LOWER CENTRAL SERIES OBSTRUCTIONS TO HOMOTOPY SECTIONS OF CURVES OVER NUMBER FIELDS

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CONTENTS

Notation. k will denote a subfield of $\mathbb{C}$. A curve over k means a finite type, separated, reduced scheme of dimension 1 over k. (So curves are not assumed to be proper or smooth.) $\overline{k}$ is the algebraic closure of k in $\mathbb{C}$. $G_k = \mathrm{Gal}(\overline{k}/k)$ denotes the absolute Galois group of k. $X_{\overline{k}} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$ denotes the base change to $\overline{k}$ of a scheme X over k. $X(k)$ denotes the maps $\mathrm{Spec}\, k \to X$, called 'k points' or 'rational points,' for such a scheme. π_1^{top} and π_1^{et} denote the topological and étale fundamental groups respectively. For a group G, $G^\wedge$ denotes the profinite completion of G.

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1. Introduction

1.1. **Some background and motivation.** We study cohomological obstructions of Jordan Ellenberg to sections of the map on étale fundamental groups induced by the structure map of a curve defined over a subfield of $\mathbb{C}$. Let $\sigma : X \to \operatorname{Spec} k$ denote such a structure map. We will be most interested in the case where k is a number field. The obstructions examined here study the role of the lower central series of π_1 in blocking sections of $\pi_1(\sigma)$, and therefore in blocking rational points. While the interest of understanding rational points is clear, the motivation for understanding sections of $\pi_1(\sigma)$ requires some explanation: Grothendieck [Gro97] conjectures the existence of a subcategory of "anabelian" schemes, including hyperbolic curves over number fields, $\operatorname{Spec} k$ for k a number field, moduli spaces of curves, and total spaces of fibrations with base and fiber anabelian, which are determined by their étale fundamental groups. (See [Sza00] for a nice description of these conjectures.) These conjectures can be viewed as follows: algebraic maps are so rigid that homotopies do not deform one into another. From this point of view, a $K(\pi, 1)$ in algebraic geometry would be a variety X such that $\operatorname{Mor}(Y, X) = \operatorname{Hom}(\pi_1(Y), \pi_1(X))$ (ignoring basepoints for now). In other words, "anabelian schemes" can be viewed as algebraic geometry's $K(\pi, 1)$'s, and Grothendieck conjectures that some familiar $K(\pi, 1)$'s from topology (not including elliptic curves or abelian varieties!) are also $K(\pi, 1)$'s in algebraic geometry. (See also [NSW08, Ch. XII].) In particular, the rational points on a hyperbolic curve over a number field are conjectured to be in bijection with the sections of π_1 of the structure map (up to certain inner automorphisms to account for basepoints). This is Grothendieck's *Section Conjecture*.

A topologist could take the point of view that Grothendieck's anabelian conjectures identify how to associate a topological space to an anabelian scheme over a number field k; instead of taking the topological space T underlying the associated complex analytic space, one should take the classifying space of the étale fundamental group. The short exact sequence

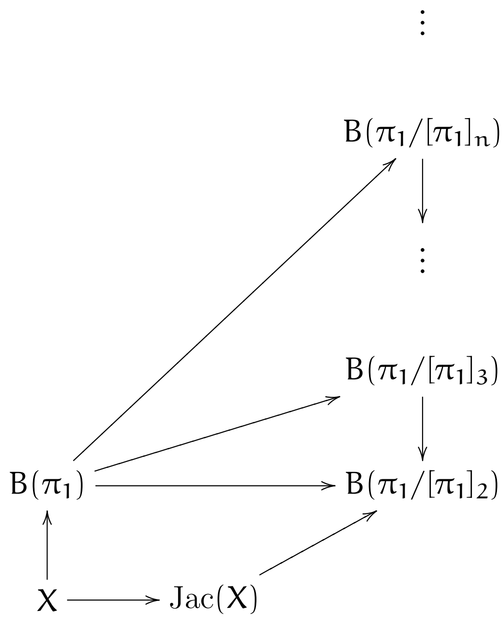
$$1 \to \pi_1^{et}(X_{\overline{k}}) \to \pi_1^{et}(X) \to G_k \to 1$$

for a (geometrically connected, quasi-compact) scheme X over k [SGAI, Exp. IX Thm. 6.1], and the isomorphism $\pi_1^{et}(X_{\overline{k}}) \cong \pi_1^{top}(T)^\wedge$ [SGAI, Exp. XII Cor. 5.2] [Méz00, Prop. 2.18] allow us to view the classifying space $B(\pi_1^{et}(X))$ as the total space of a fibration over $B(G_k)$ with fiber $B(\pi_1^{top}(T)^\wedge)$. ($B(G)$ denotes the classifying space of G. Since étale fundamental groups are profinite, care should be taken when defining and considering classifying spaces and fibrations between these classifying spaces. For our purposes, it turns out we can sidestep these issues using group cohomology, so we don't define these notions rigorously, but see [Goe95] and [Dav06]. By the heuristic that T should be a $K(\pi_1^{top}(T), 1)$ for X anabelian, an anabelian scheme X is associated to the total space a topological fibration over $B(G_k)$ whose fiber $B(\pi_1^{top}(T)^\wedge)$ is some sort of completion of T. The anabelian conjectures then assert that studying these topological spaces up to homotopy is equivalent to studying the original schemes. A broad goal of this work is to compare algebraic curves and topological approximations to them.

1.1.1. Two topological pictures of Ellenberg's obstructions. More specifically, we study obstructions of Jordan Ellenberg [Ell] which admit the following interpretation (see 1.1.3

for a precise description of these obstructions): take a hyperbolic curve X over $k \subset \mathbb{C}$, and let π_1 denote its étale fundamental group. We place ourselves in some unspecified category 'containing' both schemes and classifying spaces of profinite groups (e.g. prosimplicial sets. See [AM86] [Fri82]). Approximate X first by its Jacobian or the classifying space of the abelianization of π_1, and then by a tower of fibrations associated to π_1's lower central series. Letting $\pi_1 > [\pi_1]_2 > [\pi_1]_3 > \ldots$ denote this lower central series and $B(\pi_1/[\pi_1]_n)$ denote the classifying space of $\pi_1/[\pi_1]_n$, this tower is

(1)



and this tower lies over $\operatorname{Spec} k$ or $B(G_k)$. Maps to X are studied by lifting maps through this tower. Ellenberg's obstruction δ_n is the obstruction to lifting a homotopy section of $B(\pi_1/[\pi_1]_n) \to B(G_k)$ through the fibration $B(\pi_1/[\pi_1]_{n+1}) \to B(\pi_1/[\pi_1]_n)$. By 'homotopy section' we mean that we assume our category has some notion of homotopy and such a homotopy section is a homotopy class of maps $B(G_k) \to B(\pi_1/[\pi_1]_n)$, so that the composite $B(G_k) \to B(G_k)$ is the homotopy class of the identity. (Rigorous definitions of δ_n and homotopy sections for use in this paper are given in 1.1.3.)

The δ_n give a series of obstructions to a homotopy section of $B(\pi_1/[\pi_1]_2)$ to be the image of a homotopy section of $B(\pi_1)$, and therefore the δ_n also give a recursive procedure for eliminating k points of the Jacobian which are not k points of the curve. For the later, a k point of X or its Jacobian determines a homotopy section of $B(\pi)$ or $B(\pi_1/[\pi_1]_2)$ respectively. (This is described precisely in 1.1.3.) Starting with a k point of the Jacobian, take the corresponding homotopy section. Then apply δ_2. If δ_2 vanishes, we can lift to a homotopy section of $B(\pi_1/[\pi_1]_3)$. Apply δ_3 to all such lifts. For the lifts such that δ_3 vanishes, we obtain another set of lifts to $B(\pi_1/[\pi_1]_4)$. Apply δ_4 to all these lifts etc.

Alternatively, we have the viewpoint that we approximate $X_{\overline{k}}$ G_k-equivariantly by the tower (1) with X, $\operatorname{Jac}(X)$, and π_1 replaced by $X_{\overline{k}}$, $\operatorname{Jac}(X_{\overline{k}})$, and $\pi_1^{et}(X_{\overline{k}})$ respectively. Assuming our category admits some notion of homotopy fixed point set and π_0, we apply the homotopy fixed point functor and then π_0 to the tower. (The definition of the homotopy fixed point set of a group acting on a topological space is given in Example 1.1.2. See Daniel Davis's work, e.g. [Dav06], for a more sophisticated understanding of homotopy fixed points.) Ellenberg's obstruction δ_n is the obstruction to lifting a connected component of the homotopy fixed point set of $B(\pi_1/[\pi_1]_n)$ through the fibration $B(\pi_1/[\pi_1]_{n+1}) \to B(\pi_1/[\pi_1]_n)$. (Here π_1 denotes $\pi_1^{et}(X_{\overline{k}})$.)

