A QUADRATICALLY ENRICHED COUNT OF RATIONAL CURVES

JESSE LEO KASS, MARC LEVINE, JAKE P. SOLOMON, AND KIRSTEN WICKELGREN

Abstract. We define a quadratically enriched count of rational curves in a given divisor class passing through a collection of points on a del Pezzo surface S of degree ≥ 3 over a perfect field k of characteristic $\neq 2, 3$. When S is $\mathbb{A}^1$-connected, the count takes values in the Grothendieck-Witt group $\mathrm{GW}(k)$ of quadratic forms over k and depends only on the divisor class and the fields of definition of the points. More generally, the count is a morphism from the sheaf of connected components of tuples of points on S with given fields of definition to the Grothendieck-Witt sheaf. We also treat del Pezzo surfaces of degree 2 under certain conditions. The curve count defined in the present work recovers Gromov-Witten invariants when $k = \mathbb{C}$ and Welschinger invariants when $k = \mathbb{R}$.

To obtain an invariant curve count, we define a quadratically enriched degree for an algebraic map f of n-dimensional smooth schemes over a field k under appropriate hypotheses. For example, f can be proper, generically finite and oriented over the complement of a subscheme of codimension 2. This degree is compatible with F. Morel's $\mathrm{GW}(k)$-valued degree of an A^1-homotopy class of maps between spheres. For $k \subseteq \mathbb{C}$, this produces an enrichment of the topological degree of a map between manifolds of the same dimension.

Contents

Date: July 2023.
2020 *Mathematics Subject Classification.* Primary 14N35, 14F42; Secondary 53D45, 19G38.
Key words and phrases. Gromov–Witten invariants, $\mathbb{A}^1$-homotopy theory, degree.

1. Introduction

1.1. **Background.** A degree d rational plane curve over $\mathbf{C}$ is a map $u : \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^2_{\mathbf{C}}$ given by $u([s:t]) = [u_0(s,t), u_1(s,t), u_2(s,t)]$ where the u_i are homogeneous polynomials of degree d. A dimension count shows that one expects to have finitely many degree d rational plane curves passing through $3d-1$ points. For $d = 1$, such curves are the lines through two points. For $d = 2$, they are the conics through five. Over $\mathbf{C}$, the number of such curves, N_d, is independent of the generally chosen points, and the first values are given by

$$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = 87,304, \quad \ldots$$

The number $N_4 = 620$ was first computed by Zeuthen [Zeu73] in 1873. In the early 1990's, building on ideas of Gromov [Gro85] and Witten [Wit88, Wit91], a general approach to curve counting problems was formulated [KM94, MS94, RT94, RT95], which has come to be known as Gromov-Witten theory. An early success of Gromov-Witten theory was a simple recursive formula giving N_d for $d \geq 5$. Another road-mark was the virtual enumeration of rational curves on the quintic threefold in agreement with mirror symmetry [Kon95, Giv96, LLY97].

The power of Gromov-Witten theory stems from the topological interpretation of curve counts as intersection numbers. So, even if general position cannot be achieved, one can still make sense of the curve counts. This can be done either through symplectic or algebraic geometry. Here, we focus on the algebraic approach. Let X be a projective algebraic variety over $\mathbb{C}$ of dimension r and let $\bar{M}_{g,n}(X,\beta)$ be the space of stable maps $u : \Sigma \to X$ where Σ is a nodal curve of arithmetic genus g with n marked points $p_1, \ldots, p_n$, and the degree is $u_*[\Sigma] = \beta \in H_2(X;\mathbb{Z})$. Let $ev_i : \bar{M}_{g,n}(X,\beta) \to X$ be the evaluation map at the ith marked point, given by $(u,\Sigma,p) \mapsto u(p_i)$. In general $\bar{M}_{g,n}(X,\beta)$ is a singular Deligne-Mumford stack and the dimension of irreducible components can vary. However, it admits a virtual fundamental class $[\bar{M}_{g,n}(X,\beta)]$ of dimension $(1-g)(r-3)+n+\int_\beta c_1(TX)$. The Gromov-Witten invariant counting curves of genus g and degree β passing through cycles representing the Poincaré duals of $A_i \in H^{l_i}(X)$ is defined by

$$GW_{g,\beta}(A_1,\ldots,A_n) = \int_{[\bar{M}_{g,n}(X,\beta)]} ev_1^*A_1 \cup \cdots \cup ev_n^*A_n.$$

So, in the special case that $A_1,\ldots,A_n = pt \in H^{2r}(X)$ are the Poincaré dual of the point class, the Gromov-Witten invariant $GW_{g,\beta}(A_1,\ldots,A_n)$ is the virtual degree of the total evaluation map

$$ev = ev_1 \times \cdots \times ev_n : \bar{M}_{g,n}(X,\beta) \to X^n.$$

If we take $X = \mathbb{P}^2$ and $\ell \in H_2(X;\mathbb{Z})$ the class of a line, then $N_d = GW_{0,d\ell}(pt^{\otimes n})$.

Over the real numbers $\mathbf{R}$, it is no longer true that the number of real degree d rational plane curves passing through $3d-1$ real points is independent of the general choice of points. For example, there can be 8, 10, or 12 real rational cubics passing through 8 real points [DK00, p. 55]. However, Degtyarev-Kharlamov [DK00, Lem. 4.7.3] showed that the number of such cubics with a node where two real branches intersect minus the number with a node where two complex conjugate branches intersect is always 8. Welschinger showed the invariance of a signed count of rational plane curves over $\mathbb{R}$ of degree d passing through $3d-1-2m$ real points and $2m$ pairs of complex conjugate points. The sign with which a curve contributes to the count is given by the parity of the number of nodes where two complex conjugate branches intersect. More generally, he showed [Wel05a, Wel05b] the invariance of analogous counts of real J-holomorphic spheres on real symplectic manifolds of dimensions 4 and 6. In algebraic geometry, this corresponds to counting real rational curves on real surfaces or threefolds.

A topological approach to Welschinger's invariants was developed in the context of open Gromov-Witten theory [Cho08, Sol06]. The terminology 'open' comes from open string theory [Wit95]. Let X be a symplectic manifold of dimension $2r$, let $L \subset X$ be a Lagrangian submanifold and let J be a tame almost complex structure on X. For example, take L to be a component of the real points of a projective algebraic variety P over $\mathbb{R}$, take X to be the complex points of the base change to $\mathbb{C}$ and take J to be the standard complex structure on X. In this example, we have an anti-symplectic involution $\phi : X \to X$ given by the action of $Gal(\mathbb{C}/\mathbb{R})$ such that $L \subset Fix(\phi)$. Let $\bar{M}_{D,s,t}(X/L,\beta)$ denote the space of J-holomorphic stable maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ where Σ is a nodal disk with s boundary marked points $z_1,\ldots,z_s$ and t interior marked points $w_1,\ldots,w_t$ of degree $u_*[\Sigma,\partial\Sigma] = \beta \in H_2(X,L;\mathbb{Z})$. Let $evb_i : \bar{M}_{D,s,t}(X/L,\beta) \to L$ denote the evaluation map at the ith boundary marked point given by $(u,\Sigma,z,w) \mapsto u(z_i)$. Let $evi_j : \bar{M}_{D,s,t}(X/L,\beta) \to X$ denote the evaluation map at the jth interior marked point given by $(u,\Sigma,z,w) \mapsto u(w_j)$. In nice cases, the space $\bar{M}_{D,s,t}(X/L,\beta)$ is a manifold with corners of dimension $\mu(\beta)+r-3+s+2t$ where $\mu : H_2(X,L;\mathbb{Z}) \to \mathbb{Z}$ is the Maslov index. In general, $\bar{M}_{D,s,t}(X/L,\beta)$ is singular but nonetheless admits a virtual fundamental class with dimension given by the same formula. Let

$$ev_D = evb_1 \times \cdots \times evb_s \times evi_1 \times \cdots \times evi_t : \bar{M}_{D,s,t}(X/L,\beta) \to L^s \times X^t$$

denote the total evaluation map. Recall that a relative orientation for a map of smooth manifolds $f : M \to N$ is an isomorphism $\det(TM) \xrightarrow{\sim} f^*\det(TN)$. It was shown in [Sol06] that a Pin structure on L and an orientation if L is orientable determine a virtual relative orientation for the map ev_D when the dimensions of the domain and codomain coincide. Since $\bar{M}_{D,s,t}(X/L,\beta)$ is a manifold with corners, the degree of ev_D is not a priori defined. However, when $\dim X = 4, 6,$ and there is an anti-symplectic involution of X that fixes L, it is possible to glue together certain boundary components of $\bar{M}_{D,s,t}(X/L,\beta)$ to obtain a new manifold with corners $\widetilde{M}_{D,s,t}(X/L,\beta)$ with the following two properties.

(1) There is an induced evaluation map $\widetilde{ev}_D$ that is still relatively oriented.
(2) The image of the boundary $\widetilde{ev}_D(\partial\widetilde{M}_{D,s,t}(X/L,\beta)) \subset L^s \times X^t$ has codimension at least 2.

These two properties allow the degree of $\widetilde{ev}_D$ to be defined. There is a natural doubling map $\varpi : H_2(X,L;\mathbb{Z}) \to H_2(X;\mathbb{Z})$. The degree of $\widetilde{ev}_D$ coincides up to a factor of 2^{1-t} with the Welschinger invariant counting real J-holomorphic spheres in X representing the class $\varpi(\beta)$ and passing through s real points and t complex conjugate pairs of points. Open Gromov-Witten theory leads to efficient recursive formulas for Welschinger invariants [Che22, CZ21, HS12, Sol07, ST23] and the enumeration of disks on the quintic threefold in agreement with mirror symmetry [PSW08]. It also allows the definition of invariants in arbitrary dimension and for L not necessarily fixed by an anti-symplectic involution [ST21].

Analogous results over the real and complex numbers may indicate the presence of a common generalization in the $\mathbf{A}^1$-homotopy theory of F. Morel and V. Voevodsky [MV99] valid over more general fields or base rings. We show this to be the case here. $\mathbf{A}^1$-homotopy theory adds homotopy colimits to smooth schemes, allowing one to glue or crush them. For example, one has spheres $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$, where k is a fixed base field. Morel's $\mathbf{A}^1$-Brouwer degree theorem [Mor12, Theorem 1.23] identifies the $\mathbb{A}^1$-stable homotopy classes of maps from the sphere $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ to itself with the Grothendieck–Witt group $\mathrm{GW}(k)$ of bilinear forms over k, recalled below in Section 1.2.3. More generally, the theorem computes the $(0,0)$-stable homotopy sheaf of the sphere spectrum in $\mathbf{A}^1$-homotopy theory over k with a sheaf $\mathcal{GW}$, described more in Section 1.2.6. Given polynomial equations for a map $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ the degree is computed as a sum of local degrees in [KW19]. Morel's $\mathbf{A}^1$-Brouwer degree for maps between spheres identifies the target for the $\mathbb{A}^1$-degrees that we develop here and apply to the above evaluation maps. Away from a codimension 1 locus, the degree is the sum over points of the fiber of the same local degrees present in the degree of a map of spheres.

1.2. **Statement of results.** The present paper aims to develop certain Gromov–Witten invariants and rational curve counts over perfect fields k of characteristic not 2 or 3, by recasting the arguments of [Sol06] in $\mathbf{A}^1$-homotopy theory. A relative orientation of a morphism $f : M \to N$ of smooth k-schemes is an invertible sheaf L on M together with an isomorphism $\rho : \mathcal{H}om(\det TM, \det TN) \to L^{\otimes 2}$. Let S be a del Pezzo surface over k, in the sense that S is a geometrically connected, smooth, projective k-scheme of dimension 2 with ample anticanonical bundle $-K_S$. Let $d_S = K_S \cdot K_S$ denote the degree of S.

Let $\bar{M}_{0,n}(S,D)$ denote the space of genus zero stable maps with n marked points in the class $D \in \mathrm{Pic}(S)$ and consider the total evaluation map $ev : \bar{M}_{0,n}(S,D) \to S^n$. Let $\sigma = (L_1,\ldots,L_r)$ be an r-tuple of field extensions $k \subset L_i \subset \bar{k}$ such that $\sum_{i=1}^{k}[L_i : k] = n$. For an L-scheme X, let $\mathrm{Res}_{L/k} X$ denote the restriction of scalars to k. We construct a corresponding Galois twist (see Section 5)

$$\mathrm{ev}_\sigma : \bar{\mathrm{M}}_{0,n}(S,D)_\sigma \to (S^n)_\sigma = \prod_{i=1}^{r} \mathrm{Res}_{L_i/k} S.$$

For the rest of the introduction, we fix $n = d - 1$ and work under the following hypothesis.

Hypothesis 1. *Assume that D is not an m-fold multiple of a -1-curve for $m > 1$. Moreover, assume that $d_S \geq 4$, or $d_S = 3$ and $d := -K_S \cdot D \neq 6$, or $d_S = 2$ and $d \geq 7$.*

1.2.1. *Characteristic zero.* Assume first that k has characteristic zero. In [KLSW23, Theorem 4.5], we identify a closed subset $A \subset (S^n)_\sigma$ such that $\bar{\mathrm{M}}_{0,n}(S,D)_\sigma^{\mathrm{good}} := \bar{\mathrm{M}}_{0,n}(S,D)_\sigma \setminus \mathrm{ev}^{-1}(A)$ has the following two properties analogous to properties (1) and (2) of $\widetilde{\mathcal{M}}_{D,s,t}(X/L,\beta)$ above.

- (1′) The restriction of the total evaluation map $\mathrm{ev}_\sigma^{\mathrm{good}} : \bar{\mathrm{M}}_{0,n}(S,D)_\sigma^{\mathrm{good}} \to (S^n)_\sigma$ is relatively oriented.
- (2′) The codimension of $A \subset (S^n)_\sigma$ is at least 2.

In the case $k = \mathbb{R}$, we can make the relation between $\widetilde{\mathcal{M}}_{D,s,t}(X/L,\beta)$ and $\bar{\mathrm{M}}_{0,n}(S,D)_\sigma^{\mathrm{good}}$ precise as follows. Take $L \subset X$ the Lagrangian submanifold corresponding to S, take s the number of i such that $L_i = \mathbb{R}$ and t the number of i such that $L_i = \mathbb{C}$. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{int}\,\widetilde{\mathcal{M}}_{D,s,t}(X/L,\beta) \arrow[r, "\widetilde{ev}_D"] \arrow[d] & L^s \times X^t \arrow[d, "\wr"] \\
\bar{\mathrm{M}}_{0,n}(S,D)_\sigma^{\mathrm{good}}(k) \arrow[r, "ev_\sigma^{\mathrm{good}}"] & (S^n)_\sigma(k)
\end{tikzcd}$$

where the right vertical arrow is a bijection and the left vertical arrow is two-to-one onto a fundamental domain for an action of the group $(\mathbb{Z}/2)^t$.

Properties (1′) and (2′) allow us to define the degree of $\mathrm{ev}_\sigma^{\mathrm{good}}$. However, the degree is no longer valued in the integers $\mathbf{Z}$. Rather, we build on F. Morel's $\mathbb{A}^1$-degree [Mor04, KW19] to define a degree in the Grothendieck–Witt ring $\mathrm{GW}(k)$. We recall the definition and basic properties of $\mathrm{GW}(k)$ in Section 1.2.3 below. One of our main results is the following.

Theorem 1. *Let S, D, σ satisfy Hypothesis 1 and assume that S is $\mathbb{A}^1$-connected. Then there exists an invariant $N_{S,D,\sigma}$ in the Grothendieck–Witt ring $\mathrm{GW}(k)$ given by the degree of $\mathrm{ev}_\sigma^{\mathrm{good}}$.*

1.2.2. *Positive characteristic.* We turn to the case when k has positive characteristic. Let $\mathrm{M}_0^{\mathrm{bir}}(S,D) \subset \overline{\mathrm{M}}_0(S,D)$ be the open subscheme of maps $u : \mathbf{P} \to S$ from irreducible genus 0 curves such that $\mathbf{P} \to u(\mathbf{P})$ is birational. Such u is said to be *unramified* if $u^*T^*S \to T^*\mathbf{P}$ is surjective.

Hypothesis 2. *In addition to Hypothesis 1, assume k is perfect of characteristic not 2 or 3. If $d_S = 2$, assume additionally that for every effective $D' \in \mathrm{Pic}(S)$, there is a geometric point f in each irreducible component of $\mathrm{M}_0^{\mathrm{bir}}(S,D')$ with f unramified.*

Let Λ be a complete discrete valuation ring with reside field k and quotient field K of characteristic 0. In [KLSW23, Section 9] we construct $\tilde{S} \to \operatorname{Spec} \Lambda$ a smooth del Pezzo surface equipped with an effective $\tilde{D} \in \mathrm{Pic}(\tilde{S})$ with special fibers $\tilde{S}_k \cong S$ and $\tilde{D}_k \cong D$. We construct a Galois twist

$$\tilde{ev}_\sigma : \bar{M}_{0,n}(\tilde{S}, \tilde{D})_\sigma \to (\tilde{S}^n)_\sigma$$

that agrees with ev_σ on the special fiber. Moreover, we identify a closed subset $\tilde{A} \subset (\tilde{S}^n)_\sigma$ such that $\bar{M}_{0,n}(\tilde{S}, \tilde{D})_\sigma^{\mathrm{good}} := \bar{M}_{0,n}(\tilde{S}, \tilde{D})_\sigma \backslash ev^{-1}(\tilde{A})$ has the following two properties analogous to properties (1) and (2) of $\widetilde{M}_{D,s,t}(X/L, \beta)$ above.

- (1″) The restriction of the total evaluation map $\tilde{ev}_\sigma^{\mathrm{good}} : \bar{M}_{0,n}(\tilde{S}, \tilde{D})_\sigma^{\mathrm{good}} \to (\tilde{S}^n)_\sigma$ is relatively oriented.
- (2″) The codimension of $\tilde{A} \subset (\tilde{S}^n)_\sigma$ is at least 2.

Properties (1″) and (2″) again allow us to define the degree of $\tilde{ev}_\sigma^{\mathrm{good}}$ in $\mathrm{GW}(k)$. See Section 2.4 for the precise condition. Thus we obtain the following result.

Theorem 2. *Let S, D, σ satisfy Hypothesis 2 and assume that S is $\mathbb{A}^1$-connected. Then, there exists an invariant $N_{S,D,\sigma}$ in the Grothendieck–Witt ring $\mathrm{GW}(k)$ given by the degree of $\tilde{ev}_\sigma^{\mathrm{good}}$. It is independent of the choice of $\tilde{S}, \tilde{D}$.*

1.2.3. *The Grothendieck–Witt ring.* In order to explain the enumerative meaning of the invariants $N_{S,D,\sigma}$, we recall the definition and basic properties of the Grothendieck-Witt ring $\mathrm{GW}(k)$. The Grothendieck–Witt ring is defined as the group completion of the semi-ring of non-degenerate symmetric bilinear forms over k. Since symmetric bilinear forms over a field are stably diagonalizable, an arbitrary element of this group can be expressed as a sum of rank 1 bilinear forms. Let $\langle a \rangle$ denote the element of $\mathrm{GW}(k)$ corresponding the rank 1 bilinear form $k \times k \to k$ given by $(x, y) \mapsto axy$ for a in k^*. Replacing the basis $\{1\}$ of k by $\{b\}$ for b in k^* gives the equality $\langle a \rangle = \langle ab^2 \rangle$, and in particular for fields such that $k^*/(k^*)^2$ is trivial, $\mathrm{GW}(k)$ is isomorphic to $\mathbf{Z}$ by the homomorphism taking a bilinear form to its rank, i.e. the dimension of the underlying vector space. Applying the rank will result in the classical count of curves over the algebraic closure. For more general fields, $\mathrm{GW}(k)$ contains more information. For example,

$$\mathrm{GW}(\mathbf{R}) \cong \mathbf{Z} \oplus \mathbf{Z}, \quad \mathrm{GW}(\mathbf{F}_q) \cong \mathbf{Z} \times \mathbf{F}_q^*/(\mathbf{F}_q^*)^2, \quad \mathrm{GW}(\ \mathbf{C}((z))\ ) \cong \mathbf{Z} \times \mathbf{C}((z))^*/(\mathbf{C}((z))^*)^2,$$

$$\mathrm{GW}(\mathbf{Q}_q) \cong \frac{\mathrm{GW}(\mathbf{F}_q) \oplus \mathrm{GW}(\mathbf{F}_q)}{(\langle 1 \rangle + \langle -1 \rangle, -(\langle 1 \rangle + \langle -1 \rangle))\mathbf{Z}} \text{ for } 2 \nmid q,$$

$$\mathrm{GW}(\mathbf{Q}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \bigoplus_{\substack{p \text{ prime} \\ p \neq 2}} \frac{\mathrm{GW}(\mathbf{F}_p)}{(\langle 1 \rangle + \langle -1 \rangle)\mathbf{Z}}.$$

For finite rank field extensions $L \subseteq E$, there is an additive transfer map

$$\mathrm{Tr}_{E/L} : \mathrm{GW}(E) \to \mathrm{GW}(L),$$

which has the following simple description when $L \subseteq E$ is separable: for a symmetric, non-degenerate bilinear form $\beta : V \times V \to E$ over E, we can view V as a vector space over L and consider the composition

$$V \times V \xrightarrow{\beta} E \xrightarrow{\mathrm{Tr}_{E/L}} L$$

where $\mathrm{Tr}_{E/L}$ is the sum of the Galois conjugates. Since $L \subseteq E$ is separable, $\mathrm{Tr}_{E/L} \circ \beta$ is a non-degenerate symmetric bilinear form over L. The value of the transfer map on the class $[\beta]$ of the form β is given $\mathrm{Tr}_{E/L}[\beta] = [\mathrm{Tr}_{E/L} \circ \beta]$.

The Milnor conjecture, proven by Voevodsky and Orlov–Vishik–Voevodsky, defines a sequence of invariants beginning with the rank, discriminant, Hasse–Witt invariant, Arason invariant, which for many fields (including finite fields, number fields, complete discretely valued fields, say in residue characteristic not 2 etc.) give a terminating algorithm for determining if two elements given by sums of rank 1 forms $\langle a \rangle$ are equal [Mil70] [OVV07] [Voe03a] [Voe03b]. There are many powerful tools for working with Grothendieck–Witt groups. See for example [Lam05] [Lam06] [MH73].

1.2.4. *Enumerative meaning.* To see the enumerative meaning of the degree $N_{S,D,\sigma}$, we generalize the sign associated to a node with two complex conjugate branches over $\mathbf{R}$. Suppose $u : \mathbf{P}_{k(u)} \to S$ is a rational curve on S defined over the field extension $k(u)$ of k. Let p be a node of $u(\mathbf{P}_{k(u)})$. The two tangent directions at p define a degree 2 field extension $k(p)[\sqrt{D(p)}]$ of $k(p)$, for a unique element $D(p)$ in $k(p)^*/(k(p)^*)^2$. By [SGA73, Exposé XV Théoréme 1.2.6], the extension $k(u) \subseteq k(p)$ is separable. Let $N_{k(p)/k(u)} : k(p)^* \to k(u)^*$ denote the norm of the field extension $k(u) \subseteq k(p)$ given by the product of the Galois conjugates.

Definition 1.1. *The* mass *of* p *is defined by*

$$\mathrm{mass}(p) = \langle N_{k(p)/k(u)} D(p) \rangle \quad \textit{in } \mathrm{GW}(k(u)). \tag{1}$$

This makes sense because multiplying $D(p)$ *by a square in* $k(p)$ *multiplies the norm by a square in* $k(u)$*.*

The following is valid under the same hypotheses as Theorem 1 for k of characteristic zero and under the same hypotheses as Theorem 2 for k of positive characteristic.

Theorem 3. *If there exist* $p_1, p_2, \ldots, p_r$ *points of* S *with* $k(p_i) \cong L_i$ *in general position, we have the equality*

$$N_{S,D,\sigma} = \sum_{\substack{u \textit{ degree } D \\ \textit{rational curve} \\ \textit{through the points} \\ p_1,\ldots,p_r}} \mathrm{Tr}_{k(u)/k} \prod_{p \textit{ node of } u(\mathbf{P}^1)} \mathrm{mass}(p).$$

in $\mathrm{GW}(k)$*. So the weighted count of degree* D *rational plane curves through the points* $p_1, p_2, \ldots, p_r$ *given on the right hand side is independent of the general choice of points. When* k *is an infinite field and* S *is rational over* k*, such a general choice of points exists.*

Consequently, for $k = \mathbf{C}$ the rank of $N_{S,D,\sigma}$ coincides with the corresponding Gromov–Witten invariant. For $k = \mathbf{R}$, the signature of $N_{S,D,\sigma}$ recovers the signed counts of real rational curves of Degtyarev-Kharlamov and Welschinger. For $k = \mathbf{F}_p, \mathbb{Q}_p, \mathbb{Q}$ etc., one obtains a new Gromov–Witten invariant. Andrés Jaramillo Puentes and Sabrina Pauli have work in progress giving an enriched count of rational curves of a fixed degree through rational points on a toric surface via a tropical correspondence theorem, building on their previous work [JPP22].

