb{L}}$ is represented by

$$p_* \underline{\mathbb{Z}} \times p_* \underline{\mathbb{Z}} \longrightarrow p_* \det{}^* TM \xrightarrow{\mathrm{Tr}} \mathbb{Z}$$

12 dualizing object $\quad \mathrm{Tr} = \cap [M]$

$C^*(M;\mathbb{Z})$

Verdier duality

gives the intersection pairing

When $m = 4k$, the isomorphism

deg 0 deg 4

$$H\mathbb{Q} \wedge \mathbb{L}(\mathcal{D}^{\mathrm{Perf}}(\mathbb{Z}), \mathbb{Q}^{qrs})_m(*) \cong \deg m\,(\mathbb{Q}[[t]]) \cong \mathbb{Q}$$

Sends $\quad \Psi \qquad\qquad \Psi$

$$p_*[M]_{\mathbb{L}} \longmapsto \text{Signature } \sigma_M$$

$\Rightarrow$ Hirzebruch signature theorem:

Let M be an oriented smooth compact manifold of dimension $m = 4k$. Then

$$\sigma_M = f_{4k}(TM)[M]$$

$f_{4k} \in H^{4k}(M; \mathbb{Q})$

- f can be determined by applying the signature theorem for $M = \mathbb{C}\mathbb{P}^n$

see Lurie L25 or Milnor–Stasheff §19

<u>references</u>

Lurie L25, L24, L22, L18, L16, L15

Milnor–Stashef "characteristic classes"