1.1.2. *Example.* We compute the second approximating tower and the obstructions δ_n for $X = \mathbb{P}^1_{\mathbb{R}} - \{0, 1, \infty\}$. We work in the category of topological spaces. Since the topological fundamental group admits an action of $G_{\mathbb{R}}$ and since the étale fundamental group is $G_{\mathbb{R}}$ equivariantly the profinite completion of the topological fundamental group, we use π_1^{top} instead of π_1^{et}. (This allows us to draw pictures of the spaces in the tower, and the difference between π_1^{top} and π_1^{et} turns out to be unimportant here- see Proposition 3.1.19. Similarly, drawing the second tower is easier than the first. To form the first from the second, apply $- \times_{B(G_{\mathbb{R}})} E(G_{\mathbb{R}}) \cong - \times_{\mathbb{RP}^\infty} S^\infty$.) Let the base point of X be the tangential base point at 0 pointing along the real line towards 1. Let π_1 denote $\pi_1^{top}(X_{\overline{k}}, b)$. π_1 is the free group on the two generators α_1, α_2 shown below. Let τ denote complex conjugation, so $G_{\mathbb{R}} = \langle \tau \rangle \cong \mathbb{Z}/2$. The action of $G_{\mathbb{R}}$ on π_1 is given by $\tau\alpha_i = \alpha_i^{-1}$ for $i = 1, 2$.

Homotopy fixed point sets: for a group G acting on a topological space T, the *homotopy fixed point set* will mean $F(EG, T)^G$, where EG denotes a contractible space with a free action of G, $F(EG, T)$ denotes the space of functions $f : EG \to T$ equipped with the G action $gf = gfg^{-1}$, and $F(EG, T)^G$ denotes the fixed points. The G equivariant map from EG to the one point space with trivial G action induces a map $T^G \to F(EG, T)^G$.

For the $G_{\mathbb{R}}$ spaces in this example, $T^G \to F(EG, T)^G$ is a bijection on π_0 (see Corollary 3.1.5 and the specific models for $B(\pi_1/[\pi_1]_n)$ constructed below), so the obstruction δ_n is the obstruction to lifting a connected component of the fixed point set $B(\pi_1/[\pi_1]_n)^{G_{\mathbb{R}}}$ through the fibration $B(\pi_1/[\pi_1]_{n+1}) \to B(\pi_1/[\pi_1]_n)$.

The topological spaces in the tower are as follows: $X_{\overline{\mathbb{R}}} = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ is a model for $B(\pi_1)$. The fixed point set $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}^{G_{\mathbb{R}}}$ is $\mathbb{P}^1_{\mathbb{R}} - \{0, 1, \infty\}$. Both are shown in Figure 1.

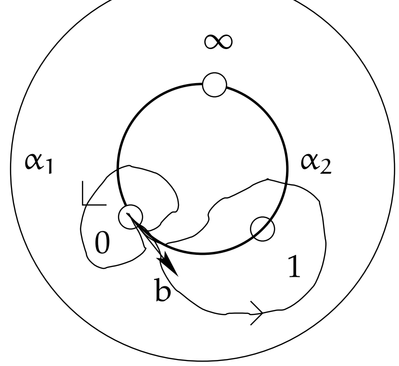


FIGURE 1. $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$

We now build models for $B(\pi_1/[\pi_1]_n)$ from free actions of $\pi_1/[\pi_1]_n$ on $\mathbb{R}^{2^{n-2}+1}$. The model for $B(\pi_1/[\pi_1]_n)$ is a quotient of the $2^{n-2} + 1$ dimensional 'cube'

$$[0, 1]^{2^{n-2}+1}$$

Start with the case $n = 2$. Any element of $\pi_1/[\pi_1]_2$ can be written uniquely in the form $\alpha_1^x\alpha_2^y$ for $x, y \in \mathbb{Z}$, so we have a bijection from $\pi_1/[\pi_1]_2$ to the set $\mathbb{Z}^2$. (This bijection is of course also a group isomorphism, but that is not important.) The free action of $\pi_1/[\pi_1]_2$ on itself by left multiplication induces a free action of $\pi_1/[\pi_1]_2$ on the set $\mathbb{Z}^2$ via this bijection, and it is easy to see that this action extends to a free action of $\pi_1/[\pi_1]_2$ on $\mathbb{R}^2$, giving the usual torus model $\mathbb{R}^2/\mathbb{Z}^2$ for $B(\pi_1/[\pi_1]_2)$. $G_{\mathbb{R}}$ acts on the model $\mathbb{R}^2/\mathbb{Z}^2$ by $\tau(x, y) = (-x, -y)$.

The case $n = 3$ follows the same pattern. Any element of $\pi_1/[\pi_1]_3$ can be written uniquely in the form $\alpha_1^x\alpha_2^y[\alpha_1, \alpha_2]^z$ for $x, y, z \in \mathbb{Z}$, giving a bijection $\pi_1/[\pi_1]_3 \to \mathbb{Z}^3$. The free action of $\pi_1/[\pi_1]_3$ on itself by left multiplication gives a free action of $\pi_1/[\pi_1]_3$ on $\mathbb{R}^3$. Explicitly, α_1,α_2, and $[\alpha_1, \alpha_2]$ act on $(x, y, z) \in \mathbb{R}^3$ by

(1) $\alpha_1(x, y, z) = (x + 1, y, z)$
(2) $\alpha_2(x, y, z) = (x, y + 1, z - x)$
(3) $[\alpha_1, \alpha_2](x, y, z) = (x, y, z + 1)$

Quotienting $\mathbb{R}^3$ by $\pi_1/[\pi_1]_3$ gives a model for $B(\pi_1/[\pi_1]_3)$ which can be obtained from $[0, 1]^3$ by first gluing the front to the back by translation, then gluing the top to the bottom by translation, and gluing the last two sides together by a 'shear.' $G_{\mathbb{R}}$ acts on the model $\mathbb{R}^3/(\pi_1/[\pi_1]_3)$ by $\tau(x, y, z) = (-x, -y, z)$. Both $B(\pi_1/[\pi_1]_3)$ and the fixed point set $B(\pi_1/[\pi_1]_3)^{G_{\mathbb{R}}}$ are shown in Figure 2. The fixed point set is composed of three vertical lines: the vertical axis, and the two shown translates: one in the middle of the back side, and the other in the middle of the left side.

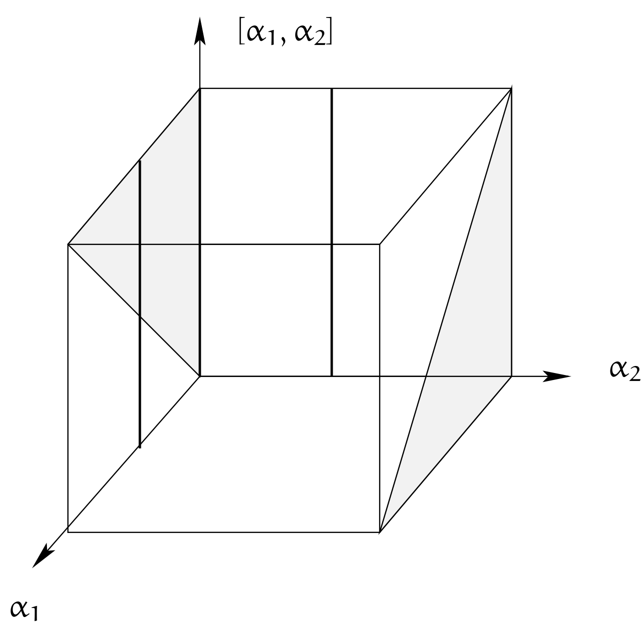


FIGURE 2. $B(\pi_1/[\pi_1]_3)$

For arbitrary n, we obtain a similar model for $B(\pi_1/[\pi_1]_n)$ by choosing bases for $[\pi_1]_k/[\pi_1]_{k+1}$ and an ordering of the elements in these bases for $k = 1, \ldots n - 1$, and then following the same procedure. Since $[\pi_1]_k/[\pi_1]_{k+1} \cong \mathbb{Z}^{2^{k-2}}$ for $k \geq 2$, the resulting model is a quotient of

$[0,1]^N$, where $N = 2 + \sum_{k=2}^{n-1} 2^{k-2} = 2^{n-2} + 1$. The action of $G_{\mathbb{R}}$ is determined by τ's action on an ordered product of basis elements raised to powers in $\hat{\mathbb{Z}}$ considered as an element of $\pi_1/[\pi_1]_n$.

The $G_{\mathbb{R}}$ equivariant tower approximating $\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$ (as in 1.1.1) is shown in Figure 3. The maps $B(\pi_1/[\pi_1]_{n+1}) \to B(\pi_1/[\pi_1]_n)$ are induced from projection maps

$$[0,1]^{2^{n-1}+1} \to [0,1]^{2^{n-2}+1}$$

(More specifically, the map $B(\pi_1/[\pi_1]_{n+1}) \to B(\pi_1/[\pi_1]_n)$ is induced from the projection map projecting onto the factors of $[0,1]$ corresponding to elements of the chosen bases of $[\pi_1]_k/[\pi_1]_{k+1}$ for $k = 1, \ldots, n-1$.) The $G_{\mathbb{R}}$ fixed points of the approximating spaces are also shown. $\mathrm{Jac}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}) = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ is redundant in the sense that it gives another model for $B(\pi_1/[\pi_1]_2)$, and it is omitted.