General position of the points $p_1, p_2, \ldots, p_r$ of S with $k(p_i) \cong L_i$ means the following. There is a dense open subset U of $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ such that for any rational point of U, the theorem holds for the corresponding r-tuple of points $p_1, p_2, \ldots, p_r$ of S with $k(p_i) \cong L_i$. The open subset U may not contain a rational point. Even for $S = \mathbb{P}^2$, this may happen over a finite field. Nonetheless, $N_{S,D,\sigma}$ is a meaningful invariant. It is the $\mathbb{A}^1$-degree of an evaluation map given in Section 5.2 and an analogue of a Gromov–Witten invariant defined over perfect fields of characteristic not 2 or 3, including finite fields. Just as Gromov–Witten invariants make sense of curve counts when general position can not be achieved, these analogues give meaning to curve counts when rational points do not exist.

This degree also retains concrete enumerative significance: The open subset U will contain many points over finite extensions of k and our constructions behave well under base change. Pick a closed point of U with field of definition L. The list σ of field extensions corresponds to a permutation representation of the Galois group $\mathrm{Gal}(k^s/k) \to \mathfrak{S}_n$, where $\mathrm{Gal}(k^s/k)$ denotes the Galois group of field isomorphisms of the separable closure k^s of k fixing k and $\mathfrak{S}_n$ denotes the symmetric group on n letters. This representation may be restricted to $\mathrm{Gal}(k^s/L)$ giving rise to σ'. We have $N_{S\otimes L, D\otimes L, \sigma'} = N_{S,D,\sigma} \otimes L$ and the equation of Theorem 3 holds in $\mathrm{GW}(L)$ for the $p'_1, \ldots, p'_{r'}$ corresponding to the chosen closed point of U. While base change to L may result in a loss of information, it frequently results in meaningful equalities. For example, $\otimes L : \mathrm{GW}(k) \to \mathrm{GW}(L)$ is injective for k a finite field and $[L : k]$ odd, resulting in infinitely many concrete enumerative equalities in the case of a finite field and $S = \mathbb{P}^2$.

1.2.5. *Examples.*

Example 1.2. *$\mathbf{A}^1$-connected del Pezzo surfaces include $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathrm{Bl}_B \mathbf{P}^1$, where B is a set of closed points $\{p_1, \ldots, p_r\}$ considered as a subscheme defined over k satisfying $|B| = \sum_{i=1}^r [k(p_i) : k] \leq 7$. In this case, $d_S = 9 - |B|$. In particular, let k be a perfect field of characteristic not 2 or 3. Then, Theorems 1 and 2 give invariants $N_{\mathbf{P}^2_k, D, \sigma}$ and $N_{\mathbf{P}^1_k \times \mathbf{P}^1_k, D, \sigma}$ in $\mathrm{GW}(k)$ for all Picard classes D. Similarly, for $|B| \leq 6$ and $S = \mathrm{Bl}_B \mathbf{P}^2_k$, we have $N_{S,D,\sigma}$ for all $D \in \mathrm{Pic}(S)$ that are not m-fold multiples of a -1-curve.*

Example 1.3. *Smooth, proper, k-rational surfaces are also $\mathbf{A}^1$-connected* [AM11, Corollary 2.3.7]*. So, Theorems 1 and 2 apply to rational del Pezzo surfaces. A smooth cubic surface over k containing two skew lines over k or two skew lines over a quadratic extension of k which are conjugate is k-rational* [KSC04, 1.33, 1.34]*. Cubic surfaces are del Pezzo surfaces with $d_S = 3$, so Theorems 1 and 2 give invariants $N_{S,D,\sigma}$ in $\mathrm{GW}(k)$ for any D with $d = -K_S \cdot D \neq 6$ and D not an m-fold multiples of a -1-curve. For example, let $S_0 \subset \mathbf{P}^3$ be the smooth cubic surface given by the zero locus of $x^2y + y^2z + z^2w + w^2x$. Then S_0 is rational* [KSC04, 1.4] *giving invariants $N_{S_0,D,\sigma}$ in $\mathrm{GW}(k)$.*

Example 1.4. *We compute $N_{S,-K_S,\sigma} = \langle -1 \rangle \chi^{\mathbb{A}^1}(S) + \langle 1 \rangle + \mathrm{Tr}_{k(\sigma)/k}\langle 1 \rangle$, where $\chi^{\mathbf{A}^1}(S)$ denotes the $\mathbb{A}^1$-Euler characteristic. See Example 9.4. For S_0 as in Example 1.3, $\chi^{\mathbf{A}^1}(S_0) = \langle -5 \rangle + 4(\langle 1 \rangle + \langle -1 \rangle)$* [LLS21]*, which gives $N_{S_0,-K_{S_0},\sigma} = \langle 5 \rangle + \langle 1 \rangle + 4(\langle 1 \rangle + \langle -1 \rangle) + \mathrm{Tr}_{k(\sigma)/k}\langle 1 \rangle$.*

1.2.6. *Without the connectedness hypothesis.* Theorems 1, 2 and 3 above are special cases of more general results that do not require that S be $\mathbb{A}^1$-connected.

The Grothendieck–Witt groups $\mathrm{GW}(E)$ discussed above for E a finite type field extension of k, together with certain boundary maps, determine a sheaf of abelian groups on smooth k-schemes

$$\mathcal{GW} : \mathbf{Sm}^{\mathrm{op}} \to \mathbf{Ab},$$

which is unramified in the sense of e.g. [Mor12, Definition 2.1]. For X a smooth k-scheme, $\mathcal{GW}(X) \subset \mathrm{GW}(k(X))$ is the subset of the Grothendieck–Witt group of its field of rational functions which is in the kernel of boundary maps indexed by the codimension 1 points of X. See [Mor12, Definition 2.1, Lemma 3.10, Section 3.2]. For a presheaf $X : \mathbf{Sm}^{\mathrm{op}} \to \mathbf{Set}$, define $\mathcal{GW}(X) := \mathrm{Map}_{\mathrm{Fun}(\mathbf{Sm}^{\mathrm{op}},\mathbf{Set})}(X, \mathcal{GW})$. This will be discussed further in Section 2.3.

To formulate our general result, we use the sheaf of $\mathbf{A}^1$-connected components, $\pi_0^{\mathbf{A}^1}$. This sheaf arises naturally when considering the degree of a morphism to a scheme that is not $\mathbf{A}^1$-connected, reflecting the classical phenomenon that a map to a disconnected manifold may have a different degree over different connected components. Unlike in classical topology, it is not possible to decompose a smooth scheme into $\mathbf{A}^1$-connected pieces; the sheaf of connected components is often a very complicated object. For a smooth scheme X, define $\pi_0^{\mathbf{A}^1}(X)$ to be the Nisnevich sheaf associated to the presheaf taking a smooth k-scheme U to $[U, X]_{\mathbf{A}^1}$, where $[U, X]_{\mathbf{A}^1}$ denotes the (unstable) $\mathbf{A}^1$-homotopy classes of maps from U to X. A smooth scheme X is said to be $\mathbf{A}^1$-connected when $\pi_0^{\mathbf{A}^1}(X)$ is trivial. This is discussed further in Section 2.5. As in topology, there is a natural map $X \to \pi_0^{\mathbf{A}^1}(X)$. For a k-point x of X and a section $\underline{N}$ in $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(X))$, let $\underline{N}(x) \in \mathrm{GW}(k)$ denote the pullback of $\underline{N}$ to x along the composition $\mathrm{Spec}\, k \xrightarrow{x} X \to \pi_0^{\mathbf{A}^1}(X)$.

For k of characteristic zero, we have the following result.

Theorem 4. *Let S, D, σ satisfy Hypothesis 1. Then there exists an invariant $\underline{N}_{S,D,\sigma}$ in $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(\prod_{i=1}^r \mathrm{Res}_{L_i/k} S))$ given by the degree of $ev_\sigma^{\mathrm{good}}$.*

For k of positive characteristic, let $\tilde{S}, \tilde{D}$ and $\tilde{ev}_\sigma^{\mathrm{good}}$ be as in Section 1.2.2.

Theorem 5. *Let S, D, σ satisfy Hypothesis 2. Then, there exists an invariant $\underline{N}_{S,D,\sigma}$ in $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(\prod_{i=1}^r \mathrm{Res}_{L_i/k} S))$ given by the degree of $\tilde{ev}_\sigma^{\mathrm{good}}$. It is independent of the choice of $\tilde{S}, \tilde{D}$.*

The following result is valid under the same hypotheses as Theorem 4 for k of characteristic zero and under the same hypotheses as Theorem 5 for k of positive characteristic.

Theorem 6. *If there exist $p_1, p_2, \ldots, p_r$ points of S with $k(p_i) \cong L_i$ in general position, we have the equality in $\mathrm{GW}(k)$,*

$$\underline{N}_{S,D,\sigma}(p_*) = \sum_{\substack{u \textit{ rational curve} \\ \textit{in class } D \\ \textit{through the points} \\ p_1,\ldots,p_r}} \mathrm{Tr}_{k(u)/k} \prod_{p \textit{ node of } u(\mathbf{P}^1)} \mathrm{mass}(p),$$

where p_* is the k-point of $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ given by $p_* = (p_1, \ldots, p_r)$. When k is an infinite field and S is rational over k, such a general choice of points exists.

We may alternatively package the invariants $\underline{N}_{S,D,\sigma}$ into a single invariant as follows. Let $\mathrm{Sym}_0^n S \subset \mathrm{Sym}^n S$ be the complement of the union of pairwise diagonals in the n-fold symmetric product. Let $\Delta_\sigma \subset (S^n)_\sigma = \prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ denote the union of pairwise diagonals. The following result is valid under the same hypotheses as Theorem 4 for k of characteristic zero and under the same hypotheses as Theorem 5 for k of positive characteristic.

Theorem 7. *There exists an invariant* $\underline{\mathrm{N}}^{\mathfrak{S}}_{S,D}$ *in* $\mathrm{GW}(\pi_0^{\mathbf{A}^1}(\mathrm{Sym}^n_0 S))$ *that pulls back to the restriction of* $\underline{\mathrm{N}}_{S,D,\sigma}$ *for each* σ *under the natural map* $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S \setminus \Delta_\sigma \to \mathrm{Sym}^n_0 S$.

Example 1.5. *Building on Example 1.2, let* S *be a twist of* $\mathrm{Bl}_B \mathbf{P}^2_k$ *and let* k *be a perfect field of characteristic not* 2 *or* 3*. Then, Theorems 4 and 5 give us invariants* $\underline{\mathrm{N}}_{S,D,\sigma}$ *in* $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(\prod_{i=1}^r \mathrm{Res}_{L_i/k} S))$ *for all* $D \in \mathsf{Pic}(S)$ *that are not* m*-fold multiples of a* -1*-curve.*

1.3. **Acknowledgements.** We thank Jean Fasel, Dan Freed, Fabien Morel, and Rahul Pandharipande for useful discussions. ML is supported by the ERC Grant QUADAG: this paper is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 832833).

JS was partially supported by ERC Starting Grant 337560 as well as ISF Grants 569/18 and 1127/22. KW was partially supported by National Science Foundation Awards DMS-1552730, DMS-2001890, and DMS-2103838. She also thanks the Newton Institute and the organizers and participants of the special program *Homotopy harnessing higher structures* for hospitality while working on this paper, and the *A Room of One's Own* initiative for focused research time.

2. Degree

2.1. **Orientations.** Define a local complete intersection morphism $f : X \to Y$ as in [Sta18, Tag 068E]. For example, let i be a closed immersion locally determined by a regular sequence and let π be a smooth map. The composition $f = \pi \circ i$ is then a local complete intersection morphism. A finite type map between regular schemes is also a local complete intersection morphism [Sta18, Lemma 37.54.11., Tag 068E]. For $f : X \to Y$ a local complete intersection morphism, the cotangent complex L_f is perfect [Sta18, Proposition 89.13.4, Tag 08SH] and we may form its determinant, which is a line bundle on X. (We could view a shift of this line bundle by some integer as the determinant viewed as an element of the derived category, but we don't do this.) Define ω_f by

$$\omega_f := \det L_f.$$

Example 2.1. *For* $f = \pi \circ i$*, there is a canonical isomorphism*

$$\omega_f \cong i^* \omega_\pi \otimes \omega_i \tag{2}$$

[Sta18, Proposition 89.7.4, Tag 08QX]. *When we additionally assume that* i *is a closed immersion determined by a regular sequence and* π *is smooth as above, we have canonical isomorphisms*

$$\omega_i \cong \det(\mathcal{I}/\mathcal{I}^2)^*,$$

$$\omega_\pi \cong \det \Omega_\pi, \tag{3}$$

where $\mathcal{I}$ denotes the ideal sheaf associated to the closed immersion $\mathfrak{i}$, and Ω_π denotes the sheaf of Kähler differentials [Sta18, Lemma 89.13.2 Tag 08SH] [Sta18, Lemma 89.9.1, Tag 08R4].[1] *A map between smooth k-schemes X and Y is a local complete intersection morphism* [Sta18, Lemma 37.54.11 Tag 068E] *and* $\omega_f = \mathrm{Hom}(\det T\mathsf{X}, f^* \det T\mathsf{Y})$.

Definition 2.2. *An orientation for a complete local intersection (lci) morphism f is the choice of an invertible sheaf L on X and an isomorphism $\rho : \omega_f \to L^{\otimes 2}$.*

2.2. **Global degree of a finite, flat map.** Suppose $f : X \to Y$ is a finite, flat, local complete intersection morphism with relative orientation $\rho : \omega_f \xrightarrow{\cong} L^{\otimes 2}$. We construct the degree of f.

Example 2.3. *If X and Y are smooth n-dimensional schemes over k, and $f : X \to Y$ is a finite map with relative orientation $\rho : \omega_f \xrightarrow{\cong} L^{\otimes 2}$, then f is flat by* [Mat89, Theorem 23.1 p.179] *and lci* [Sta18, Tag 0E9K] *and we will be able to construct the degree of f.*

Grothendieck–Serre duality produces a canonical isomorphism

$$\omega_f \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y),$$

given by identifying ω_f with $f^! \mathcal{O}_Y$ (see for example [Har66] or [BW20, Proposition A.1]) and $f^! \mathcal{O}_Y$ with $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$ [Har66, p. 165 and Section 8]. The associated trace map $\mathrm{Tr}_f : f_* \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \to \mathcal{O}_Y$ is evaluation at 1, cf. [Har66, Ideal theorem p. 6]. Since f is flat, it follows that $f_* L$ is locally free.

Definition 2.4. *Suppose $f : X \to Y$ is a finite, flat, local complete intersection morphism with relative orientation $\rho : \omega_f \xrightarrow{\cong} L^{\otimes 2}$. The* degree $\deg f$ *of f is the bilinear form $f_* L \otimes f_* L \to \mathcal{O}_Y$ given by the composition*

$$f_* L \otimes f_* L \to f_* L^{\otimes 2} \xrightarrow{f_*(\rho^{-1})} f_* \omega_f \xrightarrow{\mathrm{Tr}_f} \mathcal{O}_Y$$

When we wish to make the orientation explicit, we also write $\deg(f, \rho)$.

Proposition 2.5. $\deg f$ *is symmetric and non-degenerate.*

Proof. The swap map $L^{\otimes 2} \to L^{\otimes 2}$ defined by taking $\ell \otimes \ell'$ to $\ell' \otimes \ell$ is equal to the identity map, and it follows that $\deg f$ is symmetric.

We now prove $\deg f$ is non-degenerate. Since ρ is an isomorphism, so is the induced map $L \to \mathrm{Hom}(L, \omega_f) \cong \mathrm{Hom}(L, f^! \mathcal{O}_Y)$. Therefore the pushforward

$$f_* L \to f_* \mathrm{Hom}(L, f^! \mathcal{O}_Y)$$

is an isomorphism. Since f is proper, coherent duality as in [Har66, p. 8 Ideal Theorem c)] gives a canonical isomorphism $f_* \mathrm{Hom}(L, f^! \mathcal{O}_Y) \cong \mathrm{Hom}(f_* L, \mathcal{O}_Y)$ and the composite $f_* L \to \mathrm{Hom}(f_* L, \mathcal{O}_Y)$ is the desired isomorphism.

∎

Remark 2.6. *A finite étale map $f : X \to Y$ admits a canonical relative orientation $\omega_f \cong \mathcal{O}_X^{\otimes 2}$ and the resulting $\mathbf{A}^1$-degree is simply the classical trace form.*

[1]The references treat affine schemes, but the isomorphisms globalize.

This degree commutes with base change. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{4}$$

be a pullback diagram with f a finite, flat, local complete intersection morphism oriented by ρ. If g is flat, then f' is automatically a local complete intersection morphism by [Sta18, Lemma 37.54.6. Tag 068E]. However, in our discussion of basechange, we will not assume g to be flat, and instead assume that f' is a local complete intersection morphism. Since f is flat, the square (4) and [Sta18, Tag08QL Lemma 90.6.2] define a canonical isomorphism

$$\omega_{f'} \cong (g')^* \omega_f. \tag{5}$$

Therefore $(g')^*\rho$ determines an orientation of f'.

Proposition 2.7. *Let* (4) *be a pullback square such that* f *is a finite, flat, local complete intersection morphism oriented by* ρ*. Suppose that* f' *is a local complete intersection morphism. Then we have the equality in* $\mathcal{GW}(Y')$

$$\deg(f', (g')^*\rho) = g^* \deg(f, \rho).$$

Proof. Let L denote the line bundle on X associated to the orientation ρ, i.e., the orientation of f is the isomorphism $\rho : L^{\otimes 2} \to \omega_f$. The natural map from cohomology and base change and the isomorphism (5) determine the commutative diagram

$$\begin{tikzcd} g^*f_*L^{\otimes 2} \arrow[r, "\cong"] \arrow[d, "g^*f_*\rho"] & f'_*(g')^*L^{\otimes 2} \arrow[r, "\cong"] \arrow[d, "f'_*(g')^*\rho"] & f'_*((g')^*L)^{\otimes 2} \arrow[d, "f'_*(g')^*\rho"] \\ g^*f_*\omega_f \arrow[r, "\cong"] & f'_*(g')^*\omega_f \arrow[r, "\cong"] & f'_*\omega_{f'} \end{tikzcd},$$

where the horizontal morphisms are isomorphisms. The claim follows by the commutativity [Sta18, Lemma 48.7.1 0B6J] of

$$\begin{tikzcd} g^*f_*\omega_f \arrow[r] \arrow[d, "g^* \mathrm{Tr}_f"] & f'_*\omega_{f'} \arrow[d, "\mathrm{Tr}_{f'}"] \\ g^*\mathcal{O}_X \arrow[r, "\cong"] & \mathcal{O}_{X'} \end{tikzcd}$$

□

2.3. **Global degree of an oriented map between smooth, proper n-dimensional schemes.** We use the degree construction in Section 2.2 to construct a degree for a map $f : X \to Y$ over a field k satisfying either Assumption 2.8 or 2.13.

A map $f : X \to Y$ between integral schemes is said to be *generically finite* if f is dominant (meaning its set-theoretic image is dense) and the associated extension of function fields is finite. For example, if the differential df of a map f between connected, smooth n-dimensional k-schemes is injective at one point, then f is generically finite. If X and Y have more than one connected components, which are all integral, say that f is generically finite if f is dominant, only finitely many components of X map to each component of Y, and for each component of X, its function field is a finite extension of the function field of the component of Y containing its image. We include in the definition of f being generically finite that the connected components of X and Y are integral.

Assumption 2.8. *Let X and Y be proper schemes over a field k of dimension n, and suppose that Y/k is smooth. Let $U \subseteq Y$ be an open subset such that $Y - U$ has codimension greater than or equal to 2. Suppose $f : X \to Y$ is a generically finite map such that its restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a local complete intersection morphism equipped with a relative orientation $\rho : L^{\otimes 2} \xrightarrow{\cong} \omega_{f|_{f^{-1}(U)}}$.*

Example 2.9. *Let $f : X \to Y$ be a generically finite map between proper, smooth n-dimensional k-schemes, and let U be an open subset of Y such that $Y - U$ has codimension at least two. f is proper because X and Y are, whence the basechange $f|_{f^{-1}(U)}$ is as well, so in particular, it is finite type. The map $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a local complete intersection morphism because it is a finite type map between regular schemes* [Sta18, Lemma 37.54.11., Tag 068E]*, so it makes sense to speak of a relative orientation of $f|_{f^{-1}(U)}$. Indeed, such a relative orientation is the data of a line bundle L on $f|_{f^{-1}(U)}$ and an isomorphism $L^{\otimes 2} \cong \mathrm{Hom}(\det TX, f^* \det TY)$. A relative orientation of $f|_{f^1(U)}$ equips f with the data to satisfy Assumption 2.8.*

Moreover, a similar discussion holds if X is only assumed to be a regular, proper k-scheme of dimension n. Simply replace $\mathrm{Hom}(\det TX, f^* \det TY)$ by $\omega_{f|_{f^{-1}(U)}}$.

Remark 2.10. *Suppose that Y is smooth of dimension n and proper over k, and that X is a geometrically normal, proper scheme over k of dimension n. Let $f : X \to Y$ a generically finite map. The assumption that X is geometrically normal implies that X/k is smooth at codimension 1 points* [Sta18, 33.10 TAG 038L, Lemma 28.12.5. TAG 033P, Lemma 33.12.6. TAG 038S]*. Since the points of X where $X \to \mathrm{Spec}\, k$ is smooth is open* [Sta18, Definition 29.33.1 01V5]*, contains the points of codimension 1, and X is regular at any point where X/k is smooth, there is necessarily an open subset $U \subset Y$ such that $Y - U$ has codimension greater than or equal to 2 and $f^{-1}(U)$ is regular. So that f satisfies Assumption 2.8, we need an orientation on some such restriction.*

Generically finite maps are finite over a large open set under the following hypotheses. This will be useful to apply Section 2.2 to define the degree of a map satisfying Assumption 2.8.

Proposition 2.11. *Let Y be a smooth k scheme, and let $f : X \to Y$ be a proper, generically finite map. Then there exists a codimension 2 subset Z of Y such that f is finite and flat over the complement of Z.*

Proof. For any point x of X, the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective because f is dominant. Let x be such that $y = f(x)$ is codimension 1 in Y. Since y is codimension 1 and Y is smooth, the ring $\mathcal{O}_{Y,y}$ is a discrete valuation ring. Since the components of X are integral (this is part of the definition of f being generically finite for us), $\mathcal{O}_{X,x}$ is torsion free, and it follows that f is flat at x because $\mathcal{O}_{Y,y}$ is a principle ideal domain.

Let U be the subset of points y of Y such that f is flat at all the points x in $f^{-1}(y)$. We claim that U is open. Suppose y_0 specializes to y_1 in Y and that y_1 is in U. Let x_0 be a point of $f^{-1}(y_0)$. Let $\overline{x_0}$ denote the closure of x_0. Since f is proper, $f(\overline{x_0})$ is closed. $f(\overline{x_0})$ contains y_0 and therefore y_1 by construction. Thus we can choose x_1 such that x_0 specializes to x_1 and $f(x_1) = y_1$. We thus have a flat extension $\mathcal{O}_{Y,y_1} \subseteq \mathcal{O}_{X,x_1}$ and ideals p_{x_0} and p_{y_0} in $\mathcal{O}_{X,x_1}$ and $\mathcal{O}_{Y,y_1}$ respectively such that $p_{x_0} \cap \mathcal{O}_{Y,y_1} = p_{y_0}$. Since localization is flat, it follows that $\mathcal{O}_{Y,y_0} \subseteq \mathcal{O}_{X,x_0}$ is flat and U is open as claimed.

By the above, U contains all the points of codimension 1, whence its complement Z is closed of codimension at least 2. Let f^0 denote the restriction of f to $f^{-1}(U)$. Since proper maps are stable under base change, f^0 is proper. f^0 is flat by construction. Thus the fibers are equidimensional [Mat89, Theorem 15.1] and [Sta18, Lemma 29.17.4. 02J7]. Since f is generically finite, the fibers of f^0 are dimension 0 and therefore finite. Thus f^0 is a proper map with finite fibers and therefore finite [Sta18, Lemma 30.21.1. 02OG]. $\square$

The degree in this section will be valued in the sections $\mathcal{GW}(Y)$ of the Grothendieck–Witt sheaf $\mathcal{GW}$ at Y, so we introduce the definition of $\mathcal{GW}$ here. The Grothendieck–Witt sheaf is a sheaf on smooth k-schemes with the Nisnevich topology which can be defined as the sheafification of the functor sending Y to the group completion of the semi-ring of isomorphism classes of locally free sheaves V on Y equipped with a non-degenerate symmetric bilinear form $V \times V \to \mathcal{O}_Y$. It has a construction given in [Mor12, Chapter 3] and in an unramified sheaf by a result of Panin and Ojanguren [OP99]. In particular, suppose $U \subseteq Y$ is an open subset of Y with complement of codimension at least 2. Then a locally free sheaf on U equipped with a symmetric, non-degenerate bilinear form determines an element of $\mathcal{GW}(Y)$. (A complete definition of an unramified sheaf is in [Mor12, Chapter 2, Def 2.1, Remark 2.4].)