We can now see the obstructions δ_n: there are three elements of $\pi_0(X(\mathbb{R}))$ which inject into the four elements of $\pi_0(\mathrm{Jac}(X(\mathbb{R}))) = \pi_0(B(\pi_1/[\pi_1]_2)^{G_{\mathbb{R}}})$. δ_2 eliminates the one extra element. (And so the higher δ_n don't eliminate any further elements of $\pi_0(\mathrm{Jac}(X(\mathbb{R})))$.) This behavior turns out to be typical over $\mathbb{R}$; see Sections 3.1-3.2, and Proposition 3.2.1.

1.1.3. Ellenberg's obstructions as boundary maps in group cohomology. Let k be a subfield of $\mathbb{C}$ and let $\sigma : X \to \operatorname{Spec} k$ be a geometrically connected curve, equipped with a rational point or rational tangential point b, to be used as a base point. (See [Del89, §15] for a discussion of rational tangential base points. For a smooth k point p on X, a rational point of the tangent space to X at p is an example of a rational tangential point of $X - \{p\}$.) Let π_1 denote the étale fundamental group of $X_{\overline{k}}$ based at b. Let $\mathrm{Jac}\, X$ denote the Jacobian of X, and let $X \to \mathrm{Jac}\, X$ denote the Abel-Jacobi map corresponding to the base point b. (See [Ser88] for information on Jacobians of non-proper curves.) This data determines a commutative diagram

$$\begin{array}{ccc} H^1(G_k, \pi_1) & \longrightarrow & H^1(G_k, \pi_1/[\pi_1]_2) \\ \uparrow & & \uparrow \\ X(k) & \longrightarrow & \mathrm{Jac}\, X(k) \end{array} \tag{2}$$

discussed in more detail below. Ellenberg's obstruction δ_n is given by:

Definition $\delta_n : H^1(G_k, \pi_1/[\pi_1]_n) \to H^2(G_k, [\pi_1]_n/[\pi_1]_{n+1})$ is the boundary map associated to

$$1 \to [\pi_1]_n/[\pi_1]_{n+1} \to \pi_1/[\pi_1]_{n+1} \to \pi_1/[\pi_1]_n \to 1$$

as in [Ser02, I §5].

The δ_n give a series of obstructions to an element of $H^1(G_k, \pi_1/[\pi_1]_2)$ being the image of an element of $H^1(G_k, \pi_1)$, thereby also providing a series of obstructions to a rational point of the Jacobian coming from a rational point of the curve: to a given element x of $H^1(G_k, \pi_1/[\pi_1]_2)$, apply δ_2. If $\delta_2(x)$ is not 0, x does not come from an element of $H^1(G_k, \pi_1)$. Otherwise, x lifts to $H^1(G_k, \pi_1/[\pi_1]_3)$. Apply δ_3 to all the lifts of x. If δ_3 is never 0, x does not come from an element of $H^1(G_k, \pi_1)$. Otherwise, x lifts to $H^1(G_k, \pi_1/[\pi_1]_4)$. Apply δ_4 to all the lifts of x etc.

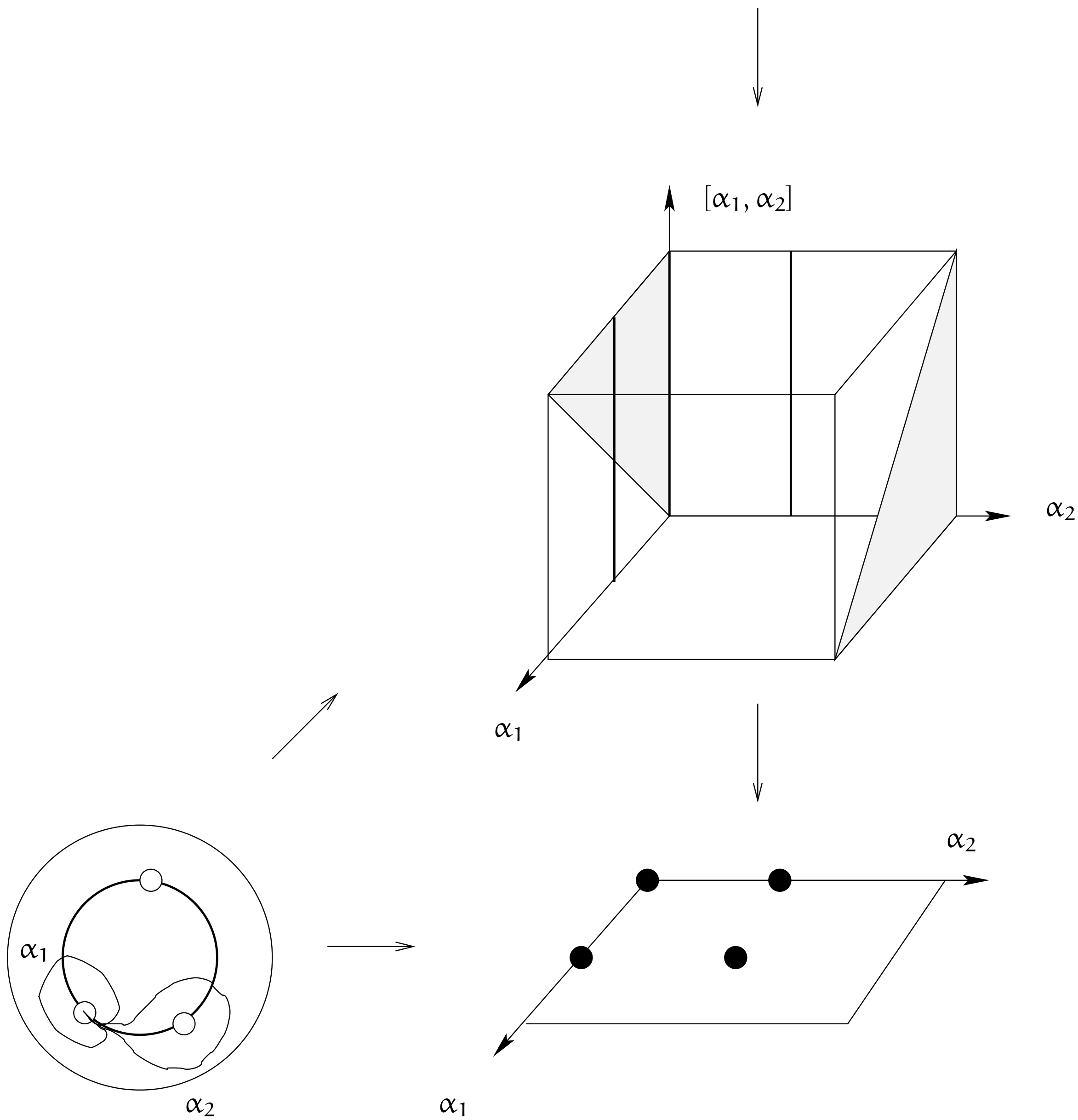


FIGURE 3. Approximating $\mathbb{P}^1_{\mathbb{R}} - \{0, 1, \infty\}$

For X geometrically connected, smooth, of Euler characteristic ≤ 0, $X(k) \to \operatorname{Jac} X(k)$ is an injection, so we can think of the rational points of X as a subset of the rational points of the Jacobian. If additionally k is a number field, it follows from the Mordell-Weil theorem that $\operatorname{Jac} X(k) \to H^1(G_k, \pi_1/[\pi_1]_2)$ is injective, at least for X proper. (We reproduce a proof of this fact below for the reader's convenience.) Thus, under these hypotheses, we can think of $X(k)$ as a subset of $H^1(G_k, \pi_1/[\pi_1]_2)$ which we wish to approximate by the images of $H^1(G_k, \pi_1/[\pi_1]_n) \to H^1(G_k, \pi_1/[\pi_1]_2)$. These images are determined by the δ_n.