Let $f : X \to Y$, and ρ be as in Assumption 2.8, so in particular, $f : X \to Y$ is a generically finite map equipped with a subset U of Y with complement of codimension at least 2 and a relative orientation ρ of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$. By Proposition 2.11, there is an open subset U' of Y with complement of codimension at least 2 such that $f|_{f^{-1}(U')}$ is finite and flat. Then $f|_{f^{-1}(U \cap U')}$ is a finite, flat, oriented, locally complete intersection morphism, and Definition 2.4 constructs a bilinear form $\deg f|_{f^{-1}(U\cap U')}$ on a locally free sheaf on $U \cap U'$. The associated section of $\mathcal{GW}(Y)$ is defined to be $\deg f$, and is independent of the choice of U' because $\mathcal{GW}$ is unramified.

Definition 2.12. *Let $f : X \to Y$, and ρ be as in Assumption 2.8.* $\deg f$ *in* $\mathcal{GW}(Y)$ *is the section determined by the bilinear form* $\deg f|_{f^{-1}(U\cap U')}$. *If we wish to make the choice of relative orientation explicit, we write* $\deg(f, \rho)$ *for* $\deg f$.

We now relax the hypothesis that X and Y are proper schemes.

Assumption 2.13. *Let Y be a smooth k-scheme. Let $U \subseteq Y$ be an open subset such that $Y - U$ has codimension greater than or equal to 2 and $f^{-1}(U)$ has integral connected components. Suppose $f : X \to Y$ is a generically étale map such that its restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a proper, local complete intersection morphism equipped with a relative orientation $\rho : L^{\otimes 2} \xrightarrow{\cong} \omega_{f|_{f^{-1}(U)}}$.*

Since f is generically étale, the restriction $\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})} : \mathsf{f}^{-1}(\mathsf{U}) \to \mathsf{U}$ is a generically finite map. By Proposition 2.11, we can find a codimension 2 subset Z of U such that $\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U}-\mathsf{Z})} : \mathsf{f}^{-1}(\mathsf{U}-\mathsf{Z}) \to (\mathsf{U}-\mathsf{Z})$ is finite and flat.

Definition 2.14. *Let* $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$, *and* ρ *be as in Assumption 2.13.* $\deg \mathsf{f}$ *in* $\mathcal{GW}(\mathsf{Y})$ *is the section determined by* $\mathcal{GW}(\mathsf{Y}) \xrightarrow{\cong} \mathcal{GW}(\mathsf{U}-\mathsf{Z})$ *and the bilinear form* $\deg \mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U}-\mathsf{Z})}$ *of Definition 2.4. If we wish to make the choice of relative orientation explicit, we write* $\deg(\mathsf{f},\rho)$ *for* $\deg \mathsf{f}$.

Definition 2.15. *The degree* $\deg(\mathsf{f},\rho)(\mathsf{y})$ *of* f *at a point* y *of* Y *is defined to be the pullback of* $\deg \mathsf{f}$ *along* $\mathsf{y} : \operatorname{Spec} \mathsf{k}(\mathsf{y}) \to \mathsf{Y}$, *so* $\deg(\mathsf{f},\rho)(\mathsf{y})$ *in* $\mathrm{GW}(\mathsf{k}(\mathsf{y}))$. *When the relative orientation is clear from context, we also write* $\deg \mathsf{f}(\mathsf{y})$.

2.4. **Global degree of a map oriented away from codimension 1 equipped with lifting data.** Let k be a field of characteristic $\mathsf{p} > 0$, $\mathsf{p} \neq 2$, and let $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ be a map to a smooth k-scheme Y. Let $\mathsf{U} \subseteq \mathsf{Y}$ be a dense open subset. Suppose that the restriction $\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})} : \mathsf{f}^{-1}(\mathsf{U}) \to \mathsf{U}$ is a proper, generically finite, local complete intersection morphism equipped with a relative orientation $\rho : \mathsf{L}^{\otimes 2} \xrightarrow{\cong} \omega_{\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})}}$. Then $\deg \mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})}$ determines a section of $\mathcal{GW}(\mathsf{U})$ by Proposition 2.11, Definition 2.4, and the ability to extend sections of $\mathcal{GW}$ over the complements of codimension 2 closed subschemes of smooth schemes. Since $\mathcal{GW}$ is unramified, the restriction map $\mathcal{GW}(\mathsf{U}) \to \mathcal{GW}(\mathsf{Y})$ is injective. We give a condition on f ensuring that $\deg \mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})}$ extends to a (necessarily unique) section of $\mathcal{GW}(\mathsf{Y})$, which we will define to be the A^1-degree.

Assumption 2.16. *Let* Y *be a smooth* k*-scheme. Let* $\mathsf{U} \subseteq \mathsf{Y}$ *be a dense open subset. Suppose* $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ *is a generically étale map such that its restriction* $\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})} : \mathsf{f}^{-1}(\mathsf{U}) \to \mathsf{U}$ *is a proper, local complete intersection morphism. Let* $\rho : \mathsf{L}^{\otimes 2} \xrightarrow{\cong} \omega_{\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})}}$ *be an orientation of* $\mathsf{f}|_{\mathsf{f}^{-1}(\mathsf{U})} : \mathsf{f}^{-1}(\mathsf{U}) \to \mathsf{U}$. *Suppose that there exists the following data: a discrete valuation ring* Λ *and a lifting of* $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ *to a generically étale map* $\mathfrak{f} : \mathcal{X} \to \mathcal{Y}$ *of* Λ *schemes with* $\mathcal{Y} \to \operatorname{Spec} \Lambda$ *smooth of finite type. Suppose there is an open subset* $\mathcal{U} \subset \mathcal{Y}$ *such that* $\mathcal{U} \cap \mathsf{Y}$ *is dense in* U, *the intersection of the complement* $\mathcal{Y} - \mathcal{U}$ *with the generic fiber is codimension* ≥ 2, *and the intersection of* $\mathfrak{f}^{-1}(\mathcal{U})$ *with the generic fiber is integral. Suppose that* $\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U})} : \mathfrak{f}^{-1}(\mathcal{U}) \to \mathcal{U}$ *is a proper, local complete intersection morphism of schemes and there is a lift of* L *to a line bundle* $\mathcal{L}$ *on* $\mathfrak{f}^{-1}(\mathcal{U}) \subset \mathcal{X}$ *and a lift of* ρ *to an isomorphism* $\mathcal{L}^{\otimes 2} \cong \omega_{\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U})}}$.

For A a commutative ring, we let $\mathsf{W}(\mathsf{A}) = \mathrm{GW}(\mathsf{A})/\mathsf{M}$ denote the Witt group, defined to be the group completion of the isomorphisms of non-degenerate symmetric bilinear forms over A, modulo the ideal of metabolic forms. See for example [MH73]. In place of constructing an unramified sheaf $\mathcal{GW}$ over a more general base, we use the following results on the Witt and Grothendieck Witt groups to extend the section of $\mathcal{GW}(\mathsf{U})$ mentioned above.

Definition 2.17. *Let* $\mathcal{O}$ *be a regular local ring with quotient field* K. *Let* $\operatorname{Spec} \mathcal{O}^{(1)}$ *be the set of height one prime ideals of* $\mathcal{O}$. *For* $\mathsf{P} \in \operatorname{Spec} \mathcal{O}^{(1)}$, *let* $\mathcal{O}_{\mathsf{P}} \subset \mathsf{K}$ *denote the localization of* $\mathcal{O}$ *at* P. *For* $\mathsf{A} \subset \mathsf{K}$ *a subring, let* $\bar{\mathsf{W}}(\mathsf{A})$ *be the image of* $\mathsf{W}(\mathsf{A})$ *in* $\mathsf{W}(\mathsf{K})$. *We say that* purity holds for $\mathsf{W}(\mathcal{O})$ *if*

$$\bar{\mathsf{W}}(\mathcal{O}) = \cap_{\mathsf{P} \in \operatorname{Spec} \mathcal{O}^{(1)}} \bar{\mathsf{W}}(\mathcal{O}_{\mathsf{P}})$$

Theorem 2.18 (Colliot-Thélène and Sansuc [CTS79, Corollaire 2.5]). *Let $\mathcal{O}$ be a regular local ring of dimension ≤ 2 containing $1/2$. Then purity holds for $\mathsf{W}(\mathcal{O})$.*

Theorem 2.19 (Knebusch [Kne70, Satz 11.1.1]). *Let $\mathcal{O}$ be a Dedekind domain with $1/2 \in \mathcal{O}$. Let K be the quotient field of $\mathcal{O}$. Then purity holds for $\mathsf{W}(\mathcal{O})$. Moreover the map $\mathsf{W}(\mathcal{O}) \to \mathsf{W}(\mathsf{K})$ is injective.*

Remark 2.20. *For a ring containing $1/2$, the metabolic forms are the same as the ideal generated by the hyperbolic form $\langle 1\rangle + \langle -1\rangle$. Since the ideal in $\mathrm{GW}(\mathcal{O})$ generated by the hyperbolic form is the same as the subgroup generated by the hyperbolic form, the two purity results Theorems 2.18 and 2.19 extend in the evident manner to purity statements about $\mathrm{GW}(\mathcal{O})$.*

Proposition 2.21. *Let Λ be a discrete valuation ring with residue field k of characteristic $\neq 2$. Let $\pi : \mathcal{Y} \to \operatorname{Spec}\Lambda$ be a smooth morphism of finite type, and let $\mathcal{U} \subset \mathcal{Y}$ be an open subscheme satisfying the properties that*

- *the intersection $\mathcal{U}_{\mathsf{k}}$ with the closed fiber $\mathcal{Y}_{\mathsf{k}}$ is dense in $\mathcal{Y}_{\mathsf{k}}$,*
- *and the intersection of the complement $\mathcal{Y} - \mathcal{U}$ with the general fiber is codimension ≥ 2.*

Let $\mathsf{q} : \mathcal{E} \times \mathcal{E} \to \mathcal{O}_{\mathcal{U}}$ be a symmetric nondegenerate bilinear form over $\mathcal{U}$. Then the restriction of q to $\mathcal{GW}(\mathcal{U}_{\mathsf{k}})$ extends uniquely to a section in $\mathcal{GW}(\mathcal{Y}_{\mathsf{k}})$.

Proof. Let x be a codimension one point of $\mathcal{Y}_{\mathsf{k}}$, which we consider as a codimension two point of $\mathcal{Y}$, and let $\mathcal{O} = \mathcal{O}_{\mathcal{Y},\mathsf{x}}$. Let $\bar{\eta}$ be a generic point of $\mathcal{U}_{\mathsf{k}}$ in the connected component containing x. Let L be the field of rational functions on $\mathcal{Y}$, and let K be the ring of rational functions on $\mathcal{Y}_{\mathsf{k}}$. Since $\mathcal{U}_{\mathsf{k}}$ is dense in $\mathcal{Y}_{\mathsf{k}}$, K is also the ring of rational functions on $\mathcal{U}_{\mathsf{k}}$. Since $\mathcal{O}$ is a regular ring of dimension two, it follows from Theorem 2.18 and Remark 2.20 that q is in the image of $\mathrm{GW}(\mathcal{O})$ in $\mathrm{GW}(\mathsf{L})$. Moreover, by Theorem 2.19, the map $\mathrm{GW}(\mathcal{O}_{\mathcal{Y},\bar{\eta}}) \to \mathrm{GW}(\mathsf{L})$ is injective, so the restriction of q to $\mathrm{GW}(\mathcal{O}_{\mathcal{Y},\bar{\eta}})$ is in the image of $\mathrm{GW}(\mathcal{O}) \to \mathrm{GW}(\mathcal{O}_{\mathcal{Y},\bar{\eta}})$. Restricting to the fiber over $\operatorname{Spec}\mathsf{k}$, this implies that the image $\bar{\mathsf{q}}$ of q in $\mathrm{GW}(\mathsf{K})$ is in the image of $\mathrm{GW}(\mathcal{O}_{\mathcal{Y}_{\mathsf{k}},\mathsf{x}})$. Since x was an arbitrary codimension one point of $\mathcal{Y}_{\mathsf{k}}$, it follows that $\bar{\mathsf{q}}$ extends uniquely to a section of $\mathcal{GW}$ over $\mathcal{Y}_{\mathsf{k}}$ because $\mathcal{GW}$ is unramified. (Here $\mathcal{GW}$ denotes the unramified sheaf on smooth k-schemes.) □

Suppose $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ as in Assumption 2.16. In particular, $\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U})} : \mathfrak{f}^{-1}(\mathcal{U}) \to \mathcal{U}$ is generically étale, and proper. We may thus find an open subset $\mathcal{V}' \subset \mathfrak{f}^{-1}(\mathcal{U})$ on which $\mathfrak{f}$ is étale. As $\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U})}$ is proper, the image of $\mathfrak{f}^{-1}(\mathcal{U}) \setminus \mathcal{V}'$ under $\mathfrak{f}$ is closed in $\mathcal{U}$, whence has open complement $\mathcal{U}'$ in $\mathcal{U}$. Thus $\mathcal{U}'$ is also open in $\mathcal{Y}$. By construction, $\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U}')} : \mathfrak{f}^{-1}(\mathcal{U}') \to \mathcal{U}'$ is étale and proper, whence finite and flat [Sta18, 03WS, 01TH, 02GS]. We may thus apply Definition 2.4 and obtain $\deg(\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U}')} : \mathfrak{f}^{-1}(\mathcal{U}') \to \mathcal{U}')$ in $\mathcal{GW}(\mathcal{U}')$.

Moreover, the data and hypotheses given in Assumption 2.16 imply that the restriction of $\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U})}$ to the generic fiber satisfies Assumption 2.13. Let η denote the generic point of Λ and let $\mathfrak{f}_\eta$ denote this restriction. We may thus apply Definition 2.14 to $\mathfrak{f}_\eta$ and obtain $\deg(\mathfrak{f}_\eta)$ in $\mathcal{Y}_\eta$.

$\deg(\mathfrak{f}_\eta)$ and $\deg(\mathfrak{f}|_{\mathfrak{f}^{-1}(\mathcal{U}')} : \mathfrak{f}^{-1}(\mathcal{U}') \to \mathcal{U}')$ have the same restriction to $\mathcal{GW}(\mathcal{U}' \cap \mathcal{Y}_\eta)$ by construction and thus determine a section of $\mathcal{GW}(\mathcal{U}' \cup Y_\eta)$ whose restriction to $\mathcal{GW}(U)$ is $\deg(f, \rho)$. By Proposition 2.21, it follows that $\deg(f, \rho)$ in $\mathcal{GW}(U)$ extends to a unique section of $\mathcal{GW}(Y)$.

Definition 2.22. *For $f : X \to Y$ as in Assumption 2.16, the* A^1*-degree,* $\deg(f, \rho)$ *in* $\mathcal{GW}(U)$ *extends to a unique section of* $\mathcal{GW}(Y)$ *as above, which we define the* A^1*-degree.*

2.5. $\mathrm{GW}(\pi_0^{\mathbb{A}^1}(Y))$**-valued global degree and** $\mathrm{GW}(k)$**-valued global degree.** The degree of a map $M \to N$ between smooth, oriented, compact n-dimensional manifolds is an integer when N is connected. Without assuming that N is connected, the degree can be viewed as an integer valued function on the connected components $\pi_0(N)$ of N. We show the analogous results in our algebraic setting. For example, the $\mathrm{GW}(Y)$-valued degree of Definition 2.14 (or that of Definition 2.4 or 2.12) is pulled back from a unique element of the Grothendieck–Witt group $\mathrm{GW}(k)$ of k when Y is appropriately connected in an algebraic sense, for example, when Y is $\mathbb{A}^1$-connected. More generally, this degree is a section of $\mathrm{GW}(\pi_0^{\mathbb{A}^1}(Y))$, where $\pi_0^{\mathbb{A}^1}(Y)$ denotes the sheaf of $\mathbf{A}^1$-connected components. We recall the needed definitions and notations.

For smooth k-schemes, or more generally, for simplicial presheaves on smooth k-schemes, X and Y, let $[X, Y]_{\mathbf{A}^1}$ denote the set of $\mathbf{A}^1$-homotopy classes of maps from X to Y [MV99]. Let $\pi_0^{\mathbb{A}^1}(X)$ denote the Nisnevich-sheafification of the presheaf taking a smooth k-scheme U to $[U, X]_{\mathbf{A}^1}$. There is a canonical map

$$\phi_X : X \to \pi_0^{\mathbb{A}^1}(X).$$

For example, $\phi_k : \mathrm{Spec}\, k \to \pi_0^{\mathbb{A}^1}(\mathrm{Spec}\, k)$ is an isomorphism [AM11, Definition 2.1.4].

Definition 2.23. X *is* $\mathbf{A}^1$-connected *if the canonical map* $\pi_0^{\mathbb{A}^1}(X) \to \pi_0^{\mathbb{A}^1}(\mathrm{Spec}\, k) \overset{\phi_k}{\cong} \mathrm{Spec}\, k$ *is an isomorphism.*

Example 2.24. [AM11, Lemma 2.2.11, Lemma 2.2.5] *If X is a smooth k-variety that is covered by finitely many affine spaces $\mathbf{A}^n_k$, whose pairwise intersections all contain a k-point, then X is $\mathbf{A}^1$-connected.*

Remark 2.25. [AM11, Example 2.1.6] *If X is a smooth $\mathbf{A}^1$-connected k-scheme, then X has a rational point. For example,* $\mathrm{Spec}\, L$ *is not an $\mathbf{A}^1$-connected k-scheme for $k \subset L$ a finite separable extension.*

A Nisnevich sheaf $\mathcal{F}$ of sets is said to be $\mathbb{A}^1$-*homotopy invariant* if the projection $U \times \mathbf{A}^1 \to U$ induces bijection $\mathcal{F}(U) \to \mathcal{F}(U \times \mathbf{A}^1)$ for all smooth schemes U. For a smooth k-scheme X, the map ϕ_X induces the map

$$\phi_X^* : \mathrm{Hom}(\pi_0^{\mathbb{A}^1}(X), \mathcal{F}) \to \mathrm{Hom}(X, \mathcal{F}),$$

where, in this expression, Hom denotes the set of maps of presheaves on smooth k-schemes. The following proposition is known to experts, but we include a proof for completeness.

Proposition 2.26. *Let $\mathcal{F}$ be an $\mathbb{A}^1$-homotopy invariant Nisnevich sheaf of sets and let X be a smooth k-scheme. Then*

(1) ϕ_X^* *is a bijection.*
(2) If in addition X *is* $\mathbb{A}^1$*-connected, then the canonical map* $\mathcal{F}(k) \to \mathcal{F}(X)$ *is an isomorphism.*

Proof. The canonical commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\phi_X"] \arrow[dr] & & \pi_0^{\mathbb{A}^1}(X) \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

gives rise to the commutative triangle

$$\begin{tikzcd} \mathcal{F}(X) & & \operatorname{Hom}(\pi_0^{\mathbb{A}^1}(X), \mathcal{F}) \arrow[ll, "\phi_X^*"'] \\ & \mathcal{F}(\operatorname{Spec} k) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Since X is $\mathbb{A}^1$-connected, $\pi_0^{\mathbb{A}^1}(X) \to \operatorname{Spec} k$ is an isomorphism (of sheaves of sets). Thus we have that (1) implies (2).

For (1), it follows from [MV99, Corollary 3.22] that the map ϕ_X is an epimorphism of Nisnevich sheaves of sets. (To use [MV99, Corollary 3.22], let $I = \mathbb{A}^1, \mathcal{X} = X$, and $\mathcal{X}'$ be the $\mathbb{A}^1$-localization of X.) Thus ϕ_X^* is injective.

For $\alpha : X \to \mathcal{F}$, the natural transformations ϕ and $\pi_0^{\mathbb{A}^1}$ give the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\phi_X"'] & \mathcal{F} \arrow[d, "\phi_{\mathcal{F}}"] \\ \pi_0^{\mathbb{A}^1}(X) \arrow[r, "\pi_0^{\mathbb{A}^1}(\alpha)"'] & \pi_0^{\mathbb{A}^1}(\mathcal{F}) \end{tikzcd}$$

Since $\mathcal{F}$ is an $\mathbb{A}^1$-homotopy invariant sheaf of sets, $\phi_{\mathcal{F}} : \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{F})$ is an isomorphism of Nisnevich sheaves of sets (Lemma 2.27).Then $\alpha = \phi_X^*(\phi_{\mathcal{F}}^{-1} \circ \pi_0^{\mathbb{A}^1}(\alpha))$. Thus ϕ_X^* is surjective. □

Lemma 2.27. *Let* $\mathcal{F}$ *be an* $\mathbb{A}^1$*-homotopy invariant Nisnevich sheaf of sets. Then* $\phi_{\mathcal{F}} : \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{F})$ *is an isomorphism of Nisnevich sheaves of sets.*

Proof. The category of simplicial presheaves on smooth k-schemes can be given the structure of a simplicial model category with the global injective model structure, the injective local model structure with the Nisnevich topology, and the $\mathbf{A}^1$-model structure. Let L_{Nis} and $L_{\mathbb{A}^1}$ denote fibrant replacement functors for the injective local model structure with the Nisnevich topology and the $\mathbf{A}^1$-model structure, respectively. (See for example, [AWW17a, 2.2] [DHI04].)

All sheaves, thought of as discrete simplicial sheaves, are globally fibrant [Jar07, pg 10 3)]. Thus $\mathcal{F}$ is globally fibrant. Since a local weak equivalence of globally fibrant sheaves is a global weak equivalence, we have that the map

$$\mathcal{F}(U) \to L_{\mathrm{Nis}}\mathcal{F}(U) \tag{6}$$

is a weak equivalence of simplicial sets for all smooth k-schemes U [Jar07, pg 10 4)]. By [MV99, Proposition 3.19], it follows that $L_{\mathrm{Nis}}\mathcal{F}$ is $\mathbb{A}^1$-local (and thus fibrant in the $\mathbb{A}^1$-model structure). By [AWW17a, Proposition 2.2.1], the map $\mathcal{F} \to L_{\mathrm{Nis}}\mathcal{F}$ factors $\mathcal{F} \to L_{\mathbb{A}^1}\mathcal{F} \to L_{\mathrm{Nis}}\mathcal{F}$. The map $L_{\mathbb{A}^1}\mathcal{F} \to L_{\mathrm{Nis}}\mathcal{F}$ is an $\mathbb{A}^1$-weak equivalence by 2-out-of-3 and $L_{\mathbb{A}^1}\mathcal{F}$ and $L_{\mathrm{Nis}}\mathcal{F}$ are both fibrant in the injective local model structure. Thus the map $L_{\mathbb{A}^1}\mathcal{F} \to L_{\mathrm{Nis}}\mathcal{F}$ is a local whence global weak equivalence.The sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{F})$ is the Nisnevich sheaf associated to the presheaf $U \mapsto \pi_0|L_{\mathbb{A}^1}\mathcal{F}(U)|$. Since $L_{\mathbb{A}^1}\mathcal{F}(U) \simeq L_{\mathrm{Nis}}\mathcal{F}(U) \simeq \mathcal{F}(U)$ and $\mathcal{F}(U)$ is a set (i.e., discrete topological space), we have that the natural map $\mathcal{F}(U) \to \pi_0|L_{\mathbb{A}^1}\mathcal{F}(U)|$ is a bijection. Since $\mathcal{F}$ is a sheaf, it follows that the natural map $\phi_{\mathcal{F}} : \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{F})$ is an isomorphism.

□

$\mathcal{GW}$ is $\mathbb{A}^1$-homotopy invariant by [Mor12, Ch 2 and 3]. Applying Proposition 2.26, elements of $\mathcal{GW}(Y)$, such as our $\mathbb{A}^1$-degrees, are pulled back from a unique element of $\mathcal{GW}(\pi_0^{\mathbb{A}^1}(Y))$ for Y a smooth k-scheme. Thus we can refine our definition of degree to lie in $\mathcal{GW}(\pi_0^{\mathbb{A}^1}(Y))$.

Definition 2.28. *Let $f : X \to Y$ and ρ be as in Assumption 2.8 (respectively Assumption 2.13, Assumption 2.16). Define* $\deg(f, \rho)$ *in* $\mathcal{GW}(\pi_0^{\mathbb{A}^1}(Y))$ *to be the unique preimage of the degree of Definition 2.12 (respectively Definition 2.14, Definition 2.22) under the canonical bijection $\mathcal{GW}(\pi_0^{\mathbb{A}^1}(Y)) \to \mathcal{GW}(Y)$ of Proposition 2.26. When there is no danger of confusion, we will simply write* $\deg f$.

In particular, when Y in $\mathbf{A}^1$-connected, the degree lies in $\mathrm{GW}(k)$.

Corollary 2.29. *Suppose X is an $\mathbf{A}^1$-connected smooth scheme over k. Then the canonical map* $\mathrm{GW}(k) \to \mathcal{GW}(X)$ *is an isomorphism.*

Proof. This follows from Proposition 2.26 and the $\mathbf{A}^1$-homtopy invariance of $\mathcal{GW}$ [Mor12, Ch 2 and 3 e.g. Theorem 3.37]. □

Definition 2.30. *Let $f : X \to Y$ and ρ be as in Assumption 2.8 and suppose that Y is $\mathbf{A}^1$-connected. Define* $\deg(f, \rho)$ *in* $\mathrm{GW}(k)$ *to be the unique preimage of the degree of Definition 2.12 under the canonical bijection* $\mathrm{GW}(k) \to \mathcal{GW}(Y)$.

There are alternate connectivity conditions in $\mathbf{A}^1$-homotopy theory which also give rise to a $\mathrm{GW}(k)$-valued degree.

The notion of $\mathbf{A}^1$-chain connected varieties was defined in [AM11, Definition 2.2.2] as follows. Let L be a finitely generated separable extension of k, which is defined to mean that there exists a subextension $k \subseteq E \subseteq L$ such that E is purely transcendental over k and L is separable and algebraic over E. Let y and y' be L-points of Y.

Definition 2.31. [AM11, Definition 2.2.2] *An* elementary $\mathbf{A}^1$-equivalence between y and y' *is a map $f : \mathbf{A}^1_L \to Y_L$ such that $f(t) = y$ and $f(t') = y'$ for some t, t' in $\mathbf{A}^1(L)$.*

Elementary $\mathbf{A}^1$-equivalence generates an equivalence relation $\sim$ on $\mathsf{Y}(\mathsf{L})$. Denote the quotient $\mathsf{Y}(\mathsf{L})/\sim$ by $\pi_0^{\mathbf{A}^1,\mathrm{ch}}(\mathsf{Y})(\mathsf{L})$.

Definition 2.32. *Y is* $\mathbf{A}^1$-chain connected *if for every finitely generated separable field extension L/k, the set of equivalence classes $\pi_0^{\mathbf{A}^1,\mathrm{ch}}(\mathsf{Y})(\mathsf{L}) = \mathsf{Y}(\mathsf{L})/\sim$ consists of exactly 1 element.*

For Y a smooth, proper variety over a field k, it is a Theorem of A. Asok and F. Morel that $\mathbf{A}^1$-chain connected and $\mathbf{A}^1$-connectedness are equivalent [AM11, Theorem 2]. More generally, there is an evident map $\psi_{\mathsf{Y},\mathsf{L}} : \pi_0^{\mathbf{A}^1,\mathrm{ch}}(\mathsf{Y})(\mathsf{L}) \to \pi_0^{\mathbf{A}^1}(\mathsf{Y})(\mathsf{L})$ sending the class of $y \in \mathsf{Y}(\mathsf{L})$ in $\pi_0^{\mathbf{A}^1,\mathrm{ch}}(\mathsf{Y})(\mathsf{L})$ to the class $[y] \in \pi_0^{\mathbf{A}^1}(\mathsf{Y})(\mathsf{L})$. Asok and Morel show that if Y is finite type and proper over k, then $\psi_{\mathsf{Y},\mathsf{L}}$ is an isomorphism for all finitely generated separable extensions L of k [AM11, Theorem 2.4.3].

Since the pullback along field extensions of finite odd degree induces an injection on Grothendieck–Witt groups and we are interested in a $\mathrm{GW}(\mathsf{k})$-valued degree, we weaken the notion of $\mathbf{A}^1$-chain connectedness as follows.

Definition 2.33. *Y is* $\mathbf{A}^1$-odd extended chain connected *if for every finitely generated separable field extension L/k, and every pair y, y' in $\mathsf{Y}(\mathsf{L})$, there exists a finite extension $\mathsf{L} \subseteq \mathsf{L}'$ of odd degree such that $y \sim y'$ in $\mathsf{Y}(\mathsf{L}')$.*

We remark that an n-dimensional smooth k-scheme Y has many closed points with separable residue field: for each y in Y there is an open neighborhood U and an étale map $\phi : \mathsf{U} \to \mathbf{A}^n_{\mathsf{k}}$ [sga03, Deféninition II.1.1]. The points of $\mathbf{A}^n_{\mathsf{k}}$ with separable residue field are dense. The image $\phi(\mathsf{U})$ is open [Sta18, Tag 01U2 Lemma 29.24.10], and therefore contains points with separable residue field. For all u of U such that $\phi(u)$ has separable residue field, $\mathsf{k} \subseteq \mathsf{k}(u)$ is separable.

Lemma 2.34. *Let Y be a smooth proper k-scheme which is $\mathbf{A}^1$-odd extended chain connected and assume that $\mathsf{Y}(\mathsf{k}) \neq \emptyset$. Then for any section β of $\mathcal{GW}(\mathsf{Y})$, there is a unique b in $\mathrm{GW}(\mathsf{k})$, such that for every point $y : \operatorname{Spec} \mathsf{k}(y) \to \mathsf{Y}$ as in the commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} \mathsf{k}(y) \arrow[rr, "y"] \arrow[dr, "p"'] & & \mathsf{Y} \arrow[dl] \\ & \operatorname{Spec} \mathsf{k} & \end{tikzcd}$$

and such that $\mathsf{k} \subseteq \mathsf{k}(y)$ is separable, we have

$$y^*\beta = p^*\mathsf{b}.$$

Proof. Choose y_0 in $\mathsf{Y}(\mathsf{k})$. We must let $\mathsf{b} = y_0^*\beta$, showing uniqueness. Let $y : \operatorname{Spec} \mathsf{k}(y) \to \mathsf{Y}$ be a closed point. By Corollary 2.29 and Example 2.24, if $i : \mathsf{k}(y) \subseteq \mathsf{L}'$ is an extension of fields, y' in $\mathsf{Y}(\mathsf{L}')$ and $y' \sim (y \circ i)$, then

$$i^*y^*\beta = (y')^*\beta.$$

We may view $y_0 \circ p$ as an element of $\mathsf{Y}(\mathsf{k}(y))$. Since Y is $\mathbf{A}^1$-odd extended chain connected, there is an extension $i : \mathsf{k}(y) \subseteq \mathsf{L}'$ of odd degree such that $y \circ i \sim y_0 \circ p \circ i$ in $\mathsf{Y}(\mathsf{L}')$. Thus

$$i^*y^*\beta = i^*p * \mathsf{b}$$

in $\mathrm{GW}(L')$. An odd degree field extension induces an injection on GW [Lam05, Chapter VII, Corollary 2.6][2], therefore $y^*\beta = p^*b$ as claimed. $\square$

Definition 2.35. *Let $f : X \to Y$ and ρ be as in Assumption 2.8. If Y is additionally $\mathbf{A}^1$-odd extended chain connected and $Y(k) \neq \emptyset$, then define the degree of f, denoted $\deg(f, \rho)$ or $\deg f$, to be the b associated by Lemma 2.34 to the section of $\mathcal{GW}(Y)$ given by the degree of f defined in Definition 2.12.*

2.6. **Connectivity and restriction of scalars.** We will have use of the degrees of maps whose targets are restrictions of scalars. Since the $\mathbf{A}^1$-degree lands in $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(-))$ applied to the target, we give a result on the connectivity of a restriction of scalars in this section.

We extend the notation of $\mathbf{A}^1$-chain connected components $\pi_0^{\mathbf{A}^1,\mathrm{ch}}(Y)(L)$ of a finite type k-scheme Y over a field L (see Definition 2.32) to products of fields by defining

$$\pi_0^{\mathbf{A}^1,\mathrm{ch}}(Y)(\prod_{i=1}^{r} L_i) := \prod_{i=1}^{r} \pi_0^{\mathbf{A}^1,\mathrm{ch}}(Y)(L_i).$$

Define $\psi_{Y,\prod_i L_i} : \pi_0^{\mathbf{A}^1,\mathrm{ch}}(Y)(\prod_i L_i) \to \pi_0^{\mathbf{A}^1}(Y)(\prod_i L_i)$ by sending the class of an r-tuple of points $\prod_{i=1}^r y_i \in Y(\prod_{i=1}^r L)$ to the class $[\prod_{i=1}^r y_i]$ in $\pi_0^{\mathbf{A}^1}(Y)(\prod_i L_i)$ as above. Clearly $(Y, L) \mapsto \pi_0^{\mathbf{A}^1,\mathrm{ch}}(Y)(L)$ is (covariantly) functorial in Y and L and $\psi_{Y,L}$ is natural in (Y, L).

For a field F, let $\mathbf{Sch}_F$ denote the category of finite type separated F-schemes.

Let $k \subset L$ be a finite separable field extension. We have the Weil restriction functor $\mathrm{Res}_{L/k} : \mathbf{Sch}_L \to \mathbf{Sch}_k$, which is right adjoint to the extension of scalars functor $X \mapsto X_L := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$; passing to the limit over suitable open subschemes, this induces the natural isomorphism $X(L \otimes_k F) \cong \mathrm{Res}_{L/k}(X)(F)$ for all finitely generated extensions F of k. The isomorphisms $X(L \otimes_k F) \cong \mathrm{Res}_{L/k}(X)(F)$ and $X(\mathbf{A}^1_{L\otimes_k F}) \cong \mathrm{Res}_{L/k}(X)(\mathbf{A}^1_F)$ are natural with respect with the 0- and 1-sections $i_0, i_1 : \mathrm{Spec}(-) \to \mathbf{A}^1_{(-)}$, and thus induce an isomorphism, natural in F and X

$$\rho_{L,X,F} : \pi_0^{\mathbf{A}^1,\mathrm{ch}}(X)(L \otimes_k F) \xrightarrow{\cong} \pi_0^{\mathbf{A}^1,\mathrm{ch}}(\mathrm{Res}_{L/k}(X)(F)) \tag{7}$$

Lemma 2.36. *Let $k \subset L$ be a finite separable field extension. For X a finite type, proper L-scheme and F a finitely generated separable extension of k, the isomorphism (7) induces an isomorphism*

$$\pi_0^{\mathbf{A}^1}(\psi_X)(L \otimes_k F) : \pi_0^{\mathbf{A}^1}(X)(L \otimes_k F) \xrightarrow{\cong} \pi_0^{\mathbf{A}^1}(\mathrm{Res}_{L/k}(X))(F),$$

natural in F and X.

Proof. If F is a finitely generated separable extension of k, then $L \otimes_k F$ is a finite product of finitely generated separable extensions of k; if X is a proper L-scheme, then $\mathrm{Res}_{L/k}(X)$ is a proper k-scheme. We apply [AM11, Theorem 2.4.3], which together with the isomorphism

[2] The reference shows the claim on Witt groups, from which the injection on GW follows from the isomorphism $\mathrm{GW} \cong \mathrm{W} \times_{\mathbb{Z}/2} \mathbb{Z}$.

(7) gives us the sequence of isomorphisms

$$\pi_0^{\mathbf{A}^1}(X)(L \otimes_k F) \xrightarrow{\psi^{-1}_{X,L\otimes_k F}} \pi_0^{\mathbf{A}^1,\mathrm{ch}}(X)(L \otimes_k F)$$
$$\xrightarrow{\rho_{k,L,X,F}} \pi_0^{\mathbf{A}^1,\mathrm{ch}}(\mathrm{Res}_{L/k}(X))(F) \xrightarrow{\psi_{\mathrm{Res}_{L/k}(X),F}} \pi_0^{\mathbf{A}^1}(\mathrm{Res}_{L/k}(X))(F),$$

natural in F and X. □

Proposition 2.37. *Let $k \subset L$ be a finite separable field extension. Let X be a smooth proper L-scheme. If X is $\mathbf{A}^1$-connected, then so is $\mathrm{Res}_{L/k}(X)$.*

Proof. By Lemma 2.36 and (7), $\mathrm{Res}_{L/k}(X)$ is $\mathbf{A}^1$-chain connected. (See Definition 2.32). Since X is a smooth, proper L-scheme, $\mathrm{Res}_{L/k}(X)$ is a smooth proper k-scheme. By [AM11, Theorem 2], it follows that $\mathrm{Res}_{L/k}(X)$ is $\mathbf{A}^1$ connected. □

We record the following well-known fact.

Proposition 2.38. *Let X and Y be smooth k-schemes which are $\mathbf{A}^1$-connected. Then $X \times_k Y$ is $\mathbf{A}^1$-connected.*

Proof. $\pi_0^{\mathbf{A}^1}(X \times_k Y)$ is the sheaf associated to the presheaf sending a smooth k-scheme U to $\pi_0(\mathrm{L}_{\mathbb{A}^1}(X \times_k Y)(U))$. The functor $\mathrm{L}_{\mathbb{A}^1}$ commutes with finite products (see e.g. [AWW17b, 2.2.1(iii)]). It follows that

$$\pi_0(\mathrm{L}_{\mathbb{A}^1}(X \times_k Y)(U)) \cong \pi_0((\mathrm{L}_{\mathbb{A}^1}(X) \times \mathrm{L}_{\mathbb{A}^1}(Y))(U)) \cong$$
$$\pi_0(\mathrm{L}_{\mathbb{A}^1}(X)(U) \times \mathrm{L}_{\mathbb{A}^1}(Y)(U)) \cong \pi_0\mathrm{L}_{\mathbb{A}^1}X(U) \times \pi_0\mathrm{L}_{\mathbb{A}^1}X(U).$$

Since sheafification preserves finite limits, it follows that $\pi_0^{\mathbf{A}^1}(X \times_k Y) \cong \pi_0^{\mathbf{A}^1}(X) \times \pi_0^{\mathbf{A}^1}(Y)$. □

3. Local degree

Now suppose that $f : X \to Y$ is a map of smooth n-dimensional schemes over k or a map satisfying one of Assumptions 2.8 2.13 or 2.16. Let x be a point of X with image $y = f(x)$. Suppose there are Zariski open neighborhoods W and U of x and y, respectively, such that $f(W) \subset U$ and the restriction $f|_W : W \to U$ is finite and oriented by $\rho : L^{\otimes 2} \xrightarrow{\cong} \omega_{f|_W}$. By Proposition 2.11 it is possible to find many x and y which admit such a U,W and ρ. We will define the local degree $\deg_x(f, \rho)$ in $\mathrm{GW}(k(y))$ under such circumstances in this section. We also use the notation $\deg_x f$ for $\deg_x(f, \rho)$ when there is no danger of confusion. The degree defined in Section 2 will be shown to be a sum of local degrees in Proposition 3.2, and we will give a formula to compute $\deg_x f$ with the Jacobian in Proposition 3.8.

3.1. **Definition and properties.** To simplify notation, we let g denote the restriction $g = f|_W : W \to U$, where f,W, and U are as in the beginning of the section. Let $g \times y : U \times_y \mathrm{Spec}\, k(y) \to \mathrm{Spec}\, k(y)$ denote the pullback of g along $y : \mathrm{Spec}\, k(y) \to U$ as in the pullback diagram

$$\begin{tikzcd} U \times_y \operatorname{Spec} k(y) \arrow[r] \arrow[d, "g\times y"] & W \arrow[d, "g=f|_W"] \\ \operatorname{Spec} k(y) \arrow[r, "y"] & U. \end{tikzcd}$$

The fiber $g^{-1}(y) \cong U \times_y \operatorname{Spec} k(y)$ is the coproduct

$$U \times_y \operatorname{Spec} k(y) \cong \coprod_{z\in W: f(z)=y} \mathcal{O}_{g^{-1}(y),z}.$$

For every z in W mapping to y, let g_z denote the composition of the inclusion $\operatorname{Spec} \mathcal{O}_{f^{-1}(y),z} \to g^{-1}(y)$ with $g \times y$.

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{f^{-1}(y),z} \arrow[r] \arrow[d, "g_z"'] & W \arrow[d, "g"] \\ \operatorname{Spec} k(y) \arrow[r, "y"] & U. \end{tikzcd}$$

Denote the fiber of the sheaf g_*L at y by $g_*L(y)$. Since g is finite, the canonical map $g_*L(y) \to (g \times y)_*L$ is an isomorphism. By a slight abuse of notation, we also let L denote its pull back to $\operatorname{Spec} \mathcal{O}_{f^{-1}(y),z}$. We have a canonical isomorphism

$$g_*L(y) \cong \oplus_{z\in W: f(z)=y} (g_z)_*L. \tag{8}$$

Recall that the pullback of the pairing $\deg(g, \rho)$ of Definition 2.4 to $\operatorname{Spec} k(y)$ is denoted $\deg(g, \rho)(y)$, as in Definition 2.15. Each of the direct summands of the isomorphism (8) are perpendicular under $\deg(g, \rho)(y)$.

Definition 3.1. *Let* $\deg_x(f, \rho)$ *in* $\mathrm{GW}(k(y))$ *be the restriction of* $\deg(g, \rho)(y)$ *to* $(g_z)_*L$. *When the relative orientation is clear from context, we also write* $\deg_x f$.

For f satisfying Assumption 2.8, there is an open subset $U \subset Y$ such that $f|_{f^{-1}(U)}$ is finite and oriented by Proposition 2.11. For any x in $f^{-1}(U)$, we may take $W = f^{-1}(U)$ and we have just shown that the global degree is the sum of local degrees.

Proposition 3.2. *For all* y *in* U*, there is an equality* $\deg f(y) = \sum_{x\in f^{-1}(y)} \deg_x f$ *in* $\mathrm{GW}(k(y))$.

Given a field extension $k(y) \subseteq E$, let $f_E|_{U_E} : W_E \to U_E$ denote $f|_W \otimes_k E$. We have the commutative diagram

$$\begin{tikzcd} W \arrow[d, "f|_W"'] & W_E \arrow[l, "\pi'"'] \arrow[d, "f_E|_{W_E}"] \\ U & U_E \arrow[l, "\pi"] \end{tikzcd}$$

There is a canonical point $\tilde{x}$ of W_E mapping to x under π', and $f_E(\tilde{x}) = \tilde{y}$, the canonical point $\tilde{y}$ of U_E mapping to y under π. The pullback of ρ determines an orientation $(\pi')^*\rho$ of $f_E|_{W_E}$ because there is a canonical isomorphism $(\pi')^*\omega_{f|_W} \cong \omega_{f_E|_{W_E}}$ [Sta18, Lemma 29.31.10 Tag 01UM]. Let $\deg_x(f, \rho) \otimes \mathrm{GW}(E)$ denote the image of $\deg_x(f, \rho)$ under the map $\mathrm{GW}(k(y)) \to \mathrm{GW}(E)$.

Proposition 3.3. $\deg_{\tilde{x}}(f_E, (\pi')^*\rho) = \deg_x(f, \rho) \otimes \mathrm{GW}(E)$ *in* $\mathrm{GW}(E)$.

Proof. Let L denote the line bundle on W of the orientation ρ. By the proof of Proposition 2.7, there is an isomorphism $(f_E|_{W_E})_*(\pi')^*L \cong (\pi^* f|_W)_*L$ identifying the forms defining $\pi^*\deg(f,\rho)$ and $\deg(f_E,(\pi')^*\rho)$. Under this isomorphism the subspaces $((f_E)_{\tilde{x}})_*L$ and $\pi^*(f_x)_*L$ are identified, proving the claim. $\square$

By Proposition 3.3, the computation of $\deg_x(f,\rho)$ reduces to the case where $y = f(x)$ is a rational, because it may be computed after basechange to $k(y)$, and in particular we may assume that y is a closed point. We now give such a method of computation for $\deg_x f$ for x a closed point with $k \subseteq k(x)$ separable.

As above, let W and U be smooth k-schemes of dimension n, let x be a point of X, and let $f|_W : W \to U$ be finite and oriented by ρ. Suppose that $y = f(x)$ is a closed point with $k \subseteq k(x)$ separable. Then $f|_W^{-1}(y) \hookrightarrow W$ is a closed immersion. Consider the pullback diagram

$$\begin{tikzcd} f|_W^{-1}(y) \arrow[r, "y'"] \arrow[d, "f'|_W"'] & W \arrow[d, "f|_W"] \\ \operatorname{Spec} k(y) \arrow[r, "y"] & U \end{tikzcd} \tag{9}$$

Lemma 3.4. *The map $f'|_W$ is a finite, flat, local complete intersection morphism.*

Proof. $f'|_W$ is a finite and flat because these properties are stable under pullback. Since $\operatorname{Spec} k(y)$ and U are regular schemes, the finite type map $y : \operatorname{Spec} k(y) \to U$ is a local complete intersection morphism [Sta18, 0E9K]. Since $f|_W$ is finite and W and U are smooth and dimension n, $f|_W$ is flat [Mat89, Theorem 23.1 p.179]. Thus the pullback y' is a local complete intersection morphism [Sta18, 069I]. Since W and U are smooth k-schemes, $f|_W$ is a local complete intersection morphism. Thus the composition $f|_W \circ y' = y \circ f'|_W$ is lci as well. Since $U \to \operatorname{Spec} k$ is smooth, it follows that the structure map for $f|_W^{-1}(y)$ over $\operatorname{Spec} k$ is lci. Since $\operatorname{Spec} k(y) \to \operatorname{Spec} k$ is smooth, it follows that $f'|_W$ is lci as claimed [Sta18, 069M] $\square$

Since $f'|_W$ is flat, the square (9) and [Sta18, Tag08QL Lemma 89.6.1.] define a canonical isomorphism

$$\omega_{f'|_W} \cong (y')^*\omega_{f|_W}. \tag{10}$$

Since $f|_W$ is finite, x determines a closed and open subscheme $\operatorname{Spec} \mathcal{O}_{f^{-1}(y),x} \hookrightarrow f|_W^{-1}(y)$ of $f|_W^{-1}(y)$. By [Sta18, Tag 0638, Lemma 31.21.5. and Lemma 31.21.3.], the inclusion of a closed and open component in a locally Noetherian scheme is a local complete intersection morphism, because Koszul-regular immersions are lci. Define $f_x : \operatorname{Spec} \mathcal{O}_{f^{-1}(y),x} \to \operatorname{Spec} k(y)$ to be the composition of $\operatorname{Spec} \mathcal{O}_{f^{-1}(y),x} \hookrightarrow f|_W^{-1}(y)$ and $f'|_W$, so in particular f_x is finite, flat, lci and fits into the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{f^{-1}(y),x} \arrow[r, "i_{f,x}"] \arrow[d, "f_x"'] & W \arrow[d, "f|_W"] \\ \operatorname{Spec} k(y) \arrow[r, "y"] & U \end{tikzcd} .$$

The isomorphism (10) defines an isomorphism $\omega_{f_x} \cong i_{f,x}^*\omega_{f|_W}$, whence $i_{f,x}^*\rho$ defines a relative orientation of f_x.