The elements of $H^1(G_k, \pi_1)$ and $H^1(G_k, \pi_1/[\pi_1]_n)$ should be thought of as homotopy sections of the spaces $B(\pi_1^{et}(X, b))$ and $B(\pi_1^{et}(X, b)/[\pi_1^{et}(X, b)]_n)$ (resp.) of the tower 1 over $B(G_k)$. By the heuristic that for anabelian schemes Y, Y should be a sort of $K(\pi_1^{et}(Y), 1) =$

$B(\pi_1^{et}(Y))$, elements of $H^1(G_k, \pi_1)$ can also be thought of as homotopy sections of X, for X hyperbolic. We adopt this terminology. Furthermore, although abelian varieties are not considered anabelian schemes, we will also call elements of $H^1(G_k, \pi_1^{et}(Y_{\overline{k}}))$ homotopy sections for abelian varieties Y over number fields. For convenience, we will even use the terminology 'homotopy section' of X to refer to an element of $H^1(G_k, \pi_1^{et}(X_{\overline{k}}))$, when k is not a number field, but this is only for convenience. (We will mainly deal with geometrically connected, smooth, hyperbolic curves, over number fields, where 'hyperbolic' means that the corresponding complex analytic space has Euler characteristic < 0.) Explicitly, we adopt the following definition:

Definition: Let Y be a geometrically connected, finite type scheme over k, where k is a subfield of $\mathbb{C}$. For simplicity, assume that Y is either a smooth curve such that the Euler characteristic of the corresponding complex analytic space is ≤ 0 or an Abelian variety. A *homotopy section* of $Y \to \operatorname{Spec} k$ is an equivalence class of sections of $\pi_1^{et}(Y) \to \pi_1^{et}(\operatorname{Spec} k)$, where two sections s_1, s_2 are considered equivalent if there is an element γ of $\pi_1^{et}(Y_{\overline{k}})$ such that $s_1(g) = \gamma s_2(g)\gamma^{-1}$ for all $g \in \pi_1^{et}(\operatorname{Spec} k)$.

The name homotopy section comes from the fact that homotopy classes of continuous unbased maps $T \to K(\varpi, 1)$ from a CW complex to a/the $K(\varpi, 1)$ for a group ϖ are in bijection with homomorphisms $\pi_1^{top}(T, t) \to \varpi$ up to conjugation by elements of ϖ. (Compare with 1.1.1.) (It would therefore be reasonable to use the above definition of 'homotopy section' for any scheme Y, expected to be a $K(\pi_1^{et}(Y), 1)$ in algebraic geometry.)

It follows from the short exact sequence

$$1 \to \pi_1^{et}(X_{\overline{k}}) \to \pi_1^{et}(X) \to G_k \to 1 \tag{3}$$

that for $\sigma : X \to \operatorname{Spec} k$, b, and π_1 as above, $H^1(G_k, \pi_1)$ and $H^1(G_k, \pi_1/[\pi_1]_2)$ are in natural bijection with the homotopy sections of X and $\operatorname{Jac} X$ respectively. In more detail, this natural bijection is described as follows: the point b gives a section of $\pi_1^{et}(\sigma) : \pi_1^{et}(X, b) \to G_k$ by functoriality, and thus an isomorphism $\pi_1^{et}(X, b) \cong \pi_1 \rtimes G_k$. The G_k action on $\pi_1 = \pi_1^{et}(X_{\overline{k}}, b)$ is via the splitting of (3), or equivalently via the action of G_k on $X_{\overline{k}}$. It follows that the set of sections of $\pi_1^{et}(\sigma) : \pi_1^{et}(X) \to G_k$ up to conjugation by π_1 equals $H^1(G_k, \pi_1)$ (see [Bro94, IV.2]. The profinite case is the same).

In this language, Ellenberg's δ_n are obstructions to a homotopy section of $\operatorname{Jac} X \to \operatorname{Spec} k$ coming from a homotopy section of $X \to \operatorname{Spec} k$.

Rational points of Y give rise to homotopy sections of $Y \to \operatorname{Spec} k$. Here are two ways to see this: the first uses functoriality of π_1^{et}. A k point is a section of the map $Y \to \operatorname{Spec} k$. If we let b denote the base point of Y and y denote a k point, $\pi_1^{et}(Y, b)$ and $\pi_1^{et}(Y, y)$ are isomorphic by an isomorphism determined up to inner automorphism. (We are being sloppy about the distinction between rational points and geometric points factoring through rational points.) By composing the map $\pi_1^{et}(\operatorname{Spec} k) \to \pi_1^{et}(Y, y)$ induced by $y : \operatorname{Spec} k \to Y$ with an isomorphism $\pi_1^{et}(Y, y) \to \pi_1^{et}(Y, b)$ we obtain a map $\pi_1^{et}(\operatorname{Spec} k) \to \pi_1^{et}(Y, b)$ such that the composite $\pi_1^{et}(\operatorname{Spec} k) \to \pi_1^{et}(Y, b) \to \pi_1^{et}(\operatorname{Spec} k)$ is an inner automorphism of $\pi_1^{et}(\operatorname{Spec} k)$. By appropriately modifying the isomorphism $\pi_1^{et}(Y, y) \to \pi_1^{et}(Y, b)$, we obtain a section of $\pi_1^{et}(Y, b) \to \pi_1^{et}(\mathrm{Spec}\, k)$ determined up to conjugation by elements of the kernel. A rational tangential point of Y gives rise to a homotopy section of $Y \to \mathrm{Spec}\, k$ by the same procedure.

A second way to associate a homotopy section to a rational point is via the boundary map $H^0 \to H^1$ in cohomology of G_k associated to $\pi_1(Y_{\overline{k}}) \to \tilde{Y} \to Y_{\overline{k}}$. Explicitly, take a k point y of Y. Choose a path γ from b to y. (A path from b to y is a natural transformation from the fiber functor of b to the fiber functor of y. See [SGAI, Exp. V §5].) G_k acts on the set of such paths because b and y are defined over k. γ determines a cocycle in $C^1(G_k, \pi_1^{et}(Y, b))$ by $g \mapsto \gamma g(\gamma^{-1})$ for $g \in G_k$. (The notation $C^1(G_k, \pi_1^{et}(Y, b))$ is as in 2.0.4.) Associating y to the cohomology class of $g \mapsto \gamma g(\gamma^{-1})$ gives the map $Y(k) \to H^1(G_k, \pi_1^{et}(Y, b))$. Both of these methods work in the category of topological spaces, as well.

So we have the commutative diagram (2).

The map from points to homotopy sections can be thought of as a substitute for a map from $X \to B(\pi_1)$ in the tower (1). A well-defined analogue for this tower is the commutative diagram

(4)

$$
\begin{array}{ccccc}
 & & & & \vdots \\
 & & & & H^1(G_k, \pi_1/[\pi_1]_n) \\
 & & & \nearrow & \downarrow \\
 & & & & \vdots \\
 & & & & H^1(G_k, \pi_1/[\pi_1]_3) \\
 & & & \nearrow & \downarrow \\
H^1(G_k, \pi_1) & & \longrightarrow & & H^1(G_k, \pi_1/[\pi_1]_2) \\
\uparrow & & & \nearrow & \\
X(k) & \longrightarrow & \mathrm{Jac}(X)(k) & &
\end{array}
$$

For section 3.4, we will need the following generalization of the definition of δ_n. Let X,b, and π_1 be as above. A filtration $\pi_1 > \varpi_2 > \varpi_3 > \ldots$ of π_1 such that the quotients ϖ_n/ϖ_{n+1} are abelian and the extensions

$$\varpi_n/\varpi_{n+1} \to \pi_1/\varpi_{n+1} \to \pi_1/\varpi_n$$

are central gives rise to a sequence of obstructions δ_n which determine which sections of $\pi_1/\varpi_n \rtimes G_k \to G_k$ admit a lift to a section of $\pi_1/\varpi_{n+1} \rtimes G_k \to G_k$. $\delta_n : H^1(G_k, \pi_1/\varpi_n) \to H^2(G_k, \varpi_n/\varpi_{n+1})$ is the boundary map in group cohomology associated to the short exact sequence of G_k groups $\varpi_n/\varpi_{n+1} \to \pi_1/\varpi_{n+1} \to \pi_1/\varpi_n$.

Lastly, we include a proof of the injectivity of $\operatorname{Jac} X(k) \to H^1(G_k, \pi_1/[\pi_1]_2)$.

1.1.4. *Proposition. — Let* k *be a number field. Let* $X \to \operatorname{Spec} k$ *be a geometrically connected, smooth, proper curve of genus* ≥ 1. *Let* π *denote* $\pi_1^{et}(\operatorname{Jac} X, 0)$. *Then* $\operatorname{Jac} X(k) \to H^1(G_k, \pi)$ *is injective.*

Proof. $\operatorname{Jac} X$ is an abelian variety over k. Let $\operatorname{Jac} X[n]$ denote the n-torsion $\overline{k}$ points. The short exact sequence of G_k modules

$$0 \longrightarrow \operatorname{Jac} X[n] \longrightarrow \operatorname{Jac} X(\overline{k}) \xrightarrow{\times n} \operatorname{Jac} X(\overline{k}) \longrightarrow 0$$

gives rise to the injection $\operatorname{Jac} X(k)/n \operatorname{Jac} X(k) \to H^1(G_k, \operatorname{Jac} X[n])$. $\pi = \varprojlim_n \operatorname{Jac} X[n]$ and $H^1(G_k, \pi) = \varprojlim_n H^1(G_k, \operatorname{Jac} X[n])$. The kernel of $\operatorname{Jac} X(k) \to H^1(G_k, \pi)$ is therefore contained in $\cap_n n \operatorname{Jac} X(k)$. By the Mordell-Weil theorem, $\operatorname{Jac} X(k)$ is a finitely generated abelian group, so $\cap_n n \operatorname{Jac} X(k) = \{0\}$. □

1.2. **Summary of results.** Section 2 is devoted to results on the structure of the δ_n. It is group theoretic and could be applied to the boundary maps $\delta_n : H^1(G, \varpi/[\varpi]_n) \to H^2(G, [\varpi]_n/[\varpi]_{n+1})$, where G and ϖ are groups or profinite groups with G acting continuously on ϖ. For the rest of the summary of Section 2, we drop the more general group theoretic δ_n, in favor of the geometric language of Section 1. The existence of the extra generality, however, means that we do not incorporate interesting aspects of the Galois action on the fundamental group. Rather, we look for structure and its origins to limit what needs to be understood about this action.