Proposition 3.5. *Let W and U be smooth k-schemes of dimension n, let x be a point of X, and let $f : W \to U$ be finite and oriented by ρ. Suppose that $y = f(x)$ is a closed point with $k(y) \subseteq k(x)$ separable. Then*

$$\deg_x(f, \rho) = \deg(f_x, i_{f,x}^*\rho).$$

Proof. f is flat by [Mat89, Theorem 23.1 p.179]. Both $\deg_x(f, \rho)$ and $\deg(f_x, i_{f,x}^*\rho)$ are bilinear forms on the $k(y)$-vector space $(f_x)_* i_{f,x}^* L$. $\deg(f_x, i_{f,x}^*\rho)$ is obtained by composing

$$((f_x)_* i_{f,x}^* L)^{\otimes 2} \to (f_x)_* i_{f,x}^*(L^{\otimes 2}) \xrightarrow{i_{f,x}^*\rho} (f_x)_*\omega_{f_x} \to k(y),$$

and $\deg_x(f, \rho)$ is the composition

$$((f_x)_* i_{f,x}^* L)^{\otimes 2} \to ((f|_W)_* L(y))^{\otimes 2} \to (f|_W)_*(L^{\otimes 2})(y) \xrightarrow{\rho} (f|_W)_*\omega_{f|_W}(y) \to k(y).$$

By the commutative diagram

$$\begin{tikzcd}
((f|_W)_* L)^{\otimes 2} \arrow[r] & y^*(f|_W)_*(L^{\otimes 2}) \arrow[r, "\rho"] & y^*(f|_W)_*\omega_{f|_W} \\
((f_x)_* i_{f,x}^* L)^{\otimes 2} \arrow[r] \arrow[u] & (f_x)_* i_{f,x}^*(L^{\otimes 2}) \arrow[r, "i_{f,x}^*\rho"] \arrow[u] & (f_x)_*\omega_{f_x} \arrow[u]
\end{tikzcd}$$

it suffices to check that the trace maps $(f_x)_*\omega_{f_x} \to k$ and $(f|_W)_*\omega_{f|_W}(y) \to k$ are compatible in the sense that the outermost rectangle in the diagram

$$\begin{tikzcd}
(f|_W)_*\omega_{f|_W}(y) \arrow[r] & k(y) \\
(f_y)_*\omega_{f_y} \arrow[r] \arrow[u] & k(y) \arrow[u, "\mathrm{id}"'] \\
(f_x)_*\omega_{f_x} \arrow[r] \arrow[u] & k(y) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

is commutative. To see this, let $f_y : f^{-1}(y) = \operatorname{Spec} \mathcal{O}_{f^{-1}(y),x} \to \operatorname{Spec} k(y) \times_U W \to \operatorname{Spec} k(y)$ be the pullback of $f|_W$ along y. The associated trace map $(f_y)_*\omega_{f(y)} \to \operatorname{Spec} k(y)$ fits in the commutative diagram above, so we may check the commutivity of the upper and lower squares. The commutativity of the lower follows from [Con00, Lemma 3.4.3 TRA1] applied to the composition $\operatorname{Spec} \mathcal{O}_{f^{-1}(y),x} \to \operatorname{Spec} k(y) \times_U W \to \operatorname{Spec} k(y)$. For the upper square, the trace map for finite flat maps commutes with base change as can be seen by the description "evaluate at 1" of [Con00, (3.4.7) p147]. In more detail, let $A \subset B$ be ring map corresponding to a finite flat map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ and let y be a point of $\operatorname{Spec} A$. View $\operatorname{Hom}_A(B, A)$ as a coherent sheaf on $\operatorname{Spec} A$. Then there is a canonical isomorphism $f_*\omega_f \cong \operatorname{Hom}(B, A)$ and the trace map $\operatorname{Tr}_{B/A} : \operatorname{Hom}_A(B, A) \to A$ is evaluation at 1. The pull back of $\operatorname{Tr}_{B/A}$ by $\operatorname{Spec} k(y) \to \operatorname{Spec} A$ is the evaluation at 1 map $\operatorname{Hom}_{k(y)}(B_y, k(y))$ as claimed. $\square$

3.2. **Computation with the Jacobian.** Let $f : X \to Y$ be a oriented map between smooth, connected schemes of dimension n. So we have a line bundle L on X, and an isomorphism

$$\rho : \mathrm{Hom}(\wedge^n TX, f^* \wedge^n TY) \to L^{\otimes 2}.$$

Then f induces a map Tf on tangent bundles and a global section

$$\det Tf \in \mathrm{Hom}(\wedge^n TX, f^* \wedge^n TY).$$

Taking the image under the ρ gives a global section $\rho(\det Tf)$ of $L^{\otimes 2}$.

Construction 3.6. A section of the *square* of a line bundle determines a canonical element of $\mathcal{O}_X/(\mathcal{O}_X^*)^2$. Namely, suppose σ is a section of $L^{\otimes 2}$. Around any point x, choose a local trivialization of L, identifying σ with an element of $\mathcal{O}_{X,x}$. Any two choices of local trivialization will change this element by the square of a unit in $\mathcal{O}_{X,x}$.

Definition 3.7. *The* Jacobian Jf *of* f *is the section of* $\mathcal{O}/(\mathcal{O}^*)^2$ *corresponding to* $\rho(\det Tf)$ *by Construction 3.6:*

$$Jf = \rho(\det Tf) \in \mathcal{O}/(\mathcal{O}^*)^2.$$

Let Jf_x *(respectively* $Jf(x)$*) be the image of* Jf *in* $\mathcal{O}_{X,x}/(\mathcal{O}_{X,x}^*)^2$ *(respectively* $k(x)/(k(x))^*$*).*

Proposition 3.8. *If* f *is étale at* x*, then* $\deg_x f = \mathrm{Tr}_{k(x)/k(y)}\langle Jf(x)\rangle$.

Proof. Since f is étale at x, the canonical map $\mathcal{O}_{f^{-1}(y),x} \to k(x)$ is an isomorphism and $k(y) \subseteq k(x)$ is a separable extension. The form $\deg(f,\rho)(y)$ on $f_* L_{f^{-1}(y)}$ is defined

$$f_* L_{f^{-1}(y)} \times f_* L_{f^{-1}(y)} \to f_* L^{\otimes 2}_{f^{-1}(y)} \overset{\rho}{\cong} f_*(\omega_f)_{f^{-1}(y)} \xrightarrow{\mathrm{Tr}} \mathcal{O}_Y(y) \cong k(y)$$

where the first map is the canonical map associated to the tensor product. The local degree $\deg_x f$ is then the restriction of $\deg(f,\rho)(y)$ to $f_* L_{f^{-1}(y),x}$. Choosing any local trivialization of L around x, we obtain an isomorphism $L_{f^{-1}(y),x} \cong \mathcal{O}_{f^{-1}(y),x} \cong k(x)$. The canonical map associated to the tensor product becomes multiplication on $k(x)$. Since f is finite when restricted to a neighborhood of x, there is a canonical isomorphism

$$(\omega_f)_x \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}) \tag{11}$$

from the adjunction $f_* \dashv f^!$. Under this identification $\mathrm{Tr} : (f_x)_*(\omega_f)_x \to \mathcal{O}_{Y,y}$ is evaluation at 1.

Since f is étale at x, the isomorphism (11) sends the section $\det Tf$ of $(\omega_f)_x$ to $\mathrm{Tr} : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ [SS75, (4.2)Satz], where here $\mathrm{Tr} : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ denotes the trace associated to the finite étale algebra $\mathcal{O}_{Y,y} \subseteq \mathcal{O}_{X,x}$. Thus for any a in $k(x) \cong \mathcal{O}_{f^{-1}(y),x}$, the composition

$$(f_x)_* L^{\otimes 2}_{f^{-1}(y),x} \overset{\rho}{\cong} (f_x)_*(\omega_f)_{f^{-1}(y),x} \xrightarrow{\mathrm{Tr}} \mathcal{O}_Y(y) \cong k(y)$$

sends $a\rho(\det Tf(x))$ to $\mathrm{Tr}_{k(x)/k(y)}\, a$, where $\mathrm{Tr}_{k(x)/k(y)} : k(x) \to k(y)$ denotes the trace of the finite étale algebra $k(y) \subseteq k(x)$. Remembering the chosen local trivialization of L around x, the bilinear form

$$k(x) \times k(x) \cong (f_x)_* L_{f^{-1}(y),x} \times (f_x)_* L_{f^{-1}(y),x} \to (f_x)_* L^{\otimes 2}_{f^{-1}(y),x} \overset{\rho}{\cong} (f_x)_*(\omega_f)_{f^{-1}(y),x} \xrightarrow{\mathrm{Tr}} \mathcal{O}_Y(y) \cong k(y) \tag{12}$$

represents the local degree $\deg_x f$. Since we have fixed a local trivialization of L around x, we have $\rho(\det Tf(x)) \in k(x)$. From the above, it follows that for any a, b in $k(x)$, the form (12) sends (a, b) in $k(x) \times k(x)$ to

$$\mathrm{Tr}_{k(x)/k(y)}(\frac{ab}{\rho(\det Tf(x))}).$$

Thus $\deg_x f = \mathrm{Tr}_{k(x)/k(y)}\langle\frac{1}{Jf(x)}\rangle = \mathrm{Tr}_{k(x)/k(y)}\langle Jf(x)\rangle$ as claimed.

□

Taking x to be the generic point of X, this shows the Jacobian can be used to compute $\deg f$.

Corollary 3.9. *Let $f : X \to Y$ be a separable map between smooth, proper connected k-schemes of dimension n. Suppose there exists a closed subset Z of Y of codimension at least 2 such that the restriction of f to $f^{-1}(Y-Z)$ is oriented. Let η denote the generic point of X. Then $\mathrm{Tr}_{k(X)/k(Y)}\langle Jf(\eta)\rangle$ is in the image of $\mathcal{GW}(Y) \subseteq \mathrm{GW}(k(Y))$ and the degree of f is given by*

$$\deg f = \mathrm{Tr}_{k(X)/k(Y)}\langle Jf(\eta)\rangle.$$

Moreover, if Y is either $\mathbf{A}^1$-connected or Y has a k-point and is $\mathbf{A}^1$-odd extended chain connected, then $\mathrm{Tr}_{k(X)/k(Y)}\langle Jf(\eta)\rangle$ is in the image of the pull-back $\mathrm{GW}(k) \to \mathrm{GW}(k(Y))$.

Proof. f is étale at η because f is a separable morphism, so we may apply Proposition 3.8 to $x = \eta$. The first statement then follows from Proposition 3.3 and the fact that $\mathcal{GW}$ is an unramified sheaf. The second assertion follows from Corollary 2.29 in the first case and Lemma 2.34 in the second case. □

Combining Proposition 3.8 with Proposition 3.2 implies:

Corollary 3.10. *Let y be a regular value of f. Then $\deg f = \sum_{x \in f^{-1} y} \mathrm{Tr}_{k(x)/k(y)}\langle J(f)\rangle$ in $\mathrm{GW}(k(y))$.*

4. Counts of Rational Curves

In the remaining sections, we give quadratically enriched counts of rational curves on del Pezzo surfaces passing through the appropriate number of points by taking the $\mathbf{A}^1$-degree of maps from moduli spaces of such curves.

4.1. Kontsevich moduli space of rational curves on del Pezzo surfaces.

We set up notation for the needed moduli spaces and maps, consistent with that of [KLSW23], to which we refer the reader for further information and references. Let k be a perfect field.

Definition 4.1. *A* del Pezzo surface over k *is a smooth and projective k-scheme S such that the anti-canonical sheaf $-K_S$ is ample. The* degree d_S *of a del Pezzo surface S is the self-intersection $K_S^{(2)}$.*

Let S be a del Pezzo surface over k of degree d_S. Fix an effective Cartier divisor D on S and let $d := \deg(-D \cdot K_S) > 0$ denote its degree with respect to $-K_S$.

Let $M_{0,n}(S, D)$ denote the moduli stack of n-pointed stable maps $u : \mathbf{P}^1 \to S$ in curve class D equipped with n points of $\mathbf{P}^1$ (meaning sections from the base). Let $\bar{M}_{0,n}(S, D)$ denote the compactified moduli stack of n-pointed, stable maps of a genus zero curve to S, in the curve class D. By definition this means that for an algebraically closed extension F of k, an F-point of $\bar{M}_{0,n}(S, D)$ corresponds to the data $(u : \mathbf{P}_F \to S_F, p_1, \ldots, p_n)$ where $\mathbf{P}_F$ is a semistable genus 0 curve over F, u is a stable map, $p_i : \operatorname{Spec} F \to \mathbf{P}_F$ are disjoint sections landing in the smooth locus and $u_*[\mathbf{P}_F] \in D$. See [AO01, Theorem 2.8, p. 90] and [dJHS11, Section 4] as well as [KLSW23] for more information.

Our counts of rational curves will be the $\mathbf{A}^1$-degrees of appropriate modifications of the following evaluation map.

Definition 4.2. *Define the* evaluation map $\mathrm{ev} : \bar{M}_{0,n}(S, D) \to S^n$ *by*

$$(u : \mathbf{P} \to S, (p_1, \ldots, p_n)) \mapsto (u(p_1), \ldots, u(p_n))$$

For a point $q_* = (q_1, \ldots, q_n)$ of S^n, note that the fiber $\mathrm{ev}^{-1}(q_*)$ of ev over q_* consists of stable maps $u : \mathbf{P} \to S$ with $u(p_i) = q_i$ for $i = 1, \ldots, n$. Thus $\mathrm{ev}^{-1}(q_*)$ consists of the rational curves passing through $(q_1, \ldots, q_n)$ together with a chosen point p_i of $u^{-1}(q_i)$ (which usually is no choice at all, as $u^{-1}(q_i)$ will be a single point for generally chosen q_i). Since the $\mathbf{A}^1$-degree of a map is a sum over the fiber $\mathrm{ev}^{-1}(q_*)$ of a local degree (Proposition 3.2), our quadratically enriched counts of rational curves will be $\mathbb{A}^1$-degrees of appropriate modifications of the map ev.

Consider a geometric point $(u : \mathbf{P}^1_F \to S_F, p_1, \ldots, p_n)$ of $M_{0,n}(S, D)$. Except in a few special cases ($d = -1$, or $d = 1, 2$ and $d_S \geq 3$ etc.), the image curve $u(\mathbf{P}^1)$ is not smooth. In [KLSW23], we study the geometry of $\bar{M}_{0,n}(S, D)$ using the singularities of the image curve. An *ordinary double point* of $u(\mathbf{P}^1)$ is a point q of $u(\mathbf{P}^1)$ such that there exist distinct points $p_1 \neq p_2$ of $\mathbf{P}^1$ such that $u(p_1) = q = u(p_2)$ and T_qS is spanned by the images of $T_{p_1}\mathbf{P}^1$ and $T_{p_2}\mathbf{P}^1$. There is an open subscheme of the moduli stack $\bar{M}_{0,n}(S, D)$ whose geometric points $(u : \mathbf{P}^1 \to f(\mathbf{P}^1), ((p_1, \ldots, p_n))$ are *unramified* in the sense that the induced map on cotangent spaces $du : u^*T^*S \to T^*\mathbf{P}^1$ is surjective and such that any singularities of $u(\mathbf{P}^1)$ are ordinary double points. It is well-known that $M^{\mathrm{odp}}_{0,n}(S, D)$ is either empty or a smooth scheme of dimension $n + d - 1$. See for example [KLSW23, Lemma 2.17]. Assume that $\operatorname{char} F \neq 2, 3$. We say that u has an *ordinary cusp* at $p \in \mathbf{P}^1$ if $T_pu \colon T_p(\mathbf{P}) \to T_{f(p)}S$ is the zero map, $u^{-1}(u(p)) = \{p\}$, and we may choose parameters x, y for $\mathcal{O}^{\wedge}_{S,q}$ and t for $\mathcal{O}^{\wedge}_{\mathbf{P}^1,p}$ so that $f^*(x) = t^2$, $f^*(y) = \nu t^3$ where ν is a unit in $\mathcal{O}^{\wedge}_{\mathbf{P}^1,p}$. We say u has an *ordinary tacnode* at distinct smooth points $p_1, p_2 \in \mathbf{P}$ if $u(p_1) = u(p_2)$ and $T_{p_1}u(T_{p_1}\mathbf{P}) = T_{p_2}u(T_{p_2}\mathbf{P})$, and we may choose parameters x, y for $\mathcal{O}^{\wedge}_{S,q}$ and t_i for $\mathcal{O}^{\wedge}_{\mathbf{P}^1,p_i}$ with $i = 1, 2$ so that $t_i := u^*_{p_i}(x)$, $u^*_{p_1}(y) = 0$ and $u^*_{p_2}(y) = t_2^2$. We say that u has an *ordinary triple point* at distinct smooth points $p_1, p_2, p_3 \in \mathbf{P}$ if $u(p_1) = u(p_2) = u(p_3) = q$ and the images of $T_{p_i}u : T_{p_i}\mathbf{P} \to T_qS$ are pairwise linearly independent subspaces.

Lemma 4.3. *(*[KLSW23, Lemma 2.27]*) Let* k *be a perfect field,* S *a del Pezzo surface over* k *and* D *an effective Cartier divisor on* S*. Let* $n = d - 1$*. Then* $\mathrm{ev} : M^{\mathrm{odp}}_{0,n}(S, D) \to S^n$ *is étale.*

We introduce a list of assumptions, which will be convenient for future reference, but which are not running assumptions throughout the remainder of the paper.

Assumptions 4.4. *(1)* $\operatorname{char} k = 0$.
(2) D *is not an* m*-fold multiple of a* -1*-curve for* $m > 1$.
(3) One the the following holds.
- $d_S \geq 4$
- $d_S = 3$ *and* $d \neq 6$
- $d_S = 2$ *and* $d \geq 7$

We remind the reader that $M_0^{\mathrm{bir}}(S, D) \subseteq M_0(S, D)$ represents the locus of stable maps with irreducible domain curve which are birational onto their images.

Theorem 4.5. *(*[KLSW23, Theorem 4.5]*.) Suppose Basic Assumptions 4.4*(1)(2)(3) *hold for* k, S, D. *Then there is a closed subset* $A \subset S^n$ *with* codim $A \geq 2$ *such that the inverse image* $\bar{M}_{0,n}(S, D)^{\mathrm{good}} := \bar{M}_{0,n}(S, D) \setminus \mathrm{ev}^{-1}(A)$ *satisfies the following.*

(1) $\bar{M}_{0,n}(S, D)^{\mathrm{good}} = \emptyset$ *if and only if* $M_0^{\mathrm{bir}}(S, D) = \emptyset$. *If* $M_0^{\mathrm{bir}}(S, D) \neq \emptyset$, *then the moduli space* $\bar{M}_{0,n}(S, D)^{\mathrm{good}}$ *is a geometrically irreducible smooth finite-type* k*-scheme, and the restriction of* ev *to* $\mathrm{ev} : \bar{M}_{0,n}(S, D)^{\mathrm{good}} \to S^n \setminus A$ *is a finite, flat, dominant morphism.*
(2) The evaluation map ev *is étale in a neighborhood of each* $f \in \bar{M}_{0,n}(S, D)^{\mathrm{good}}$ *with* f *unramified.*
(3) $\bar{M}_{0,n}(S, D)^{\mathrm{good}}$ *contains a dense open subset of* $M_{0,n}^{\mathrm{odp}}(S; D)$.
(4) Geometric points f *of* $\bar{M}_{0,n}(S, D)^{\mathrm{good}}$ *correspond to birational maps.*
(5) Let f *be a geometric point of* $\bar{M}_{0,n}(S, D)^{\mathrm{good}} \setminus M_{0,n}^{\mathrm{odp}}(S; D)$, *which we consider as a morphism* $f : \mathbf{P} \to S$ *for some genus 0 semi-stable curve* $\mathbf{P}$. *Then* f *satisfies:*
  (i) If $\mathbf{P} = \mathbf{P}^1$ *is irreducible, then the image curve* $C := f(\mathbf{P}^1)$ *has one singular point* q *that is not an ordinary double point, and* C *has either an ordinary cusp, an ordinary tacnode or an ordinary triple point at* q. *Moreover, the marked points do not map to* q *and* f *is free.*
  (ii) If $\mathbf{P}$ *is not irreducible, then* $\mathbf{P} = \mathbf{P}_1 \cup \mathbf{P}_2$, *with* $\mathbf{P}_i \cong \mathbf{P}^1$. *The image curve* $C := f(\mathbf{P})$ *has only ordinary double points as singularities. Moreover, if* n_i *of the* n *marked points of* $\mathbf{P}$ *are in* $\mathbf{P}_i$, *and* $C_i := f(\mathbf{P}_i)$ *has degree* $d_i := -K_S \cdot C_i$, *then* $d_i - 1 \leq n_i \leq d_i$ *for* $i = 1, 2$.

Definition 4.6. *Define* $D_{\mathrm{cusp}} \subset \bar{M}_{0,n}(S, D)^{\mathrm{good}} \setminus M_{0,n}^{\mathrm{odp}}(S; D)$ *(respectively* D_{tac}*) to be the closure of the locus of those* u *in* $\bar{M}_{0,n}(S, D)^{\mathrm{good}}$ *such that* C *has an ordinary cusp (respectively tacnode).* D_{cusp} *and* D_{tac} *are divisors by* [KLSW23, Theorem 6.1].

In [KLSW23, Section 5], we define the *double point locus* $\pi : \mathcal{D}^{\mathrm{odp}} \to M_{0,n}^{\mathrm{odp}}(S, D)$ based on [Ful98, Chapter 9.3]. For k, S, D satisfying Basic Assumptions 4.4(1)(2)(3), we furthermore define the double point locus $\pi : \mathcal{D}^{\mathrm{good}} \to \bar{M}_{0,n}(S, D)^{\mathrm{good}}$. See [KLSW23, Definition 5.3]. The loci $\mathcal{D}^{\mathrm{odp}}$ and $\mathcal{D}^{\mathrm{good}}$ are smooth k-schemes, constructed as a closure of a locus of geometric points $(u : \mathbf{P} \to S, (p_1, \ldots, p_n), (p_{n+1}, p_{n+2}))$ where $(u : \mathbf{P} \to S, (p_1, \ldots, p_n))$ is a geometric point of $M_{0,n}^{\mathrm{odp}}(S, D)$ or $\bar{M}_{0,n}(S, D)^{\mathrm{good}}$ respectively, and $p_{n+1} \neq p_{n+2}$ are smooth points of $\mathbf{P}$ such that $u(p_{n+1}) = u(p_{n+2})$. These double point loci are certain closed subschemes in the fiber product of universal curves. They contain an open subscheme of stable maps

$u : \mathbf{P}^1 \to S$ together with a pair of distinct points of $\mathbf{P}^1$ mapping to the same point q of S, and n additional marked points of $\mathbf{P}^1$. Note that q is a double point of $u(\mathbf{P}^1)$, and the double point loci are introduced to have a useful moduli space of such double points.

Proposition 4.7. *Let S be a smooth del Pezzo surface over a perfect field k equipped with an effective Cartier divisor D. Then $\pi : \mathcal{D}^{\mathrm{odp}} \to \overline{\mathcal{M}}_{0,n}(S, D)^{\mathrm{odp}}$ is finite étale of degree $\delta = \frac{1}{2}D \cdot (K_S + D) + 1$.*

Proof. π is proper by construction and is shown to be étale in [KLSW23, Lemma 5.4]. The arithmetic genus of $u(\mathbf{P}^1)$ is $\delta = \frac{1}{2}D \cdot (K_S + D) + 1$. See for example [Har77, V 1. Prop 1.5]. Thus points of $M_{0,n}^{\mathrm{odp}}(S, D)$ correspond to maps where $u(\mathbf{P}^1)$ has δ ordinary double points. □

Remark 4.8. *Let $f : Y \to Z$ be a finite, flat morphism of smooth k-schemes that is étale over each generic point of Z. Then $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_Z$-module. The multiplication map on $\mathcal{O}_Y$ gives the morphism of $\mathcal{O}_Z$-modules $m : f_*\mathcal{O}_Y \otimes_{\mathcal{O}_Z} f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y$. Since $f_*\mathcal{O}_Y$ is a finite locally free $\mathcal{O}_Z$-module, we have the trace map $\mathrm{Tr}_f : f_*\mathcal{O}_Y \to \mathcal{O}_Z$ defined by sending $s \in f_*\mathcal{O}_Y(U)$ to the trace of the multiplication map $\times s : f_*\mathcal{O}_Y(U) \to f_*\mathcal{O}_Y(U)$. The trace form*

$$f_*\mathcal{O}_Y \otimes f_*\mathcal{O}_Y \to \mathcal{O}_Z$$
$$(b, b') \mapsto \mathrm{Tr}(bb')$$

defines a map $\tau : f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y^{-1}$. The determinant

$$\det \tau : \det f_*\mathcal{O}_Y \to \det f_*\mathcal{O}_Y^{-1}$$

determines a section $\mathrm{disc}(f)$ of $\det f_*\mathcal{O}_Y^{-2}$ and a canonical isomorphism

$$\mathcal{O}(D(\mathrm{disc}(f))) \cong (\det f_*\mathcal{O}_Y)^{\otimes 2}. \tag{13}$$

Since Tr_f is a surjection if f is étale, we see that the divisor of $\mathrm{disc}(f)$ is supported on the branch locus of f. The associated element of $\mathcal{O}_Z/(\mathcal{O}_Z^*)^2$ (see Construction 3.6) has the property that at every closed point z of Z,

$$\mathrm{disc}(f) = \det(\mathrm{Tr}(b_i b_j)_{i,j})$$

where b_i runs over a basis of $f_*\mathcal{O}_Y$ as an $\mathcal{O}_Z$-module.

The map $\pi : \mathcal{D}^{\mathrm{good}} \to \bar{M}_{0,n}(S, D)^{\mathrm{good}}$ is finite, flat and étale over the generic points of $\bar{M}_{0,n}(S, D)^{\mathrm{good}}$ by [KLSW23, Corollary 5.14]. Thus $\pi_*\mathcal{O}_{\mathcal{D}^{\mathrm{good}}}$ is locally free and we have a discriminant map

$$\mathrm{disc}\,\pi : \mathcal{O}_{\bar{M}_{0,n}(S,D)^{\mathrm{good}}} \to \det(\pi_*\mathcal{O}_{\mathcal{D}^{\mathrm{good}}})^{\otimes -2}.$$

Theorem 4.9. *([KLSW23, Theorem 6.2].) Suppose Basic Assumptions 4.4(1)(2)(3) hold for k, S, D. Let $n = d - 1$. Let $\mathcal{L}$ be the invertible sheaf on $\bar{M}_{0,n}^{\mathrm{good}}(S, D)$ given by*

$$\mathcal{L} = (\det \pi_*\mathcal{O}_{\mathcal{D}}(-D_{\mathrm{tac}}))^{-1}$$

Then the composition $\det d\,\mathrm{ev} \circ \mathrm{disc}_\pi^{-1} : \mathcal{L}^{\otimes 2} \to \omega_{\mathrm{ev}}$ is an isomorphism on $\bar{M}_{0,n}^{\mathrm{good}}(S, D)$.