Let X, b, and δ_n be as in 1.1.3. Let π abbreviate $\pi_1^{et}(X_{\overline{k}}, b)$.

Ellenberg observes in [Ell] that δ_2 is quadratic, and that the associated bilinear form is the cup product composed with the map on H^2 induced from the commutator

$$\pi/[\pi]_2 \otimes \pi/[\pi]_2 \to [\pi]_2/[\pi]_3,$$

$$\gamma \otimes \eta \mapsto \gamma\eta\gamma^{-1}\eta^{-1}$$

Call this composition the *bracket cup product*. Two proofs that δ_2 is quadratic with this bilinear form are included- see 2.1.3 and 2.3.9. It follows that δ_2 is a linear perturbation of the bracket cup product after inverting two. Ellenberg points out that an attractive feature of this decomposition is that the 'highest order' bracket cup product term only depends on the Galois action on the Tate module ($= \pi/[\pi]_2$).

This behavior of δ_2 extends to higher δ_n, with certain *bracket Massey products* replacing the bracket cup product, at least under the assumption that X is non-proper.

An arbitrary element of $H^2(\pi/[\pi]_n \rtimes G_k, [\pi]_n/[\pi]_{n+1})$ determines an obstruction map $H^1(G_k, \pi/[\pi]_n) \to H^2(G_k, [\pi]_n/[\pi]_{n+1})$ (2.2.1), and under this association, δ_n is determined by the element classifying

$$1 \to [\pi]_n/[\pi]_{n+1} \to \pi/[\pi]_{n+1} \rtimes G_k \to \pi/[\pi]_n \rtimes G_k \to 1$$

We show the filtration of H^2 coming from the spectral sequence

$$H^i(G_k, H^j(\pi/[\pi]_n, [\pi]_n/[\pi]_{n+1})) \Rightarrow H^{i+j}(\pi/[\pi]_n \rtimes G_k, [\pi]_n/[\pi]_{n+1})$$

controls linearity properties of the corresponding obstruction maps (Proposition 2.2.5). More specifically, this filtration is

$$H^2(G_k) \subset H^2(G_k) \oplus H^1(G_k, H^1(\pi/[\pi]_n)) \subset H^2(\pi/[\pi]_n \rtimes G_k),$$

where $H^i(G)$ abbreviates $H^i(G, [\pi]_n/[\pi]_{n+1})$ for any group G. Proposition 2.2.5 shows that elements of $H^2(G_k)$ correspond to constant maps, and elements of $H^1(G_k, H^1(\pi/[\pi]_n))$ correspond to maps with a certain linearity property. More generally, Proposition 2.2.5 relates the spectral sequence filtration of H^2 of a semi-direct product to the linearity of the corresponding map.

For the structure of δ_n, this implies that if we can lift the class in $H^2(\pi/[\pi]_n)$ classifying

$$1 \to [\pi]_n/[\pi]_{n+1} \to \pi/[\pi]_{n+1} \to \pi/[\pi]_n \to 1$$

to a class ω in $H^2(\pi/[\pi]_n \rtimes G)$, we decompose δ_n into the sum of a linear term and the obstruction associated to ω.

Since π is the profinite completion of a surface group (= the fundamental group of a closed surface of genus ≥ 1) or a finitely generated free group, the extensions

$$1 \to [\pi]_n/[\pi]_{n+1} \to \pi/[\pi]_{n+1} \to \pi/[\pi]_n \to 1 \tag{5}$$

and their classifying elements of cohomology are approachable computationally.

For $n = 2$, a simple calculation of the element of H^2 classifying (5) reproves the above decomposition of δ_2 into the sum of a linear term and the bracket cup product. (This is done in 2.3.9.) This element is expressed in terms of the (usual) cup product.

Now assume that X is non-proper, so that π is the profinite completion of a free group. From work of Dwyer and well known results on the lower central series of free groups, we express the element ω_n of H^2 classifying (5) in terms of order n Massey products. (Proposition 2.3.8.)

We lift ω_n to an element of $H^2(\pi/[\pi]_n \rtimes G_k, [\pi]_n/[\pi]_{n+1})$ for small n using *bracket Massey products*, defined in 2.4.10-2.4.19. This implies the existence of a decomposition of δ_n into a bracket Massey product term plus a linear term, for small n. (See Theorem 2.4.28.) Like the usual Massey products, bracket Massey products require the existence of defining systems, which tautologically force the vanishing of lower order bracket Massey products. Theorem 2.4.28 relies on the vanishing of δ_m for $m < n$ to obtain these conditions, and obtains its decomposition of δ_n only when the linear terms of δ_m for $m < n$ vanish. Conversely, Theorem 2.4.28 obtains its decomposition when the elements of cohomology corresponding to the linear terms of δ_m for $m < n$ vanish.

The bracket Massey product term of the decomposition of δ_n of Theorem 2.4.28 depends only on the action of G_k on $\pi/[\pi]_n$, i.e. it is independent of the lift of this action to the G_k action on $\pi/[\pi]_{n+1}$, even though δ_n itself does depend on this lift. So the 'highest order' term of δ_n requires less understanding of the Galois action than expected. (See

2.4.30.) This is the generalization of Ellenberg's observation that the bracket cup product is determined by the Galois action on the Tate module.

Paragraph 2.4.31 records how the decomposition of δ_2 depends on the choice of base point b. More specifically, the bracket cup product term is independent of b, while the linear term changes by bracket cup product with the class of the new base point in $H^1(G_k, \pi_1)$. The question "for which X, b, and m does the linear term of δ_m vanish ?" is linked to obtaining the decomposition of Theorem 2.4.28. For example, if the linear term of δ_2 does not vanish, we do not obtain a decomposition of δ_3.

Proposition 2.2.6 turns Proposition 2.2.5 into a tool to search for aspects of Ellenberg's obstructions controlled by the topology of the underlying surface of X. It says that up to 'linear perturbation,' in the sense of Proposition 2.2.5, δ_n is determined by π considered as a profinite group, without its G_k action; or less optimistically (but more precisely), δ_n up to linear perturbation is determined by the preimage of the element classifying (5) as an extension of profinite groups without G_k action, under the map $H^2(\pi/[\pi]_n \rtimes G_k, [\pi]_n/[\pi]_{n+1}) \to H^2(\pi/[\pi]_n, [\pi]_n/[\pi]_{n+1})$. The domain of this map is dependent on the G_k action on $\pi/[\pi]_{n+1}$, but the range as well as the element classifying (5) are entirely independent of the Galois action on the fundamental group.

Section 3 is devoted to computing the δ_n in examples.

The first two subsections consider the case $k = \mathbb{R}$, and subsection 3.1 starts with a summary of the contained results. The main result is for smooth proper curves. It combines a (deep!) theorem of Gunnar Carlsson on homotopy fixed points with geometric information about Jacobians over $\mathbb{R}$ to show that the homotopy sections of the curve are precisely those of the Jacobian in the kernel of δ_2 (Proposition 3.2.1). The section conjecture over $\mathbb{R}$ (which is known to be true- see [Pál], and which we wind up reproving in the special case at hand) identifies homotopy sections with connected components of real points. In this sense, not only does the tower (4) succeed in describing the connected components of real points of the curve, but the first two levels suffice.

The next subsection considers δ_3 for $X = \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. (The base point is the same as in example 1.1.2.) Ellenberg showed that the linear term of δ_2 vanishes. We obtain a decomposition of δ_3 into Massey products and a linear term. Theorem 3.3.4 computes the linear term in terms of a well-known cocycle associated to the action of $G_{\mathbb{Q}}$ on $\pi_1^{et}(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$.

Subsection 3.4 evaluates a local mod 2 version of δ_3 on an infinite subset of rational points of $\operatorname{Jac} X$. (Proposition 3.4.7) To do this, the lower central series is replaced by the exponent 2 lower central series, so that computations can be carried out in the Brauer group. An exponential formula of Anderson, Coleman, Deligne, Ihara, Kaneko and Yukinari is used to compute the necessary projection of the above cocycle.