Suppose Basic Assumptions 4.4(1)(2)(3) hold for k, S, D. Then the isomorphism of Theorem 4.9 is an orientation on $\mathrm{ev} : \bar{M}_{0,n}^{\mathrm{good}}(S, D) \to S^n$. It follows from Theorem 4.5 that this map satisfies Assumption 2.13, where $U \subseteq Y$ (in the notation of Assumption 2.13) is $S^n \setminus A \subseteq S^n$. We therefore have

$$\deg(\mathrm{ev} : \bar{M}^{\mathrm{good}}_{0,n}(S, D) \to S^n) \in \mathcal{GW}(S^n)$$

as in Definition 2.14.

4.2. **Positive characteristic.** Let k be a field of characteristic $p > 3$ and let S be a smooth del Pezzo surface over k with an effective Cartier divisor D.

Assumption 4.10. *For every effective Cartier divisor D' on S, there is a geometric point f in each irreducible component of $M^{\mathrm{bir}}_0(S, D')$ with f unramified.*

Remark 4.11. *Assumption 4.10 is automatically satisfied in characteristic 0* [KLSW23, Lemma 2.31] *and in characteristic $p > 3$ when $d_S \geq 3$* [KLSW23, Theorem A.1].

Suppose (k, S, D) satisfies Basic Assumptions 4.4(2)(3) and Assumption 4.10. Let $d = -K_S \cdot D$ and $n = d - 1$. If $M_{0,n}(S, D)^{\mathrm{odp}}$ is empty, we consider the degree of $\mathrm{ev}_k : M_{0,n}(S, D)^{\mathrm{odp}} \to S^n$ to be zero. Suppose $M_{0,n}(S, D)^{\mathrm{odp}}$ is not empty. In this section, we construct the data of Assumption 2.16 for $\mathrm{ev}_k : M_{0,n}(S, D)^{\mathrm{odp}} \to S^n$.

Remark 4.12. *The ordinary double point locus $M_{0,n}(S, D)^{\mathrm{odp}}$ is an open subcheme of $\bar{M}_{0,n}(S, D)$ (see e.g.* [KLSW23, Lemma 2.14]*). By* [KLSW23, Proposition 2.32]*, $M_{0,n}(S, D)^{\mathrm{odp}}$ is empty if and only if there are no $u : \mathbf{P}^1_F \to S_F$ in curve class D with $\mathbf{P}^1_F \to u(\mathbf{P}^1_F)$ birational (where F is some algebraically closed extension of k). For S a twisted form of a smooth toric del Pezzo, it is possible to give a complete list of (S, D) for which $M_{0,n}(S, D)^{\mathrm{odp}}$ is empty, although we do not include this result here.*

The evaluation map $\bar{M}_{0,n}(S, D) \to S^n$ being proper, $\mathrm{ev}_k(\bar{M}_{0,n}(S, D) \setminus M_{0,n}(S, D)^{\mathrm{odp}})$ is closed in S^n. By [KLSW23, Corollary 3.15], $\mathrm{ev}_k(\bar{M}_{0,n}(S, D) \setminus M_{0,n}(S, D)^{\mathrm{odp}})$ this closed set has positive codimension in S^n. Letting $U \subset S^n$ denote its complement, we have $\mathrm{ev}_k : \mathrm{ev}_k^{-1}(U) \to U$ a proper map between smooth k-schemes, which is furthermore étale by Lemma 4.3. By Remark 2.6, $\mathrm{ev}_k : \mathrm{ev}_k^{-1}(U) \to U$ has a canonical relative orientation. This constructs the data of Assumption 2.16 over the special fiber.

It remains to construct the data of Assumption 2.16 over a lift to characteristic 0. Let Λ be a complete discrete valuation ring with reside field k and quotient field K of characteristic 0. In [KLSW23, Lemma 9.3] we construct $\tilde{S} \to \operatorname{Spec} \Lambda$ a smooth del Pezzo surface equipped with an effective Cartier divisor $\tilde{D}$ with special fiber $\tilde{S}_k \cong S$ and $\tilde{D}_k \cong D$. The general fiber of $(\tilde{S}, \tilde{D}, \Lambda)$ gives rise to $(\tilde{S}_K, \tilde{D}_K, K)$ satisfying Basic Assumptions 4.4(1)(2)(3). See [KLSW23, Lemma 9.4]. Set $\mathcal{Y} = \tilde{S}^n$ in the notation of Assumption 2.16.

There is a compactified moduli stack $\bar{M}_{0,n}(\tilde{S}, \tilde{D})$ of n-pointed, stable maps of a genus zero curve to $\tilde{S}$, in the curve class $\tilde{D}$ by [AO01, Theorem 2.8, and p.90]. There is more discussion in [KLSW23, Section 2]. As before, the moduli $\bar{M}_{0,n}(\tilde{S}, \tilde{D})$ admits the evaluation map

$$\mathrm{ev} : \bar{M}_{0,n}(\tilde{S}, \tilde{D}) \to \tilde{S}^n$$

In [KLSW23, Construction 9.6], we construct a closed subset $\tilde{A} \subset \tilde{S}^n$ such that

(1) the special fiber $\tilde{A}_k$ contains $\mathrm{ev}_k(\bar{M}_{0,n}(S,D)\setminus \bar{M}_{0,n}(S,D)^{\mathrm{odp}})$ and is codimension ≥ 1 in S^n.
(2) $\tilde{A}$ is codimension 2 in $\tilde{S}^n$.
(3) $\mathrm{ev}_K^{-1}(\tilde{S}_K^n \setminus \tilde{A}_K)$ can be taken to be $\bar{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\mathrm{good}}$ in Theorem 4.5
(4) $\bar{M}_{0,n}(\tilde{S},\tilde{D})^{\mathrm{good}} := \mathrm{ev}^{-1}(\tilde{S}^n \setminus \tilde{A})$ is a smooth Λ-scheme (see in particular [KLSW23, Proposition 9.9])

In the notation of Assumption 2.16, set $\mathcal{Y} = \tilde{S}^n \setminus \tilde{A}$ and let $\mathcal{X} = \bar{M}_{0,n}(\tilde{S},\tilde{D})^{\mathrm{good}}$. In [KLSW23, Theorem 9.13] we construct an orientation on the proper map $\bar{M}_{0,n}(\tilde{S},\tilde{D})^{\mathrm{good}} \to \mathcal{U}$ restricting to the constructed orientation on $\mathrm{ev}_k : \mathrm{ev}_k^{-1}(U) \to U$. We therefore have the data of Assumption 2.16 on the map $\mathrm{ev}_k : M_{0,n}(S,D)^{\mathrm{odp}} \to S^n$ and may define

$$\deg(\mathrm{ev}_k : M_{0,n}(S,D)^{\mathrm{odp}} \to S^n) \in \mathcal{GW}(S^n)$$

to be the degree of Definition 2.22.

5. Twists of ev

Let S be a smooth del Pezzo over a field k equipped with a relative Cartier divisor D. Let $d = -K_S \cdot D$. Let $k \subseteq k^s$ denote a separable closure of k. Let

$$\sigma = (L_1, \dots, L_r)$$

be an r-tuple of subfields $L_i \subset \overline{k}$ containing k for $i = 1, \dots, r$ subject to the requirement that $\sum_{i=1}^{k}[L_i : k] = n$. We think of σ as the fields of definition of a list of points of S that our curves will be required to pass through.

The list σ is used to define twists ev_σ of the evaluation map in the following manner. The Galois group $\mathrm{Gal}(k^s/k)$ acts on the k^s-points of k-schemes. Thus σ gives rise to a canonical homomorphism $\mathrm{Gal}(\sigma) : \mathrm{Gal}(k^s/k) \to \mathfrak{S}_{\overline{\mathcal{P}}(\sigma)}$, where $\overline{\mathcal{P}}(\sigma)$ denotes the k^s-points of $\coprod_{i=1}^r \operatorname{Spec} L_i$ and $\mathfrak{S}_{\overline{\mathcal{P}}(\sigma)} \cong \mathfrak{S}_n$ denotes the symmetric group. For convenience, we fix an identification $\overline{\mathcal{P}}(\sigma) = \{1, 2, \dots, n\}$ and thus a canonical isomorphism $\mathfrak{S}_{\overline{\mathcal{P}}(\sigma)} = \mathfrak{S}_n$.

Permuting the factors of S defines an inclusion of $\mathfrak{S}_n$ into $\mathrm{Aut}(S^n)$. We include $\mathfrak{S}_n$ into $\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}(S,D))$ by permutation of the marked points, and acting trivially on the underlying curve and the morphism to S: for τ in $\mathfrak{S}_n$, set

$$\tau(u : C \to S, p_1, \dots, p_n) = (u : C \to S, p_{\tau^{-1}(1)}, \dots, p_{\tau^{-1}(n)}).$$

The 1-cocycle

$$g \mapsto \mathrm{Gal}(\sigma)(g) \times g \tag{14}$$

$$\mathrm{Gal}(k^s/k) \to \mathrm{Aut}(X^n_{k^s})$$

for $X = S^n$, $X = \overline{\mathcal{M}}_{0,n}(S,D)$, $X = \overline{\mathcal{M}}_{0,n}^{\mathrm{odp}}(S,D)$, or $X = \mathcal{D}^{\mathrm{odp}}$ determines twists X_σ. Since ev_{k^s} and π_{k^s} are Galois equivariant for the twisted action, they descends to a k-maps

$$\mathrm{ev}_\sigma : \overline{\mathcal{M}}_{0,n}(S,d)_\sigma \to (S^n)_\sigma$$

$$\pi_\sigma : \mathcal{D}_\sigma^{\mathrm{odp}} \to \overline{\mathcal{M}}_{0,n}^{\mathrm{odp}}(S,d)_\sigma$$

Lemma 5.1. *There is a natural isomorphism*

$$\phi_\sigma : (S^n)_\sigma \cong \prod_{i=1}^{r} \mathrm{Res}_{L_i/k} S \tag{15}$$

Proof. We may assume that each L_i is a subextension of k in $\bar{k}$. Fix a Galois subextension M of k in $\bar{k}$ containing each of the L_i. Let G denote the Galois group of M over k.

Using the universal property of the restriction of scalars, it suffices to define for each finitely generated k-algebra A, an isomorphism

$$\phi_A : (S^n)_\sigma(A) \to \prod_{i=1}^{r} S(L_i \otimes_k A),$$

natural in A.

For this, it follows from the theory of descent that $(S^n)_\sigma(A)$ is naturally in bijection with the G-invariant subset of $S^n(M \otimes_k A) = S(M \otimes_k A)^{\overline{\mathcal{P}}(\sigma)}$, where $g \in G$ acts on $M \otimes_k A$ through its action on M and on $S(M \otimes_k A)^{\overline{\mathcal{P}}(\sigma)}$ by

$$g \cdot (\iota \mapsto x_\iota \in S(M \otimes_k A)) := (\iota \mapsto x^g_{g^{-1}\cdot\iota} \in S(M \otimes_k A)).$$

Let $x_* := (x_\iota \in S(M \otimes_k A))_{\iota \in \overline{\mathcal{P}}(\sigma)}$ be a G-invariant element. Let $\iota_i : L_i \to \bar{k}$ be the inclusion as a subextension of k, and take $g \in G$ with $g \cdot \iota_i = \iota_i$. Note that $\mathrm{Gal}(M/L_i)$ is exactly the isotropy group of ι_i under the G-action on $\overline{\mathcal{P}}(\sigma)$ and that

$$(M \otimes_k A)^{\mathrm{Gal}(M/L_i)} = L_i \otimes_k A$$

since A is flat over k. Looking at the ι_i component of x_*, we see that x_{ι_i} is invariant under $\mathrm{Gal}(M/L_i)$, so x_{ι_i} is in $L_i \otimes_k A$. Thus we have a well-defined map

$$\phi_A : (S(M \otimes_k A))^G \to \prod_{i=1}^{r} S(L_i \otimes_k A)$$

sending x_* to $(x_{\iota_1}, \dots, x_{\iota_r})$. To map in the other direction, start with $(x_1, \dots, x_r) \in \prod_{i=1}^{r} S(L_i \otimes_k A)$ and take an arbitrary $\iota \in \overline{\mathcal{P}}(\sigma)$, corresponding to an embedding $\iota : L_i \hookrightarrow M \subset \bar{k}$ over k for a unique i. Then there is an element $g \in G$, unique modulo $\mathrm{Gal}(M/L_i)$, with $\iota = g \cdot \iota_i$. We then take $x_\iota \in S(M \otimes_k A)$ to be x_i^g. It follows directly that $x_* := (x_\iota \in S(M \otimes_k A))_\iota$ is G-invariant and that $\phi_A(x_*) = (x_1, \dots, x_r)$. Thus ϕ_A is a split surjection. The injectivity of ϕ_A follows from the identity $x_\iota = x^g_{\iota_i}$ if $\iota = g \cdot \iota_i$ and $x_* = (x_\iota \in S(M \otimes_k A))_\iota$ is G-invariant. The naturality of ϕ_A in A is clear, which completes the proof.

□

We may thus view ev_σ as a map with codomain $\prod_{i=1}^{r} \mathrm{Res}_{L_i/k} S$. Similarly to the fibers of ev, the fibers of ev_σ consist of rational curves passing through chosen points. For simplicity of notation, consider a k-point q_* of $(S^n)_\sigma$. By (15), q_* is given by an r-tuple $(q_1, \dots, q_r)$ where q_i is an L_i point of S. In particular, q_* gives rise to a canonical geometric point of S^n (use the above identification of $\overline{\mathcal{P}}(\sigma)$ with $\{1, \dots, n\}$) which we will also denote by q_*. The geometric points of the fibers of ev_σ and ev over q_* are canonically identified.

5.1. **Characteristic 0.** Let $n = d-1$. Let k be a perfect field, S a del Pezzo surface over k and D an effective Cartier divisor on S satisfying Assumptions 4.4(1)(2)(3). We then have $\mathrm{ev} : \bar{M}_{0,n}(S, D)^{\mathrm{good}} \to S^n$ and $\pi : \mathcal{D}^{\mathrm{good}} \to \bar{M}_{0,n}(S, D)^{\mathrm{good}}$ by Theorem 4.5 and [KLSW23, Definition 5.3] (recalled in Section 4.1) respectively. The cocyle (14) defines twists

$$\mathrm{ev}_\sigma : \bar{M}_{0,n}(S, D)^{\mathrm{good}}_\sigma \to S^n_\sigma$$

$$\pi : \mathcal{D}^{\mathrm{good}}_\sigma \to \bar{M}_{0,n}(S, D)^{\mathrm{good}}_\sigma$$

In [KLSW23, Theorem 8.1], it is shown that ev_σ is a map between smooth k-schemes and setting $\mathcal{L}_\sigma := [\det(\pi_\sigma)_*\mathcal{O}_{\mathcal{D}_\sigma}(-D_{\mathrm{tac}})]^{-1}$, we have the isomorphism

$$\det d\,\mathrm{ev}_\sigma \circ \mathrm{disc}^{-1}_{\pi_\sigma} : (\mathcal{L}_\sigma)^{\otimes 2} \to \omega_{\mathrm{ev}_\sigma}.$$

Note that ev_σ is generically étale because its base change to k^s is by Theorem 4.5 This gives the data of Assumption 2.13. Let

$$\deg \mathrm{ev}_\sigma \in \mathcal{GW}(S^n_\sigma)$$

denote the degree of Definition 2.14.

5.2. **Positive characteristic.** Place ourselves in the situation of Section 4.2, which is to say k is a field of characteristic $p > 3$. S is a smooth del Pezzo surface over k with an effective Cartier divisor D. Suppose (k, S, D) satisfies Basic Assumptions 4.4(3) and Assumption 4.10 and that $M_{0,n}(S, D)^{\mathrm{odp}} \neq \emptyset$. (Note Remark 4.12 on the condition $M_{0,n}(S, D)^{\mathrm{odp}} \neq \emptyset$. As the condition $M_{0,n}(S, D)^{\mathrm{odp}} \neq \emptyset$ is equivalent to $M_{0,n}(S, D)^{\mathrm{bir}} \neq \emptyset$, the condition Basic Assumptions 4.4(2) is also satisfied.) In this section, we construct the data of Assumption 2.16 for $\mathrm{ev}_\sigma : M_{0,n}(S, D)^{\mathrm{odp}}_\sigma \to S^n_\sigma$, thereby defining

$$\deg \mathrm{ev}_\sigma \in \mathcal{GW}(S^n).$$

(If $M_{0,n}(S, D)^{\mathrm{odp}} = \emptyset$, we consider the degree of ev_σ to be 0 as above.)

Let $U \subset S^n_\sigma$ be the complement $\mathrm{ev}_\sigma(\bar{M}_{0,n}(S, D)_\sigma \setminus M_{0,n}(S, D)^{\mathrm{odp}}_\sigma)$. As in Section 4.2, U is a dense open subset of S^n_σ; $\mathrm{ev}_\sigma^{-1}(U) \to U$ is a proper, étale, map between smooth k-schemes, and therefore has a canonical relative orientation.

Take $(\Lambda, \tilde{S}, \tilde{D})$ as in Section 4.2. Enlarging the closed subset $\tilde{A} \subset \tilde{S}^n$ to be $\mathfrak{S}_n$ invariant, we construct in [KLSW23, Theorem 9.15] a map between smooth Λ-schemes $\mathrm{ev}_\sigma : \bar{M}_{0,n}(\tilde{S}, \tilde{D})^{\mathrm{good}}_\sigma \to \tilde{S}^n_\sigma$, a line bundle $\mathcal{L}_\sigma := [\det(\tilde{\pi}_\sigma)_*\mathcal{O}_{\tilde{\mathcal{D}}_\sigma}(-D_{\mathrm{tac}})]^{-1}$, and an isomorphism

$$\det d\,\mathrm{ev}_\sigma \circ \mathrm{disc}^{-1}_{\pi_\sigma} : (\mathcal{L}_\sigma)^{\otimes 2} \to \omega_{\mathrm{ev}_\sigma}.$$

on $\bar{M}_{0,n}(\tilde{S}, \tilde{D})^{\mathrm{good}}_\sigma$ restricting to the canonical relative orientation on the special fiber. This constructs the data of Assumption 2.16 for $\mathrm{ev}_\sigma : M_{0,n}(S, D)^{\mathrm{odp}}_\sigma \to S^n_\sigma$ and the degree $\deg \mathrm{ev}_\sigma$ in $\mathcal{GW}(S^n_\sigma)$.

6. The symmetrized moduli space

Let S be a smooth del Pezzo surface over a field k equipped with an effective Cartier divisor D with $\deg -K_S \cdot D \geq 1$. The symmetric group $\mathfrak{S}_n$ acts on S^n by permuting the factors, and acts freely $\bar{M}_{0,n}(S,D)$ by permuting the marked points and acting trivially on the underlying curve and morphism to S. We obtain a $\mathfrak{S}_n$-equivariant diagram

$$\bar{M}_{0,n+1}(S,D) \to \bar{M}_{0,n}(S,D) \xrightarrow{\mathrm{ev}} S^n$$

projecting from the universal curve to the moduli stack, followed by the evaluation map. Passing to quotients gives rise to a symmetrized evaluation map

$$\mathrm{ev}^{\mathfrak{S}} : \bar{M}_{0,n}(S,D)^{\mathfrak{S}} \to \mathrm{Sym}^n S.$$

Because symmetric powers of surfaces are not necessarily smooth, it will be useful to let $\mathrm{Sym}^n S^0 \subset \mathrm{Sym}^n S$ be the open subscheme formed as the quotient of $S^n \setminus \{\text{diagonals}\}$ by $\mathfrak{S}_n$. Rational points of $\mathrm{Sym}^n S^0$ correspond to $\{p_1, \dots, p_r\}$ where p_i is a closed point of S and $\sum_{i=1}^r [L_i : k] = n$, where $L_i = k(p_i)$. Fibers of $\mathrm{ev}^{\mathfrak{S}}$ correspond to rational curves passing through the p_i. So by symmetrizing, we are allowing for curve counts through non-rational points. The choice of the field extensions L_i is not fixed. This will assemble the degrees of Section 5 into a section of $\mathcal{GW}(\mathrm{Sym}^n S^0)$.

6.1. **Characteristic 0.** Suppose (k, S, D) satisfies Assumptions 4.4(1)(2)(3). Let $d = -K_S \cdot D \geq 1$ and $n = d-1$. We then have $\mathrm{ev} : \bar{M}_{0,n}(S,D)^{\mathrm{good}} \to S^n$ and $\pi : \mathcal{D}^{\mathrm{good}} \to \bar{M}_{0,n}(S,D)^{\mathrm{good}}$ by Theorem 4.5 and [KLSW23, Definition 5.3] (recalled in Section 4.1) respectively. By enlarging $A \subset S^n$ of Theorem 4.5, we may assume that $\bar{M}_{0,n}(S,D)^{\mathrm{good}}$ and $\mathcal{D}^{\mathrm{good}}$ inherit the action of $\mathfrak{S}_n$ from $\bar{M}_{0,n}(S,D)$ and $\bar{M}_{0,n+1}(S,D) \times_{\bar{M}_{0,n}(S,D)} \bar{M}_{0,n+1}(S,D)$, respectively. By Theorem 4.5(4), there are no contracted components in the stable maps corresponding to geometric points of $\bar{M}_{0,n}(S,D)^{\mathrm{good}}$, whence $S^n \setminus A \subset S^n \setminus \{\text{diagonals}\}$. Furthermore, $\bar{M}_{0,n}(S,D)^{\mathrm{good}}$, $\mathcal{D}^{\mathrm{good}}$, $S^n \setminus \{\text{diagonals}\}$ and $S^n \setminus A$ are all quasi-projective because S is projective over k and $\mathrm{ev} : \bar{M}_{0,n}(S,D)^{\mathrm{good}} \to S^n \setminus A$ is finite (Theorem 4.5(1)), and $\pi : \mathcal{D}^{\mathrm{good}} \to \bar{M}_{0,n}(S,D)^{\mathrm{good}}$ is finite by [KLSW23, Corollary 5.14]. We may thus take their quotients in the category of quasi-projective k-schemes by the action of $\mathfrak{S}_n$. Since the actions are free, the quotients, denoted $\bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n}$, $\mathcal{D}^{\mathrm{good}}_{\mathfrak{S}_n}$, $\mathrm{Sym}^n_0 S$, $U = (S^n \setminus A)/\mathfrak{S}_n$ respectively, are smooth quasi-projective k-schemes. We have maps

$$\mathcal{D}^{\mathrm{good}}_{\mathfrak{S}_n} \xrightarrow{\pi_{\mathfrak{S}_n}} \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \xrightarrow{\mathrm{ev}^{\mathrm{good}}_{\mathfrak{S}_n}} U \subseteq \mathrm{Sym}^n_0 S$$

The composition of $\mathrm{ev}^{\mathrm{good}}_{\mathfrak{S}_n}$ with the inclusion is denoted

$$\mathrm{ev}_{\mathfrak{S}_n} : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S$$

We will construct the data of Assumption 2.13 on $\mathrm{ev}_{\mathfrak{S}_n}$. By [KLSW23, Theorem 7.1], we have a line bundle $\mathcal{L}_{\mathfrak{S}_n} := [\det \pi_{\mathfrak{S}_n *} \mathcal{O}_{\mathcal{D}^{\mathrm{good}}_{\mathfrak{S}_n}}(-D_{\mathrm{tac}})]^{-1}$ on $\bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n}$ together with an isomorphism

$$\det d\, \mathrm{ev}^{\mathrm{good}}_{\mathfrak{S}_n} \circ \mathrm{disc}^{-1}_{\pi_{\mathfrak{S}_n}} : (\mathcal{L}_{\mathfrak{S}_n})^{\otimes 2} \to \omega_{\mathrm{ev}^{\mathrm{good}}_{\mathfrak{S}_n}}.$$

This constructs the data of Assumption 2.13, defining

$$\deg \mathrm{ev}_{\mathfrak{S}_n} \in \mathcal{GW}(\mathrm{Sym}^n_0 S)$$

by Definition 2.14.

Remark 6.1. $\mathrm{Sym}^n S$ *is not smooth by purity of the branch locus* [Sta18, 0BMB] [Zar58] *applied to the quotient map* $S^n \to \mathrm{Sym}^n S$. *So even though the complement of* $\mathrm{Sym}^n S^0 \subset$

$\mathrm{Sym}^n S$ *is codimension* 2*, purity results on Witt groups do not give that the associated restriction map of* $\mathcal{GW}$ *is an isomorphism. In fact, the sections* $\deg \mathrm{ev}^{\mathfrak{S}}$ *of* $\mathcal{GW}(\mathrm{Sym}^n S^0)$ *do not extend in general: for* $d = 3$ *and* $S = \mathbf{P}^2$ *the value at for example* 8 *real points has a different weighted count than the value at* 6 *real points and one complex conjugate pair.*

6.2. **Positive characteristic.** As in Sections 4.2, 5.2, let S be a smooth del Pezzo surface over a field k of characteristic $p > 3$. Let D be an effective Cartier divisor on S. Suppose (k, S, D) satisfies Basic Assumptions 4.4(2)(3) and Assumption 4.10. Noting Remark 4.12, we will assume $M_{0,n}(S, D)^{\mathrm{odp}} \neq \emptyset$. (This also implies Basic Assumptions 4.4(2).)

Let $d = -K_S \cdot D$ and $n = d - 1$. Note that $M_{0,n}(S, D)^{\mathrm{odp}}$ is stable under the free action of $\mathfrak{S}$. As in Section 6.1, we will take a the quotient in quasi-projective schemes by $\mathfrak{S}$ of an evaluation map. It is convenient to pass to a dense open subset of $M_{0,n}(S, D)^{\mathrm{odp}}$ first. By [KLSW23, Theorem 3.15], the closed set $A_k := \mathrm{ev}(\bar{M}_{0,n}(S, D) \setminus M_{0,n}(S, D)^{\mathrm{odp}})$ has positive codimension. (A_k is closed becasue ev is proper and $M_{0,n}(S, D)^{\mathrm{odp}}$ is open.) Let $M_{0,n}(S, D)^{\mathrm{odp,good}} := \mathrm{ev}^{-1}(S^n \setminus A_k)$. By [KLSW23, Lemma 2.27], $\mathrm{ev} : M_{0,n}(S, D)^{\mathrm{odp}} \to S^n$ is étale, whence locally quasi-finite. Since $M_{0,n}(S, D)^{\mathrm{odp,good}}$ is proper over $S^n \setminus A$, it follows that $M_{0,n}(S, D)^{\mathrm{odp,good}}$ is quasi-projective. Let $M_{0,n}(S, D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} := M_{0,n}(S, D)^{\mathrm{odp,good}}/\mathfrak{S}_n$ denote the quotient in quasi-projective schemes, which is smooth because $M_{0,n}(S, D)^{\mathrm{odp,good}}$ is and the action is free. We thus have a map between smooth quasi-projective k-schemes

$$\mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n} : M_{0,n}(S, D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S$$

where as before $\mathrm{Sym}^n_0 S$ denotes the the quotient of S^n minus the diagonals by $\mathfrak{S}_n$. In this section, we construct the data of Assumption 2.16 for $\mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}$.

Since $\mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}$ is a proper, local complete intersection morphism, which is étale, $\mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}$ has a canonical orientation (Remark 2.6), so it remains to construct the data of Assumption 2.16 over a discrete valuation ring Λ.

Take $(\Lambda, \tilde{S}, \tilde{D})$ as in Section 4.2. We enlarge $\tilde{A} \subset \tilde{S}^n$ to be $\mathfrak{S}_n$ invariant and obtain the smooth Λ-scheme $\bar{M}_{0,n}(\tilde{S}, \tilde{D})^{\mathrm{good}} := \mathrm{ev}^{-1}(\tilde{S}^n \setminus \tilde{A})$ as in Section 4.2. Taking quotients by $\mathfrak{S}_n$, we construct in [KLSW23, Theorem 9.14] a map

$$\mathrm{ev}_{\mathfrak{S}_n} : \bar{M}_{0,n}(\tilde{S}, \tilde{D})^{\mathrm{good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 \tilde{S}$$

between smooth Λ-schemes, a line bundle $\mathcal{L}_{\mathfrak{S}_n} := [\det \tilde{\pi}_{\mathfrak{S}_n,*} \mathcal{O}_{\mathcal{D}_{\mathfrak{S}_n}}(-D_{\mathrm{tac}})]^{-1}$, and an isomorphism

$$\det d\,\mathrm{ev}_{\mathfrak{S}_n} \circ \mathrm{disc}^{-1}_{\tilde{\mathrm{pi}}_{\mathfrak{S}_n}} : (\mathcal{L}_{\mathfrak{S}_n})^{\otimes 2} \to \omega_{\mathrm{ev}_{\mathfrak{S}_n}}.$$

This constructs the data of Assumption 2.16 for $\mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}$ defining

$$\deg \mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n} \in \mathcal{GW}(\mathrm{Sym}^n_0 S).$$

by Definition 2.22.

7. Local degree of ev

In Sections 5.1 5.2 6.1 6.2, respectively, we have constructed $\mathbf{A}^1$-degrees

$$\deg(\mathrm{ev}_\sigma : \bar{M}_{0,n}(S, D)^{\mathrm{good}}_\sigma \to S^n_\sigma) \in \mathcal{GW}(S^n_\sigma)$$

$$\deg(\mathrm{ev}_\sigma : M_{0,n}(S,D)^{\mathrm{odp}}_\sigma \to S^n_\sigma) \in \mathcal{GW}(S^n_\sigma)$$
$$\deg(\mathrm{ev}_{\mathfrak{S}_n} : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S) \in \mathcal{GW}(\mathrm{Sym}^n_0 S)$$
$$\deg \mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}(M_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S) \in \mathcal{GW}(\mathrm{Sym}^n_0 S)$$
under appropriate hypotheses. In this section, we compute the local degrees of these maps at a point of the locus of parametrized curves with only ordinary double points, e.g. $M_{0,n}(S,D)^{\mathrm{odp}}$ in the untwisted, unsymmetrized version.

We first compare the twisted and symmetrized degrees and local degrees. There are pullback diagrams [KLSW23, (8.3)]

$$\begin{tikzcd}
\bar{M}_{0,n}(S,D)^{\mathrm{odp}}_\sigma \subset \bar{M}_{0,n}(S,D)^{\mathrm{good}}_\sigma \arrow[r, "\mathfrak{S}"] \arrow[d, "\mathrm{ev}_\sigma"'] & \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \arrow[d, "\mathrm{ev}_{\mathfrak{S}_n}"'] \\
S^n_\sigma \arrow[r, "\mathfrak{S}_S"'] & \mathrm{Sym}^n S
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
\bar{M}_{0,n}(S,D)^{\mathrm{odp}}_\sigma \arrow[r, "\mathfrak{S}"] \arrow[d, "\mathrm{ev}_\sigma"'] & \bar{M}_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} \arrow[d, "\mathrm{ev}_{\mathfrak{S}_n}"'] \\
S^n_\sigma \arrow[r, "\mathfrak{S}_S"'] & \mathrm{Sym}^n S
\end{tikzcd},$$

where the latter is the characteristic $p > 0$ version of the former. Let $S^n_{\sigma,0}$ denote the inverse image under $\mathfrak{S}_S$ of the open subset $\mathrm{Sym}^n_0 S$ of $\mathrm{Sym}^n S$, and let

$$i_{S^n,0} : S^n_{\sigma,0} \to S^n$$

denote the inclusion. Pulling back (16) by $\mathrm{Sym}^n_0 S \to \mathrm{Sym}^n S$ produces the diagrams

$$\begin{tikzcd}
\bar{M}_{0,n}(S,D)^{\mathrm{odp}}_\sigma \subset \bar{M}_{0,n}(S,D)^{\mathrm{good}}_\sigma \arrow[r, "\mathfrak{S}"] \arrow[d, "\mathrm{ev}_\sigma"'] & \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \arrow[d, "\mathrm{ev}_{\mathfrak{S}_n}"'] \\
S^n_{\sigma,0} \arrow[r, "\mathfrak{S}_{S,0}"'] & \mathrm{Sym}^n_0 S
\end{tikzcd} \tag{17}$$

$$\begin{tikzcd}
\bar{M}_{0,n}(S,D)^{\mathrm{odp}}_\sigma \arrow[r, "\mathfrak{S}"] \arrow[d, "\mathrm{ev}_\sigma"'] & \bar{M}_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} \arrow[d, "\mathrm{ev}_{\mathfrak{S}_n}"'] \\
S^n_{\sigma,0} \arrow[r, "\mathfrak{S}_{S,0}"'] & \mathrm{Sym}^n_0 S
\end{tikzcd},$$

Proposition 7.1. *We have equalities of (global) degrees*

$$\mathfrak{S}^*_{S,0} \deg(\mathrm{ev}_{\mathfrak{S}_n} : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S) = i^*_{S^n,0} \deg(\mathrm{ev}_\sigma : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_\sigma \to S^n_\sigma)$$

$$\mathfrak{S}^*_{S,0} \deg \mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}(M_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S) = i^*_{S^n,0} \deg(\mathrm{ev}_\sigma : M_{0,n}(S,D)^{\mathrm{odp}}_\sigma \to S^n_\sigma) \in \mathcal{GW}(S^n_{\sigma,0})$$

in $\mathcal{GW}(S^n_{\sigma,0})$

Proof. Follows from (17) and Proposition 2.7. ☐

Consider a point $(u : \mathbf{P} \to S_{k(u)}, p_1, \ldots, p_n)$ in the twisted locus of parametrized curves with only ordinary double points $\bar{M}_{0,n}(S,D)^{\mathrm{odp}}_{\sigma}$. Let $\mathfrak{S}(u)$ denote the image of u in $\bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n}$ or $\bar{M}_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n}$ depending on if the characteristic of k is 0 or positive respectively. Let $k(\mathfrak{S}(u)) \subseteq k(u)$ be the associated field extension. By Propositions 3.3 and 3.5, the local degree $\deg_{\mathfrak{S}(u)} \mathrm{ev}_{\mathfrak{S}_n}$ of $\mathrm{ev}_{\mathfrak{S}_n}$ and the local degree of ev_{σ} at u are related by

$$\deg_{\mathfrak{S}(u)} \mathrm{ev}_{\mathfrak{S}_n} \otimes_{k(\mathfrak{S}(u))} k(u) = \deg_u \mathrm{ev}_{\sigma}.$$

Note that this holds both in characteristic p and in characteristic 0.

For any closed point u' of the symmetrized locus of parametrized curves with only ordinary double points, there is an associated k^s point of $\bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n,k^s}$ such that the action of $\mathrm{Gal}(k^s/k(u'))$ acts by permuting the marked points. This defines a map $\mathrm{Gal}(k^s/k(u')) \to \mathfrak{S}_n$ which determines a list $\sigma = (L_1, \ldots, L_r)$ of intermediate fields $k(u') \subseteq L_i \subseteq k^s$ by the Galois correspondence. (So, L_1 is the fixed field of the stabilizer of 1, L_2 is the fixed field of the stabilizer of the smallest integer not in the orbit of 1, etc.) We obtain

$$\mathfrak{S} : \bar{M}_{0,n}(S,D)^{\mathrm{odp}}_{\sigma} \to \bar{M}_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n,k(u')}$$

The k^s point corresponding to u' determines a point u of $\bar{M}_{0,n}(S,D)^{\mathrm{odp}}_{\sigma}$ such that $\mathfrak{S}(u) = u'$ and the induced extension of residue fields $k(u') \subseteq k(u)$ is an isomorphism. Basechange to $k(u')$ does not affect $\deg_{u'} \mathrm{ev}_{\mathfrak{S}_n}$. We have therefore reduced the calculation of the local degree of $\mathrm{ev}_{\mathfrak{S}_n}$ at a closed point to the calculation of the local degree of ev_{σ}.

Remark 7.2. *It follows similarly that for any closed point y' of $\mathrm{Sym}^n_0 S$, we can compute the fiber of*

$$\deg(\mathrm{ev}_{\mathfrak{S}_n} : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S)$$

in characteristic 0 (or $\deg \mathrm{ev}^{\mathrm{odp,good}}_{\mathfrak{S}_n}(M_{0,n}(S,D)^{\mathrm{odp,good}}_{\mathfrak{S}_n} \to \mathrm{Sym}^n_0 S)$ in characteristic p) at y' by choosing an appropriate σ and y in S^n_{σ} and computing the fiber of

$$\deg(\mathrm{ev}_{\sigma} : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_{\sigma} \to S^n_{\sigma})$$

at y (respectively the fiber of $\deg(\mathrm{ev}_{\sigma} : M_{0,n}(S,D)^{\mathrm{odp}}_{\sigma} \to S^n_{\sigma})$ at y).

As above, let $\pi_{\sigma} : \mathcal{D}^{\mathrm{odp}} \to M_{0,n}(S,D)^{\mathrm{odp}}$ denote the finite étale map from the twisted double point locus. (See Proposition 4.7.) Let $\langle \mathrm{disc}\, \pi_{\sigma} \rangle$ in $\mathrm{GW}(k(u))$ denote the corresponding discriminant. See Remark 4.8.

Proposition 7.3. *Let u be in $M_{0,n}(S,D)^{\mathrm{odp}}$. Then the local degree of ev_{σ} at u is computed $\deg_u \mathrm{ev} = \mathrm{Tr}_{k(u)/k(\mathrm{ev}(u))} \langle \mathrm{disc}\, \pi_{\sigma} \rangle$.*

Proof. On $M^{\mathrm{odp}}_{0,n}(S,D)_{\sigma}$, the composition

$$\omega_{\mathrm{ev}_{\sigma}} \xrightarrow{\cong} \mathcal{O}(D_{\mathrm{cusp}}) \to \mathcal{O}(D_{\mathrm{cusp}} + 2D_{\mathrm{tac}}) = \mathcal{O}(\mathrm{div}\, \mathrm{disc}(\pi_{\sigma})) \xrightarrow{\cong} (\det \pi_{\sigma,*} \mathcal{O}_{\mathcal{D}^{\mathrm{odp}}})^{\otimes 2} \tag{18}$$

constructed in Sections 4 and 5 defines the orientation of ev_{σ}. Let $D\, \mathrm{ev}_{\sigma}$ denote the section of $\mathrm{Hom}(\mathrm{ev}^*_{\sigma} T^* S^n, T^* M^{\mathrm{odp}}_{0,n}(S,D))$ given by the differential of ev_{σ}, and let $\det D\, \mathrm{ev}_{\sigma}$ be its determinant, so $\det D\, \mathrm{ev}_{\sigma}$ is a section of $\omega_{\mathrm{ev}_{\sigma}}$. By construction, the first isomorphism of (18) takes $\det D\, \mathrm{ev}_{\sigma}$ to the section 1 of $\mathcal{O}(D_{\mathrm{cusp}})$, and the last isomorphism takes 1 to $\mathrm{disc}(\pi_{\sigma})$. Since the map $\mathcal{O}(D_{\mathrm{cusp}}) \to \mathcal{O}(D_{\mathrm{cusp}} + 2D_{\mathrm{tac}})$ takes the function 1 to the function 1, we have that $\det D\, \mathrm{ev}_{\sigma} \mapsto \mathrm{disc}(\pi_{\sigma})$. Therefore $\deg_u \mathrm{ev}_{\sigma} = \mathrm{Tr}_{k(u)/k(\mathrm{ev}(u))} \langle \mathrm{disc}\, \pi_{\sigma} \rangle$ by Proposition 3.8. □

The discriminant $\langle \operatorname{disc} \pi_\sigma \rangle$ of Proposition 7.3 has a concrete geometric description. The point u corresponds to a map $u : \mathbf{P} \to S_{k(u)}$ from a smooth genus 1 curve over $k(u)$ together with points $p_1, \ldots, p_n$ of $\mathbf{P}_{k^s}$ permuted appropriately under the Galois action. Over k^s, the image curve $u_{k^s}(\mathbf{P}_{k^s})$ has $\delta = \frac{1}{2} D \cdot (K_S + D) + 1$ nodes permuted under the action of $\operatorname{Gal}(k^s/k(u))$. Over $k(u)$ these nodes p consist of points of S with various residue fields $k(p)$. At each node, there are exactly two tangent directions of $u(\mathbf{P})$ defining a degree 2 field extension $k(p) \subseteq k(p)[\sqrt{m(p)}]$ where $m(p)$ is called the *mass* of the node. See Definition 1.1.

Proposition 7.4. *Suppose* u *is a point in* $M^{odp}_{0,n}(S, D)_\sigma$*, and let* $u : \mathbf{P} \to S_{k(u)}$ *be the corresponding map. Let* Nodes *denote the set of nodes of* $u(\mathbf{P})$*. For* $p \in$ Nodes*, let* $m(p)$ *denote the mass of the node* p*. Then*

$$\deg_u \operatorname{ev}_\sigma = \operatorname{Tr}_{k(u)/k(\operatorname{ev}_\sigma(u))} \prod_{p \in \text{Nodes}} m(p)$$

Proof. We show $\operatorname{disc} \pi_\sigma(u) = \prod_{p \in \text{Nodes}} m(p)$ in $k(u)^*/(k(u)^*)^2$, which is sufficient by Proposition 7.3. The double point locus $\mathcal{D}^{odp} \hookrightarrow X \times_{M^{odp}_{0,n}(S,D)} X$, where $X = M^{odp}_{0,n+1}(S, D)$ is the universal curve, inherits an action of $\mathbb{Z}/2$ from the involution on $X \times_{\overline{\mathcal{M}}_{0,n}} X$. Let $\mathcal{N}$ denote the quotient, which will be called the universal node. We obtain a factorization of π_σ

$$\mathcal{D}^{odp}_\sigma \to \mathcal{N} \to \overline{\mathcal{M}}_{0,n}(S, D)_\sigma.$$

Pulling back over $\operatorname{Spec} k(u) \to M^{odp}_{0,n}(S, D)$, the universal node splits as the disjoint union of the nodes p of $u(\mathbf{P})$ as u is in the ordinary double point locus, giving $\mathcal{N} \otimes k(u) \cong \coprod_{p \text{ nodes}} \operatorname{Spec} k(p)$. The pullback of double point locus $\mathcal{D}^{odp}$ over $\operatorname{Spec} k(p)$ is a degree 2 extension $\mathcal{D}_{k(p)} \to \operatorname{Spec} k(p)$, whence of the form $\operatorname{Spec} k(p)[\sqrt{D(p)}] \to \operatorname{Spec} k(p)$ or $\operatorname{Spec} k(p) \coprod \operatorname{Spec} k(p) \to \operatorname{Spec} k(p)$ (a split node) because the characteristic of k is not 2. Since the discriminant is multiplicative over products of rings, it follows that

$$\operatorname{disc} \pi_\sigma(u) = \prod_{p \text{ nodes}} \operatorname{disc}(\mathcal{D}_{k(p)} \to k(u))$$

By Lemma 7.5,

$$\begin{aligned} \operatorname{disc}(\mathcal{D}_{k(p)} \to k(u)) &= \operatorname{disc}(k(p)/k(u))^2 N_{k(p)/k(u)} \operatorname{disc}(\mathcal{D}_{k(p)} \to \operatorname{Spec} k(p)) \\ &= N_{k(p)/k(u)} D(p), \end{aligned}$$

where we set $D(p) = 1$ in the case of a split node. $\square$

We include the following well-known lemma for completeness.

Lemma 7.5. *Let* $K \subset L \subset M$ *be a tower of finite degree field extensions. Then,*

$$\operatorname{disc}(M/K) = \operatorname{disc}(L/K)^{[M:L]} N_{L/K}(\operatorname{disc}(M/L)).$$

Proof. Let $\{x_i\}_{i \in S}$ be a basis for L over K and let $\{y_j\}_{j \in T}$ be a basis for M over L. Define matrices A, B, by

$$A_i^j := \operatorname{Tr}_{L/K}(x_i x_j), \qquad B_i^j := \operatorname{Tr}_{M/L}(y_i y_j).$$

Observe that $\{x_i y_j\}_{i \in S, j \in T}$ is a basis for M over K. Define a matrix C by

$$C_{ij}^{kl} := \operatorname{Tr}_{M/K}(x_i y_j x_k y_l), \qquad i, k \in S, \quad j, l \in T.$$

So, we have

$$\mathrm{disc}(M/K) = [\det(C)] \in K^\times/(K^\times)^2, \qquad \mathrm{disc}(M/L) = [\det(B)] \in L^\times/(L^\times)^2,$$
$$\mathrm{disc}(L/K) = [\det(A)] \in K^\times/(K^\times)^2.$$

Calculate

$$C^{kl}_{ij} = \mathrm{Tr}_{L/K}(x_i x_k \,\mathrm{Tr}_{M/L}(y_j y_l)) = \mathrm{Tr}_{L/K}(x_i x_k B^l_j).$$

Write $B^l_j x_i = \sum_{m\in S} D^{ml}_{ij} x_m$ for $D^{ml}_{ij} \in K$. Then

$$\mathrm{Tr}_{L/K}(x_i x_k B^l_j) = \sum_{m\in S} D^{ml}_{ij} \,\mathrm{Tr}_{L/K}(x_m x_k) = \sum_{m\in S} D^{ml}_{ij} A^k_m.$$

Writing the result of the preceding calculation in terms of matrix multiplication, we have

$$C = D \circ (\mathrm{Id}_{[M:L]} \otimes A).$$

Taking determinants, we obtain

$$\det(C) = \det(A)^{[M:L]} \det(D) = \det(A)^{[M:L]} N_{L/K}(\det(B)).$$

A reference for the last equality is [KSW99]. The lemma follows. □

8. Enumerative theorems

Theorem 8.1. *Let k be a perfect field of characteristic not 2 or 3. Let S be a del Pezzo surface with an effective Cartier divisor D, satisfying Hypothesis 1. If k is of positive characteristic, assume additionally that Hypothesis 2 is satisfied. Fix a list $\sigma = (L_1, L_2, \ldots, L_r)$ of field extensions $k \subseteq L_i \subseteq \bar{k}$ such that $\sum_{i=1}^r [L_i : k] = n := \deg(-D \cdot K_S) - 1$. Then, there is $\underline{N}_{S,D,\sigma}$ in $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(\prod_{i=1}^r \mathrm{Res}_{L_i/k} S))$ such that for any generally chosen points p_i of S, $i = 1, \ldots, r$, with $k(p_i) \cong L_i$, we have the equality in* $\mathrm{GW}(k)$

$$\underline{N}_{S,D,\sigma}(p_*) = \sum_{\substack{u \text{ rational curve} \\ \text{on } S \\ \text{in class } D \\ \text{through the points} \\ p_1,\ldots,p_r}} \mathrm{Tr}_{k(u)/k} \prod_{p \text{ node of } u(\mathbf{P}^1)} \mathrm{mass}(p)$$

where p_* is the k-point of $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ given by $p_* = (p_1, \ldots, p_r)$.

Proof. In Sections 5.1 and 5.2 we constructed

$$\deg(\mathrm{ev}_\sigma : \bar{M}_{0,n}(S,D)^{\mathrm{good}}_\sigma \to S^n_\sigma) \in \mathcal{GW}(S^n_\sigma)$$
$$\deg(\mathrm{ev}_\sigma : M_{0,n}(S,D)^{\mathrm{odp}}_\sigma \to S^n_\sigma) \in \mathcal{GW}(S^n_\sigma)$$

in the case of characteristic 0 and positive characteristic, respectively. By (15), $S^n_\sigma \cong \prod_{i=1}^r \mathrm{Res}_{L_i/k} S$. In particular, the k-points p_* of $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ correspond to points p_i of $\mathbf{P}^2$, $i = 1, \ldots, r$, with $k(p_i) \cong L_i$. We may therefore define $\underline{N}_{S,D,\sigma}$ in $\mathcal{GW}(\pi_0^{\mathbf{A}^1}(\prod_{i=1}^r \mathrm{Res}_{L_i/k} S))$ to be $\deg^{\mathbf{A}^1} \mathrm{ev}_\sigma$ as in Definition 2.28. By Proposition 3.2, for any generally chosen k-point p_* of $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$, we have $\underline{N}_{S,D,\sigma}(p_*) = \sum_{u\in \mathrm{ev}_\sigma^{-1}(p_*)} \deg_u \mathrm{ev}_\sigma$. By construction, $\mathrm{ev}_\sigma^{-1}(p_*)$ is the set of rational curves on S in class D passing through the points $p_* = (p_1, \ldots, p_r)$. By [KLSW23, Corollay], for a generally chosen p_*, the rational curves u on S in class D passing through$(p_1, \ldots, p_r)$ determine points in $M_{0,n}(S,D)^{\mathrm{odp}}_\sigma$. By Proposition 7.4, the local degree $\deg_u \mathrm{ev}_\sigma$ is given by $\deg_u \mathrm{ev}_\sigma = \prod_{p \text{ node of } u(\mathbf{P}^1)} \mathrm{mass}(p)$, completing the proof.