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2. δ_n AND MASSEY PRODUCTS

We establish some notation and recall certain constructions that will be useful:

2.0.1. For a group or profinite group π the *lower central series* filtration $\pi = [\pi]_1 > [\pi]_2 > [\pi]_3 > \ldots$ is defined by $\pi = [\pi]_1$, $[\pi]_{n+1}$ is the closure of the subgroup generated by elements of the form $[x, y] = xyx^{-1}y^{-1}$ with $x \in [\pi]_n$ and y arbitrary. In other words, $[\pi]_n$ is the closure of the subgroup generated by all order n brackets. π is said to be n-*nilpotent* if that $[\pi]_{n+1} = 0$

2.0.2. For a central extension of G groups $1 \to N \to M \to Q \to 1$, there is an exact sequence

$$1 \to N^G \to M^G \to Q^G \to H^1(G, N) \to H^1(G, M) \to H^1(G, Q) \to H^2(G, N)$$

(See for instance [Ser02, I 5.7]) G, N, M, and Q may also be profinite.

2.0.3. *Definition.* Let G be a profinite group acting on a profinite group π. (Or simply assume that G and π are groups with G acting on π.) By analogy with 1.1.3, let $\delta_n : H^1(G, \pi/[\pi]_n) \to H^2(G, [\pi]_n/[\pi]_{n+1})$ be the boundary map of the cohomology exact sequence associated to the short exact sequence of G groups

$$1 \to [\pi]_n/[\pi]_{n+1} \to \pi/[\pi]_{n+1} \to \pi/[\pi]_n \to 1$$

as in 2.0.2.

2.0.4. *Notation.* For a profinite group G and a profinite abelian group A with a continuous action of G, let $(C^*(G, A), D)$ be the complex of inhomogeneous cochains of G with coefficients in A as in [NSW08, I.2 p. 14]. For $c \in C^p(G, A)$ and $d \in C^q(G, A)$, let $c \cup d$ denote the cup product $c \cup d \in C^{p+q}(G, A \otimes A)$

$$(c \cup d)(g_1, \ldots, g_{p+q}) = c(g_1, \ldots, g_p) \otimes ((g_1 \cdots g_p)d(g_{p+1}, \ldots, g_{p+1q})).$$

This product induces a well defined map $H^p(G, A) \otimes H^q(G, A) \to H^{p+q}(G, A \otimes A)$. When A is a commutative ring, the multiplication map $A \otimes A \to A$ and the cup product give $C^*(G, A)$ the structure of a differential graded algebra. For a profinite group Q, no longer assumed to be abelian, let $C^1(G, Q)$ denote the set of continuous twisted homomorphisms $\{s : G \to Q | s \text{ is continuous}, s(g_1g_2) = s(g_1)g_1s(g_2)\}$.

2.0.5. Let $1 \to N \to M \to Q \to 1$ be a short exact sequence of groups with N abelian. A set-theoretic section $s : Q \to M$ of the quotient map gives rise to a cocycle

$$(q_1, q_2) \mapsto s(q_1)s(q_2)s(q_1q_2)^{-1} \tag{6}$$

in $C^2(Q, N)$ such that the associated cohomology class classifies the extension. (See [Bro94, IV §3] for the classification of extensions by H^2.) If N, M, Q, are profinite, and s is a continuous map of topological spaces, we likewise have the cocycle (6) which classifies the extension. By [RZ00, Prop. 2.2.2], a continuous section s always exists.

2.1. δ_2 **is quadratic.** Let G be a profinite group acting on a profinite group π. Let δ_n be as in 2.0.3.

We give a proof of Ellenberg's observation that δ_2 is quadratic and identify the corresponding bilinear form, as in [Ell, Prop. 1].

2.1.1. Define a commutator or bracket map $[-,-] : \pi/[\pi]_2 \otimes \pi/[\pi]_2 \to [\pi]_2/[\pi]_3$ by

$$[-,-](m_1 \otimes m_2) = [\tilde{m_1}, \tilde{m_2}] = \tilde{m_1}\tilde{m_2}\tilde{m_1}^{-1}\tilde{m_2}^{-1},$$

where $\tilde{m_1}$ and $\tilde{m_2}$ are elements of $\pi/[\pi]_3$ mapping to m_1 and m_2 respectively. $[\tilde{m_1}, \tilde{m_2}]$ is independent of the choice of $\tilde{m_1}$ and $\tilde{m_2}$ because different choices differ by elements in the center.

2.1.2. Let τ be the involution $\tau : \pi/[\pi]_2 \otimes \pi/[\pi]_2 \to \pi/[\pi]_2 \otimes \pi/[\pi]_2$ which switches the two factors of $\pi/[\pi]_2$. Recall that $c \cup d = (-1)^{pq}\tau_*(d \cup c)$ in $H^{p+q}(G, \pi/[\pi]_2 \otimes \pi/[\pi]_2)$. Note that $[-,-](m_1 \otimes m_2) = -[-,-](m_2 \otimes m_1) = -[-,-](\tau(m_1 \otimes m_2))$, i.e. $[-,-]\tau = -[-,-]$. It follows that in $H^{p+q}(G, [\pi]_2/[\pi]_3)$, $[-,-]_*(c \cup d) = [-,-]_*((-1)^{pq}\tau_* d \cup c) = (-1)^{pq+1}[-,-]_*(d \cup c)$. For x, y in $H^1(G, \pi/[\pi]_2)$, it follows that $[-,-]_*(x \cup y) = [-,-]_*(y \cup x)$ in $H^2(G, [\pi]_2/[\pi]_3)$. $[-,-]_*(x \cup y)$ is the symmetric bilinear form associated to δ_2.

2.1.3. *Proposition [Ellenberg]* [Ell, Prop. 1]. — *For* x, y *in* $H^1(G, \pi/[\pi]_2)$,

$$\delta_2(x + y) - \delta_2(x) - \delta_2(y) = [-,-]_*(x \cup y),$$

in $H^2(G, [\pi]_2/[\pi]_3)$.

Proof. Notice that elements of $\pi/[\pi]_3$ with the same image in $\pi/[\pi]_2$ commute (because they differ by an element of $[\pi]_2/[\pi]_3$ which is in the center). Similarly elements of $\pi/[\pi]_3$ whose product is in $[\pi]_2/[\pi]_3$ commute.

Let c, d be cocycles in $C^1(G, \pi/[\pi]_2)$ representing x and y respectively.

Choose a continuous section of $\pi/[\pi]_3 \to \pi/[\pi]_2$ as in 2.0.5. Denote the image of m in $\pi/[\pi]_2$ under this section by $\tilde{m}$. Let $\omega : \pi/[\pi]_2 \times \pi/[\pi]_2 \to [\pi]_2/[\pi]_3$ be the cocycle of 2.0.5 corresponding to this section, i.e. $\omega(m_1, m_2) = \tilde{m_1}\tilde{m_2}\widetilde{m_1 m_2}^{-1}$. By the second sentence of this proof, we have that $\omega(m_1, m_2) = \widetilde{m_1 m_2}^{-1}\tilde{m_1}\tilde{m_2}$.

Define γ in $C^1(G, [\pi]_2/[\pi]_3)$ by $\gamma(g) = \omega(c(g), d(g))$ for all g in G.

$\delta_2(c + d)$, $\delta_2(c)$, $\delta_2(d)$, and $[-,-]_*(d \cup c)$ are elements of $C^2(G, [\pi]_2/[\pi]_3)$. We have the equalities:

$$(7)\quad (\delta_2(c + d) - \delta_2(c) - \delta_2(d))(g_1, g_2) = \widetilde{(c + d)(g_1)}(g_1\widetilde{(c + d)(g_2)})\widetilde{(c + d)(g_1 g_2)}^{-1}$$
$$(\widetilde{c(g_1)}(g_1\widetilde{c(g_2)})\widetilde{c(g_1 g_2)}^{-1})^{-1}$$
$$(\widetilde{d(g_1)}(g_1\widetilde{d(g_2)})\widetilde{d(g_1 g_2)}^{-1})^{-1}.$$

(8) $$[-,-]_*(d\cup c)(g_1,g_2)=[\widetilde{d(g_1)},g_1\widetilde{c(g_2)}].$$

We will show that

(9) $$D\gamma(g_1,g_2)=[-,-]_*(d\cup c)(g_1,g_2)-(\delta_2(c+d)-\delta_2(c)-\delta_2(d))(g_1,g_2),$$

which will prove the propostition.

Because $(\widetilde{d(g_1)}(g_1\widetilde{d(g_2)})\widetilde{d(g_1g_2)}^{-1})^{-1}=\widetilde{d(g_1g_2)}g_1(\widetilde{d(g_2)})^{-1}(\widetilde{d(g_1)})^{-1}$ is in the center of $\pi/[\pi]_3$, equation (7) can be rewritten

$$\begin{aligned}(\delta_2(c+d)-\delta_2(c)-\delta_2(d))(g_1,g_2) &= \widetilde{(c+d)(g_1)}(g_1\widetilde{(c+d)(g_2)})\\ &\quad \widetilde{(c+d)(g_1g_2)}^{-1}\widetilde{c(g_1g_2)}\widetilde{d(g_1g_2)}\\ &\quad (g_1\widetilde{d(g_2)}^{-1})\widetilde{d(g_1)}^{-1}(g_1\widetilde{c(g_2)}^{-1})\widetilde{c(g_1)}^{-1}.\end{aligned}$$

We now repeatedly use that $[\pi]_2/[\pi]_3$ is contained in the center of $\pi/[\pi]_3$, and we use equation (8), and we use our two expressions for ω, namely $\omega(m_1,m_2)=\tilde{m}_1\tilde{m}_2\widetilde{m_1m_2}^{-1}=\widetilde{m_1m_2}^{-1}\tilde{m}_1\tilde{m}_2$, to obtain the following calculation of $\delta_2(c+d)-\delta_2(c)-\delta_2(d)$:

$$\begin{aligned}(\delta_2(c+d)-\delta_2(c)-\delta_2(d))(g_1,g_2) &= \omega(c(g_1g_2),d(g_1g_2))\\ &\quad \widetilde{(c+d)(g_1)}(g_1\widetilde{(c+d)(g_2)})(g_1\widetilde{d(g_2)}^{-1})\widetilde{d(g_1)}^{-1}(g_1\widetilde{c(g_2)}^{-1})\widetilde{c(g_1)}^{-1}\\ &= \omega(c(g_1g_2),d(g_1g_2))\\ &\quad \widetilde{(c+d)(g_1)}\\ &\quad (g_1\widetilde{(c+d)(g_2)})(g_1\widetilde{d(g_2)}^{-1})(g_1\widetilde{c(g_2)}^{-1})\widetilde{d(g_1)}^{-1}\\ &\quad [-,-]_*(d\cup c)(g_1,g_2)\\ &\quad \widetilde{c(g_1)}^{-1}\\ &= \omega(c(g_1g_2),d(g_1g_2))[-,-]_*(d\cup c)(g_1,g_2)\\ &\quad \widetilde{(c+d)(g_1)}\\ &\quad g_1(\omega(c(g_2),d(g_2))^{-1})\\ &\quad \widetilde{d(g_1)}^{-1}\widetilde{c(g_1)}^{-1}\\ &= [-,-]_*(d\cup c)(g_1,g_2)\\ &\quad \omega(c(g_1g_2),d(g_1g_2))\\ &\quad g_1(\omega(c(g_2),d(g_2))^{-1})\\ &\quad \omega(c(g_1),d(g_1))^{-1}\\ (10)\qquad &= [-,-]_*(d\cup c)(g_1,g_2)-D\gamma(g_1,g_2).\end{aligned}$$

This gives equation (9) as desired. □

Proposition 2.1.3 shows that after inverting 2, δ_2 is a linear perturbation of a cup product:

2.1.4. *Proposition. — For any* $x \in H^1(G, \pi/[\pi]_2)$, $2\delta_2(x) = [-,-]_*(x \cup x) + \mathcal{L}(x)$, *where* $\mathcal{L} : H^1(G, \pi/[\pi]_2) \to H^2(G, [\pi]_2/[\pi]_3)$ *is a homomorphism.*

Proof.

$$\begin{aligned}&2\delta_2(x+y) - [-,-]_*((x+y) \cup (x+y)) = \\ &2\delta_2(x) + 2\delta_2(y) + 2[-,-]_*(x \cup y) - [-,-]_*((x+y) \cup (x+y)) = \\ &2\delta_2(x) - [-,-]_*(x \cup x) + 2\delta_2(y) - [-,-]_*(y \cup y)\end{aligned}$$

The last equality follows from the symmetry of the bilinear form $x \otimes y \mapsto [-,-]_*(x \cup y)$, shown in 2.1.2. □

(Note that after inverting 2, the linearity of $x \mapsto 2\delta_2(x) - [-,-]_*(x \cup x)$ is equivalent to $\delta_2(x+y) - \delta_2(x) - \delta_2(y) = [-,-]_*(x \cup y)$.)

2.2. **Decomposing obstruction maps.** We establish an interesting relationship between the spectral sequence filtration of H^2 of a semi-direct product and linearity properties of associated obstruction maps. (Proposition 2.2.5.) Proposition 2.2.5 is then used to obtain a type of decomposition of an obstruction map. (Proposition 2.2.6)

2.2.1. *Notation.* Let π' and N be groups with G actions, and assume that N is abelian. An element $s \in H^1(G, \pi')$ determines a homomorphism $G \to \pi' \rtimes G$ up to post-composition with an inner automorphism of $\pi' \rtimes G$, where this inner automorphism is given by conjugation by an element of π'. Since π' acts trivially on N, such an inner automorphism determines an automorphism of $H^*(\pi' \rtimes G, N)$. It is well-known that this map is in fact the identity. Thus, s determines a homomorphism $s^* : H^*(\pi' \rtimes G, N) \to H^*(G, N)$. For any $\eta \in H^2(\pi' \rtimes G, N)$, denote by $\wp_\eta$ the map $H^1(G, \pi') \to H^2(G, N)$ defined by $\wp_\eta(s) = s^*(\eta)$.

2.2.2. For a central extension of G groups $1 \to N \to M \to Q \to 1$, the associated boundary map in group cohomology $H^1(G, Q) \to H^2(G, N)$ is $\wp_\eta$, where $\eta \in H^2(Q \rtimes G, N)$ is the element of cohomology classifying the short exact sequence $1 \to N \to M \rtimes G \to Q \rtimes G \to 1$.

2.2.3. Recall that morphisms of short exact sequences relate the classifying elements of cohomology as follows: suppose that:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle f(N')} & & \uparrow{\scriptstyle f(M')} & & \uparrow{\scriptstyle f(Q')} & & \\ 1 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & Q' & \longrightarrow & 1 \end{array}$$

is a commutative diagram of groups such that the rows are exact and N, N' are abelian. Let $\omega \in H^2(Q, N)$ classify the top short exact sequence, and $\omega' \in H^2(Q', N')$ classify the bottom short exact sequence. Then, $f(Q)^*\omega = f(N)_*\omega'$.

2.2.4. To describe linearity properties of the obstruction maps 2.2.1, we use the following definition. Let π' and G be profinite groups with G acting continuously on π'. (In particular π' may not be abelian and $H^1(G, \pi')$ may not be an abelian group.) We say that $s_1, s_2 \in H^1(G, \pi')$ *admit a sum* if they can be represented by cocycles $\sigma_1, \sigma_2 \in C^1(G, \pi')$ (respectively) such that $g \mapsto \sigma_1(g)\sigma_2(g)$ is a cocycle. A cohomology class $s_3 \in H^1(G, \pi')$ is *a sum* of s_1 and s_2 if s_3 can be represented by a cocycle of the form $g \mapsto \sigma_1(g)\sigma_2(g)$ If π' abelian or if s_1 or s_2 is represented by a cocycle with values in the center of π', then s_1 and s_2 admit a sum. A sum of s_1 and s_2 will be denoted $s_1 + s_2$ in the next proposition, but this is not meant to imply that such a sum is unique.

2.2.5. *Proposition*[1]. — *Let π' and N be profinite groups with continuous G actions, and assume that N is abelian. Consider N to be equipped with the trivial action of π' and with the action of $\pi' \rtimes G$ induced by the action of G. The filtration of $H^2(\pi' \rtimes G, N)$ from the spectral sequence*

$$H^i(G, H^j(\pi', N)) \Rightarrow H^{i+j}(\pi' \rtimes G, N)$$

has the form:

$$H^2(G, N) \subset H^2(G, N) \oplus H^1(G, H^1(\pi', N)) \subset H^2(\pi' \rtimes G, N)$$

Let η be an element of $H^2(\pi' \rtimes G, N)$.

- *If η is in $H^2(G, N)$, then $\wp_\eta$ is constant.*
- *If η is in $H^1(G, H^1(\pi', N))$, then $\wp_\eta$ is linear in the sense that whenever $s_1, s_2 \in H^1(G, \pi')$ admit a sum, denoted $s_1 + s_2$ as in 2.2.4, then $\wp_\eta(s_1 + s_2) = \wp_\eta(s_1) + \wp_\eta(s_2)$.*

Proof. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset H^2(\pi' \rtimes G, N)$ be the filtration from the Serre spectral sequence $H^i(G, H^j(\pi', N)) \Rightarrow H^{i+j}(\pi' \rtimes G, N)$.

$\mathcal{F}_0$ is the image of the map $p^* : H^2(G, N) \to H^2(\pi' \rtimes G, N)$ induced by the projection $p : \pi' \rtimes G \to G$. The canonical inclusion $s : G \to \pi' \rtimes G$ is a section of p, whence $p^* : H^2(G, N) \to \mathcal{F}_0$ is an isomorphism.