□

Corollary 8.2. *In the notation of Theorem 8.1, suppose that* S *is additionally* $\mathbb{A}^1$*-connected. There is* $N_{S,D,\sigma}$ *in* $\mathrm{GW}(k)$ *such that for any generally chosen points* p_i *of* S, $i = 1, \ldots, r$ *, with* $k(p_i) \cong L_i$*, we have the equality in* $\mathrm{GW}(k)$

$$N_{S,D,\sigma} = \sum_{\substack{u \textit{ rational curve} \\ \textit{on } S \\ \textit{in class } D \\ \textit{through the points} \\ p_1,\ldots,p_r}} \mathrm{Tr}_{k(u)/k} \prod_{p \textit{ node of } u(\mathbf{P}^1)} \mathrm{mass}(p)$$

Proof. When S is $\mathbb{A}^1$-connected, the twisted product $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ is as well by Proposition 2.37. By Corollary 2.29, the section $\underline{N}_{S,D,\sigma}$ of Theorem 8.1 is pulled back from a unique element $N_{S,D,\sigma}$ in $\mathrm{GW}(k)$, which has the claimed property by Theorem 8.1.

□

Remark 8.3. *By construction, the invariants* $\deg(\mathrm{ev}_\sigma)$ *in* $\mathrm{GW}(k)$ *only depend on the list of field extensions* $\{L_1, \ldots, L_r\}$*. Thus the multi-set* $\{k(p_i) : i = 1, \ldots, r\}$ *of the fields of definition the* p_i *counted with multiplicity determines the count of the degree* D *rational curves through points with the same multi-set of field extensions is independent of the chosen points. This strengthens* [Lev18, Example 3.9] *where this statement is proven for* $[L_i : k] \leq 2$.

9. Examples

Let S be an $\mathbb{A}^1$-connected del Pezzo surface over a field k and D a Cartier divisor on S. Let $\sigma = (L_1, L_2, \ldots, L_r)$ be a list of separable field extensions $k \subseteq L_i \subseteq \overline{k}$ such that $\sum_{i=1}^r [L_i : k] = n := \deg(-D \cdot K_S) - 1$. Let

$$k(\sigma) := \prod_{i=1}^r L_i.$$

So $\mathrm{Tr}_{k(\sigma)/k}\langle 1 \rangle = \sum_{i=1}^r \mathrm{Tr}_{L_i/k}\langle 1 \rangle$ is the sum of the trace forms of the field extensions $k \subseteq L_i$. For example, for k of characteristic not dividing $2d$, $d \in k$, and $\sigma = (k, k, \ldots, k, k\sqrt{d})$, we have

$$\mathrm{Tr}_{k(\sigma)/k}\langle 1 \rangle = (n-2)\langle 1 \rangle + \langle 2 \rangle + \langle 2d \rangle.$$

Table 1 computes some values of $N_{S,D,\sigma}$. Justifications follow below. (Note in particular that for appropriate σ and S, many of the $N_{S,D,\sigma}$ in Table 1 are not only sums of $\langle \pm 1 \rangle$'s; a lot more is happening here than over $\mathbf{R}$ and $\mathbf{C}$.) For S a hypersurface, for example an $\mathbb{A}^1$-connected cubic surface, the $\mathbb{A}^1$-Euler characteristic can be computed explicitly using [LLS21].

9.1. $\mathbb{A}^1$-connected del Pezzo surfaces.

The following is a theorem of Asok and Morel [AM11, Corollary 2.3.7].

Theorem 9.1. *A smooth proper surface over* k *which is rational over* k *is* $\mathbb{A}^1$*-connected.*

Example 9.2. [KSC04, Example 1.33, 1.35] *A smooth cubic surface is rational over* k *if it contains two skew lines over* k *or two conjugate skew lines defined over* $k(\sqrt{a})$ *for some degree* 2 *extension* $k \subset k(\sqrt{a})$*. It then follows* [KSC04, Exercise 1.34, Example 1.35] *that*

Table 1. $\mathrm{GW}(k)$-enriched counts of rational curves

S	D	σ	$N_{S,D,\sigma}$ = count of rational curves
$\mathbf{P}^2$	$\mathcal{O}(1)$	all σ	$\langle 1\rangle$
$\mathbf{P}^2$	$\mathcal{O}(2)$	all σ	$\langle 1\rangle$
$\mathbf{P}^2$	$\mathcal{O}(3)$	all σ	$2(\langle 1\rangle+\langle -1\rangle)+\mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle$
$\mathbf{P}^1\times\mathbf{P}^1$	$\mathcal{O}(1)\boxtimes\mathcal{O}(d)$	all σ	$\langle 1\rangle$
$\mathbf{P}^1\times\mathbf{P}^1$	$\mathcal{O}(2)\boxtimes\mathcal{O}(2)$	all σ	$2(\langle 1\rangle+\langle -1\rangle)+\langle 1\rangle+\mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle$
Fermat Cubic Surface $x^3+y^3+z^3+w^3=0$	$\mathcal{O}_{\mathbf{P}^3}(1)$	all σ	$\langle -3\rangle+4(\langle 1\rangle+\langle -1\rangle)+\langle 1\rangle+\mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle$
the cubic surface $xy^2+y^2z+z^2w+w^2x=0$	$\mathcal{O}_{\mathbf{P}^3}(1)$	all σ	$\langle 5\rangle+4(\langle 1\rangle+\langle -1\rangle)+\langle 1\rangle+\mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle$
S	$-K_S$	all σ	$\langle -1\rangle\chi^{\mathbb{A}^1}(S)+\langle 1\rangle+\mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle$
...	...	...	...

$x^2y+y^2z+z^2w+w^2x=0$ *and* $x^3+y^3+z^3+w^3=0$ *determine* $\mathbb{A}^1$*-connected smooth cubic surfaces over fields of characteristic not* 2 *or* 3.

9.2. **$N_{S,-K_S,\sigma}$ for $d_S\geq 3$.** For $D=-K_S$, we have $n=d_S-1$. Since $d_S\geq 3$, a choice of basis for $H^0(S,-K_S)$ determines an embedding $S\hookrightarrow\mathbf{P}_k^{d_S}$. For a general choice of points $(p_1,\ldots,p_r)$ of S such that $k(p_i)\cong L_i$, the $\sum_{i=1}^r[k(p):k]=n$ linear conditions on $H^0(\mathbf{P}_k^{d_S},\mathcal{O}(1))$ corresponding to vanishing on p_i for $i=1,\ldots,r$ are independent. (As before, the meaning of the phrase "a general choice of points $(p_1,\ldots,p_r)$ of S such that $k(p_i)\cong L_i$" is that there is a nonempty open set U of $\prod_{i=1}^r\mathrm{Res}_{L_i/k}S$ such that the claim holds for rational points of U. Moreover, this U is stable under base change, so there will be rational points after some finite extension of k, giving rise to potentially different Galois representation and list $\sigma=(L_1,\ldots,L_{r'})$ for which the result holds.) Thus

$$\{f\in H^0(\mathbf{P}_k^{d_S},\mathcal{O}(1)):f(p_i)=0\text{ for }i=1,\ldots r\}$$

is a 2-dimensional vector space over k. Choose a basis $\{f,g\}$ and let

$$X=\{[s,t]\times x:tf(x)+sg(x)=0\}\subset\mathbf{P}^1\times S$$

be the corresponding pencil. The baselocus $B=\{f=g=0\}\hookrightarrow S$ of the pencil has degree d_S by Bézout's theorem. By construction, the points p_i lie in B, whence $B=\{p_1,\ldots,p_i\}\cup\{p_0\}$ where p_0 is a k-rational point of S.

Let $\pi:X\to\mathbf{P}_k^1$ denote the projection. By construction, the fibers of π are precisely the curves in class $-K_S$ passing through $\{p_1,\ldots,p_i\}$. (These all then also pass through p_0.)

Let $C\hookrightarrow S$ be a general fiber of π. By adjunction, C has canonical class $K_C=K_S\otimes\mathcal{O}(C)$. Since C is in class $-K_S$, we have $\mathcal{O}(C)\cong -K_S$, whence $K_C=\mathcal{O}$ and C has arithmetic genus 1. It follows that the fibers of π are either smooth or rational with a single node. Thus

$$N_{S,D,\sigma}=\sum_{\substack{u\text{ rational curve}\\ \text{on }S\\ \text{in class }-K_S\\ \text{through the points}\\ p_1,\ldots,p_r}}\mathrm{Tr}_{k(u)/k}\,\mathrm{mass}(p(u)),$$

where $p(u)$ denotes the note of $u(\mathbf{P}^1)$. We will compute the right hand side directly using the A^1-Euler characteristic.

The projection $X \to S$ realizes X as the blow-up

$$X \cong \mathrm{Bl}_B S.$$

It follows from [Lev20, Prop 1.4] that the A^1-Euler characteristic $\chi^{\mathrm{A}^1}(X)$ is computed

$$\begin{aligned}
\chi^{\mathrm{A}^1}(X) &= \chi^{\mathrm{A}^1}(S) + (\chi^{\mathrm{A}^1}(\mathbf{P}^1) - \langle 1\rangle)\chi^{\mathrm{A}^1}(B) \\
&= \chi^{\mathrm{A}^1}(S) + \langle -1\rangle\chi^{\mathrm{A}^1}(\{p_0, \dots, p_i\}) \\
&= \chi^{\mathrm{A}^1}(S) + \langle -1\rangle + \langle -1\rangle\chi^{\mathrm{A}^1}(\{p_1, \dots, p_i\}) \\
&= \chi^{\mathrm{A}^1}(S) + \langle -1\rangle + \sum_{i=1}^{i} \langle -1\rangle \,\mathrm{Tr}_{k(p_i)/k}\langle 1\rangle,
\end{aligned}$$

whence

$$\chi^{\mathrm{A}^1}(X) = \chi^{\mathrm{A}^1}(S) + \langle -1\rangle + \langle -1\rangle \,\mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle. \tag{19}$$

We have a second calculation of $\chi^{\mathrm{A}^1}(X)$ using π and the work of the second named author M. Levine [Lev20, Section 10]. Comparing the two will compute $N_{S,-K_S,\sigma}$. Here is the second calculation. An isomorphism $T\mathbf{P}^1 \cong \mathcal{O}(1)^{\otimes 2}$ defines a relative orientation of $\mathrm{Hom}(\pi^* T^*\mathbf{P}^1, T^*X)$, where T^*X denotes the cotangent bundle, or Kähler differentials. We may thus let $n(\mathrm{Hom}(\pi^* T^*\mathbf{P}^1, T^*X))$ be the Euler number. The morphism π determines a section $d\pi$ of the bundle $\mathrm{Hom}(\pi^* T^*\mathbf{P}^1, T^*X)$ and the Euler number can be computed as a sum

$$n(\mathrm{Hom}(\pi^* T^*\mathbf{P}^1, T^*X)) = \sum_{x : d\pi(x)=0} \mathrm{ind}_x\, d\pi.$$

See [Lev20, Section 1] or [KW17, Section 4] and [BW20] for compatibility checks. In the Witt group $W(k) := \mathrm{GW}(k)/\mathbf{Z}(\langle 1\rangle + \langle -1\rangle)$, we have equalities

$$\chi^{\mathrm{A}^1}(X) = n(T^*X) = n(\mathrm{Hom}(\pi^* T^*\mathbf{P}^1, T^*X)) = \sum_{x : d\pi(x)=0} \mathrm{ind}_x\, d\pi, \tag{20}$$

where the first equality is [Lev20, Theorem 3.1, Theorem 7.1] and and the second is [Lev20, Theorem 9.1]. Comparison with the classical computation (where χ is multiplicative and the general fiber has Euler characteristic 0), shows (20) is also valid in $\mathrm{GW}(k)$.

Lemma 9.3. *For general $(p_1, \dots, p_i)$, the zeros of $d\pi$ are the nodes in the fibers of π, and for a node p, the local index $\mathrm{ind}_p\, d\pi$ is computed $\mathrm{ind}_p\, d\pi = \mathrm{Tr}_{k(p)/k}\langle -1\rangle m(p)$.*

Proof. Let $U \subset X$ denote the open subset of the pencil given by $U = \{s \neq 0\} \cong \mathbf{A}^1 \times S$ and let t be the coordinate on $\mathbf{A}^1$. Choose (t, p) in U, and local analytic coordinates (x, y) on S for the completion of $\mathcal{O}_S$ at p. In these coordinates, U is given by

$$\{t \times (x, y) : tF(x, y) + G(x, y) = 0\}.$$

The point (t, p) is a zero of $d\pi$ if and only if $\pi^* dt(t, p) = 0$. Since $tF(x, y) + G(x, y) = 0$, we have that

$$dtF + t\partial_x F dx + t\partial_y F dy + \partial_x G dx + \partial_y G dy = 0 \tag{21}$$

at p. Since we assume the pencil is smooth, we can not have $F = t\partial_x F + \partial_x G = t\partial_y F + \partial_y G = 0$. It follows that $\pi^* dt(t, p) = 0$ if and only if

$$(t\partial_x F + \partial_x G)dx + (t\partial_y F + \partial_y G)dy = 0,$$

which occurs if and only if $t\partial_x F + \partial_x G = t\partial_y F + \partial_y G = 0$. This latter condition occurs if and only if p is a node of $tF + G = 0$. Thus the zeros of $d\pi$ are the nodes in the fibers of π as claimed. Note that we have also shown that if p is a node in the fiber at t, then $F(p) \neq 0$.

$d\pi$ is a section of the vector bundle $\mathrm{Hom}(\pi^* T^* \mathbf{P}^1, T^* X)$. Without loss of generality, we may assume that a zero of π is in U, i.e. the zero is of the form (t, p). Consider the local trivialization of $\mathrm{Hom}(\pi^* T^* \mathbf{P}^1, T^* X)$ corresponding to the basis $\{dt \mapsto dx, dt \mapsto dy\}$. This local trivialization is compatible with the local coordinates and the canonical relative orientation of $\mathrm{Hom}(\pi^* T^* \mathbf{P}^1, T^* X)$ (coming from the orientability of $\mathbf{P}^1$). Using these local coordinates and local trivialization, the section $d\pi$ corresponds to

$$\left(\frac{t\partial_x F + \partial_x G}{F}, \frac{t\partial_y F + \partial_y G}{F}\right)$$

because $d\pi(dt) = (t\partial_x F + \partial_x G)/F dx + (t\partial_y F + \partial_y G)/F dy$ by (21). Since $t = -G/F$, this $d\pi$ likewise corresponds to the function

$$\left(\frac{-G\partial_x F + F\partial_x G}{F^2}, \frac{-G\partial_y F + F\partial_y G}{F^2}\right) = \left(\partial_x \frac{G}{F}, \partial_y \frac{G}{F}\right).$$

The Hessian $\mathrm{Hess}(G/F)$ as a function of x and y equals the Hessian $\mathrm{Hess}(t + G/F)$ because t is a fixed scalar. Moreover, since $t + G/F$ and its partials vanish at p, there is an equality of $\mathrm{Hess}(tF + G)$ and $\mathrm{Hess}(t + G/F)$ evaluated at p by the chain rule. By genericity, the fibers $\{(x, y) : tF + G = 0\}$ of π have only nodes. The $\mathbf{A}^1$-Milnor number of $\{tF + G = 0\}$ at p is $\langle \mathrm{Hess}(tF + G)(p)\rangle$ and $\mathrm{Hess}(tF + G)(p) \neq 0$. Thus $\mathrm{Hess}(G/F)(p) \neq 0$. Since p is a node $k \subseteq k(p)$ is a separable extension [SGA73, Exposé XV, Théorème 1.2.6] and we we have that $\mathrm{ind}_p\, d\pi = \mathrm{Tr}_{k(p)/k}\langle \mathrm{Hess}(G/F)(p)\rangle$ by [KW17, Proposition 34], whence $\mathrm{ind}_p\, d\pi = \mathrm{Tr}_{k(p)/k}\langle \mathrm{Hess}(tF + G)(p)\rangle$, proving the claim. □

Combining (19), (20) and Lemma 9.3 we have:

Example 9.4. $N_{S,-K_S,\sigma} = \langle -1\rangle \chi^{\mathrm{A}^1}(S) + \langle 1\rangle + \mathrm{Tr}_{k(\sigma)/k}\langle 1\rangle$

Note that it is not necessary to assume that S is A^1-connected for this computation to be valid.

References

[AM11] Aravind Asok and Fabien Morel. Smooth varieties up to $\mathbb{A}^1$-homotopy and algebraic h-cobordisms. *Adv. Math.*, 227(5):1990–2058, 2011.

[AO01] Dan Abramovich and Frans Oort. Stable maps and Hurwitz schemes in mixed characteristics. In *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, volume 276 of *Contemp. Math.*, pages 89–100. Amer. Math. Soc., Providence, RI, 2001.

[AWW17a] Aravind Asok, Kirsten Wickelgren, and Ben Williams. The simplicial suspension sequence in $\mathbb{A}^1$-homotopy. *Geom. Topol.*, 21(4):2093–2160, 2017.

[AWW17b] Aravind Asok, Kirsten Wickelgren, and Ben Williams. The simplicial suspension sequence in $\mathbb{A}^1$-homotopy. *Geom. Topol.*, 21(4):2093–2160, 2017.

[BW20] T. Bachmann and K. Wickelgren. $\mathbb{A}^1$-Euler classes: six functors formalisms, dualities, integrality, and linear subspaces of complete intersections. Preprint, available at `https://arxiv.org/abs/2002.01848`, 2020.

[Che22] Xujia Chen. Steenrod pseudocycles, lifted cobordisms, and Solomon's relations for Welschinger's invariants. *Geom. Funct. Anal.*, 32(3):490–567, 2022.

[Cho08] Cheol-Hyun Cho. Counting real J-holomorphic discs and spheres in dimension four and six. *J. Korean Math. Soc.*, 45(5):1427–1442, 2008.

[Con00] Brian Conrad. *Grothendieck duality and base change*, volume 1750 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[CTS79] J.-L. Colliot-Thélène and J.-J. Sansuc. Fibrés quadratiques et composantes connexes réelles. *Math. Ann.*, 244(2):105–134, 1979.

[CZ21] Xujia Chen and Aleksey Zinger. WDVV-type relations for disk Gromov-Witten invariants in dimension 6. *Math. Ann.*, 379(3-4):1231–1313, 2021.

[DHI04] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. *Math. Proc. Cambridge Philos. Soc.*, 136(1):9–51, 2004.

[dJHS11] A. J. de Jong, Xuhua He, and Jason Michael Starr. Families of rationally simply connected varieties over surfaces and torsors for semisimple groups. *Publ. Math. Inst. Hautes Études Sci.*, (114):1–85, 2011.

[DK00] A. I. Degtyarev and V. M. Kharlamov. Topological properties of real algebraic varieties: Rokhlin's way. *Uspekhi Mat. Nauk*, 55(4(334)):129–212, 2000.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Giv96] Alexander B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Har66] Robin Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HS12] A. Horev and J. P. Solomon. The open Gromov-Witten-Welschinger theory of blowups of the projective plane. *arXiv e-prints*, October 2012.

[Jar07] J.F. Jardine. Fields lectures: Simplicial presheaves. Preprint, available at `https://www.uwo.ca/math/faculty/jardine/courses/fields/fields-01.pdf`, 2007.

[JPP22] Anrés Jaramillo Puentes and Sabrina Pauli. Quadratically enriched tropical intersections. 2022. *Preprint*, available at `http://arxiv.org/abs/2208.00240`.

[KLSW23] Jesse Leo Kass, Marc Levine, Jake P. Solomon, and Kirsten Wickelgren. A relative orientation for the moduli space of stable maps to a del pezzo surface. *Preprint*, 2023.

[KM94] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.

[Kne70] Manfred Knebusch. Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen. *S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl.*, 1969/70:93–157, 1969/1970.

[Kon95] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[KSC04] János Kollár, Karen E. Smith, and Alessio Corti. *Rational and nearly rational varieties*, volume 92 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.

[KSW99] Istvan Kovacs, Daniel S. Silver, and Susan G. Williams. Determinants of commuting-block matrices. *Amer. Math. Monthly*, 106(10):950–952, 1999.

[KW17] Jesse Kass and Kirsten Wickelgren. An arithmetic count of the lines on a smooth cubic surface. *Preprint*, available at `https://arxiv.org/abs/1708.01175`, 2017.

[KW19] Jesse Kass and Kirsten Wickelgren. The class of Eisenbud–Khimshiashvili–Levine is the local A1-brouwer degree. *Duke Mathematical Journal*, 168(3):429–469, 2019.

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Lam06] T. Y. Lam. *Serre's problem on projective modules*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.

[Lev18] Marc Levine. Toward an algebraic theory of Welschinger invariants. *Preprint*, available at `https://arxiv.org/abs/1808.02238`, 2018.

[Lev20] Marc Levine. Aspects of enumerative geometry with quadratic forms. *Doc. Math.*, 25:2179–2239, 2020.

[LLS21] Marc Levine, Simon Pepin Lehalleur, and Vasudevan Srinivas. Euler characteristics of homogeneous and weighted-homogeneous hypersurfaces. *Preprint*, available at `https://arxiv.org/abs/2101.00482`, 2021.

[LLY97] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[MH73] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[Mil70] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970.

[Mor04] Fabien Morel. On the motivic π_0 of the sphere spectrum. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 219–260. Kluwer Acad. Publ., Dordrecht, 2004.

[Mor12] Fabien Morel. $\mathbb{A}^1$*-algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[MS94] Dusa McDuff and Dietmar Salamon. J*-holomorphic curves and quantum cohomology*, volume 6 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1994.

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[OP99] Manuel Ojanguren and Ivan Panin. A purity theorem for the witt group. *Ann. Sci. École Norm. Sup.*, (4) 32(1):71–86, 1999.

[OVV07] D. Orlov, A. Vishik, and V. Voevodsky. An exact sequence for $\mathsf{K}^{\mathsf{M}}_*/2$ with applications to quadratic forms. *Ann. of Math. (2)*, 165(1):1–13, 2007.

[PSW08] R. Pandharipande, J. Solomon, and J. Walcher. Disk enumeration on the quintic 3-fold. *J. Amer. Math. Soc.*, 21(4):1169–1209, 2008.

[RT94] Yongbin Ruan and Gang Tian. A mathematical theory of quantum cohomology. *Math. Res. Lett.*, 1(2):269–278, 1994.

[RT95] Yongbin Ruan and Gang Tian. A mathematical theory of quantum cohomology. *J. Differential Geom.*, 42(2):259–367, 1995.

[SGA73] *Groupes de monodromie en géométrie algébrique. Exposés X à XXII.* Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7II), Dirigé par P. Deligne et N. Katz.

[sga03] *Revêtements étales et groupe fondamental (SGA 1).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

[Sol06] Jake P. Solomon. Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions. *Thesis*, available at `https://arxiv.org/abs/math/0606429`, 2006.

[Sol07] Jake P. Solomon. A differential equation for the open Gromov-Witten potential. preprint, October 2007.

[SS75] Günter Scheja and Uwe Storch. Über Spurfunktionen bei vollständigen Durchschnitten. *J. Reine Angew. Math.*, 278/279:174–190, 1975.

[ST21] Jake P. Solomon and Sara B. Tukachinsky. Point-like bounding chains in open Gromov-Witten theory. *Geom. Funct. Anal.*, 31(5):1245–1320, 2021.

[ST23] Jake P. Solomon and Sara B. Tukachinsky. Relative quantum cohomology. *J. Eur. Math. Soc.*, 2023.

[Sta18] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

[Voe03a] Vladimir Voevodsky. Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (98):59–104, 2003.

[Voe03b] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, (98):1–57, 2003.

[Wel05a] Jean-Yves Welschinger. Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry. *Invent. Math.*, 162(1):195–234, 2005.

[Wel05b] Jean-Yves Welschinger. Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants. *Duke Math. J.*, 127(1):89–121, 2005.

[Wit88] Edward Witten. Topological sigma models. *Comm. Math. Phys.*, 118(3):411–449, 1988.

[Wit91] Edward Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

[Wit95] E. Witten. Chern-Simons gauge theory as a string theory. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 637–678. Birkhäuser, Basel, 1995.

[Zar58] Oscar Zariski. On the purity of the branch locus of algebraic functions. *Proc. Nat. Acad. Sci. U.S.A.*, 44:791–796, 1958.

[Zeu73] H. G. Zeuthen. Common properties in systems of plane curves. *Kjobenhavn. Vidensk. Selsk.*, 5(4):287–393, 1873.

Current: J. L. Kass, Dept. of Mathematics, University of California, Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, United States of America

Email address: jelkass@ucsc.edu

URL: https://www.math.ucsc.edu/people/faculty.php?uid=jelkass

Current: M. Levine, University of Duisburg-Essen, Germany

Email address: marc.levine@uni-due.de

URL: https://www.esaga.uni-due.de/marc.levine/

Current: J. P. Solomon, Institute of Mathematics, Hebrew University, Givat Ram Jerusalem, 91904, Israel

Email address: jake@math.huji.ac.il

URL: http://www.ma.huji.ac.il/~jake/

Current: K. Wickelgren, Department of Mathematics, Duke University, 120 Science Drive Room 117 Physics, Box 90320, Durham, NC 27708-0320, USA

Email address: kirsten.wickelgren@duke.edu

URL: https://services.math.duke.edu/~kgw/