Let $i : \pi' \to \pi' \rtimes G$ be the canonical inclusion, and $i^* : H^2(\pi' \rtimes G, N) \to H^2(\pi', N)$ the induced map on cohomology. $\mathcal{F}_1 = \operatorname{Ker} i^*$. Furthermore, the inclusion $\mathcal{F}_0 \subset \mathcal{F}_1$ is split by the restriction of s^* to $\mathcal{F}_1$. Thus $\mathcal{F}_1$ is the direct sum of $H^2(G, N)$ and $\operatorname{Ker} s^* \cap \operatorname{Ker} i^*$. $\operatorname{Ker} s^* \cap \operatorname{Ker} i^*$ is isomorphic to $E_\infty^{(1,1)} = \operatorname{Ker}(E_2^{(1,1)} \to E_2^{(3,0)})$, where $E_r^{(*,*)}$ denotes the r^{th} page of the spectral sequence. Since $s^*p^* = \text{id}$, $p^* : H^3(G, N) \to H^3(\pi' \rtimes G, N)$ is an injection. Thus $E_2^{(1,1)} \to E_2^{(3,0)}$ is the zero map. Thus $\operatorname{Ker} s^* \cap \operatorname{Ker} i^* \cong H^1(G, H^1(\pi', N))$.

[1]This result looks like it may be folklore, but I am unaware of a prior such statement.

It is clear that if η is in $H^2(G,N) \subset H^2(\pi' \rtimes G, N)$, then $\wp_\eta$ is constant. (Just in case a more explicit argument would be helpful: since $\eta \in H^2(G,N)$, we can choose a cocycle $n \in C^2(\pi' \rtimes G, N)$ representing η such that n is the image of $n' \in C^2(G,N)$ under the map $C^2(G,N) \to C^2(\pi' \rtimes G, N)$. Given $s \in H^1(G,\pi')$, let $\nu \in C^1(G,\pi')$ represent s. Then, $\wp_\eta(s) \in H^2(G,N)$ is represented by the cocycle $\wp_\eta(\nu)(g_1,g_2) = n(\nu(g_1) \rtimes g_1, \nu(g_2) \rtimes g_2) = n'(g_1,g_2)$.)

Let η be an element of $H^1(G, H^1(\pi', N)) \subset H^2(\pi' \rtimes G, N)$, and let

$$1 \to N \to E' \to \pi' \rtimes G \to 1, \tag{11}$$

be a/the short exact sequence corresponding to η. Since $s^*\eta = 0$, there exists a continuous homomorphism $s_{E'} : G \to E'$ lifting s as in the commutative diagram:

$$\begin{array}{ccc} & & E' \\ & \overset{s_{E'}}{\nearrow} & \downarrow \\ G & \xrightarrow{s} & \pi' \rtimes G \end{array}$$

Let $p_{E'}$ be the surjection $E' \to \pi' \rtimes G \to G$, and let $E = \operatorname{Ker} p_{E'}$. $s_{E'}$ splits $p_{E'}$, giving E a continuous action of G, and an isomorphism $E \rtimes G \cong E'$. The map $E \to \pi'$ induced from $E' \to \pi' \rtimes G$ is a surjection respecting the action of G with kernel N. We therefore have a short exact sequence of profinite groups with a continuous G action

$$1 \to N \to E \to \pi' \to 1, \tag{12}$$

and an isomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & E \rtimes G & \longrightarrow & \pi' \rtimes G & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\ 1 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & \pi' \rtimes G & \longrightarrow & 1 \end{array} \tag{13}$$

N is also in the center of E because the action of $\pi' \rtimes G$ on N induced from the short exact sequence (11) is the G action on N.

Thus $\eta \in H^2(\pi' \rtimes G, N)$ is the element of cohomology classifying $1 \to N \to E \rtimes G \to \pi' \rtimes G \to 1$.

Proposition 2.2.3 implies that $s^*(\eta)$ classifies (12) (where (12) is viewed as a short exact sequence of profinite groups–the G structure is ignored).

Since $s^*(\eta) = 0$, we have an isomorphism of profinite groups $e : \pi' \times N \overset{\cong}{\to} E$. Define $\gamma : G \times \pi' \to N$ by $\gamma(g, \varpi') = e(g\varpi')^{-1} g e(\varpi')$ for $g \in G$ and $\varpi' \in \pi'$. Since G acts on E,

$$\gamma(g, \varpi'_1 \varpi'_2) = \gamma(g, \varpi'_1) + \gamma(g, \varpi'_2), \tag{14}$$

for all $\varpi'_1, \varpi'_2 \in \pi'$.

Let $\sigma : \pi' \rtimes G \to E \rtimes G$ be the section of (13) given by $\sigma(\varpi' \rtimes g) = e(\varpi' \times 0) \rtimes g$, so $\sigma = e|_{\pi'} \rtimes \mathrm{id}$, where $e|_{\pi'}$ denotes the restriction of e to π'. Let $n \in C^2(\pi' \rtimes G, N)$ be the cocyle corresponding to σ and (13) as in 2.0.5. (In particular, n represents η.) A short calculation shows:

$$n(\varpi'_1 \rtimes g_1, \varpi'_2 \rtimes g_2) = \gamma(g_1, \varpi'_2). \tag{15}$$

(14) and (15) imply that $\wp_\eta$ is linear in the sense of the Proposition: more explicitly, suppose that $s_1, s_2 \in H^1(G, \pi')$ are represented by cocycles $\nu_1, \nu_2 \in C^1(G, \pi')$ respectively, such that the cochain $\nu_1 + \nu_2$ defined by $(\nu_1 + \nu_2)(g) = \nu_1(g)\nu_2(g)$ is a cocycle. Let $s_1 + s_2 \in H^1(G, \pi')$ denote the corresponding cohomology class.

$\wp_\eta(s_j) \in H^2(G, N)$ is represented by the cocycle $\wp_\eta(\nu_j) \in C^2(G, N)$ defined

$$\wp_\eta(\nu_j)(g_1, g_2) = n(\nu_j(g_1) \rtimes g_1, \nu_j(g_2) \rtimes g_2),$$

for $j = 1, 2$. By (15),

$$\wp_\eta(\nu_j)(g_1, g_2) = \gamma(g_1, \nu_j(g_2)).$$

Similarly,

$$\wp_\eta(\nu_1 + \nu_2)(g_1, g_2) = \gamma(g_1, (\nu_1 + \nu_2)(g_2)) = \gamma(g_1, \nu_1(g_2)\nu_2(g_2)).$$

By (14), we have $\wp_\eta(s_1 + s_2) = \wp_\eta(s_1) + \wp_\eta(s_2)$, as desired. □

Let X, π_1, G_k, and δ_n be as in 1.1.3.

Proposition 2.2.5 partially separates the topology of the surface underlying $X(\mathbb{C})$ from the complicated Galois action in the determination of δ_n from $\pi_1^{et}(X, b)$. More explicitly, the short exact sequence $1 \to \pi_1/[\pi_1]_n \to \pi_1/[\pi_1]_n \rtimes G_k \to G_k \to 1$ decomposes $\pi_1/[\pi_1]_n \rtimes G_k$ into the group $\pi_1/[\pi_1]_n$, which comes from this surface, and the Galois group G_k. (This short exact sequence is the substitute for viewing the approximating spaces in the tower (1) as total spaces of fibrations over $B(G_k)$ whose fibers approximate the surface alone.) Let i denote the inclusion

$$i : \pi_1/[\pi_1]_n \to \pi_1/[\pi_1]_n \rtimes G_k,$$

and let $H^2(i) : H^2(\pi_1/[\pi_1]_n \rtimes G_k, [\pi_1]_n/[\pi_1]_{n+1}) \to H^2(\pi_1/[\pi_1]_n, [\pi_1]_n/[\pi_1]_{n+1})$ be the induced map on cohomology. Proposition 2.2.5 marks obstructions coming from classes in the kernel of $H^2(i)$ as subject to a linearity condition. On the other hand, by 2.2.3, the image under $H^2(i)$ of the class determining δ_n (as in 2.2.2) is controlled entirely by the topology of the surface. We record this in the following Proposition.

2.2.6. *Proposition. — Let* ω_n *in* $H^2(\pi_1/[\pi_1]_n, [\pi_1]_n/[\pi_1]_{n+1})$ *denote the element classifying*

$$1 \to [\pi_1]_n/[\pi_1]_{n+1} \to \pi_1/[\pi_1]_{n+1} \to \pi_1/[\pi_1]_n \to 1$$

Then for any ϵ *in* $H^2(i)^{-1}(\omega_n)$*,*

$$\delta_n = \wp_\epsilon + \mathcal{L},$$

with $\mathcal{L} : H^1(G_k, \pi_1/[\pi_1]_n) \to H^2(G_k, [\pi_1]_n/[\pi_1]_{n+1})$ *linear in the sense of Proposition 2.2.5*

2.3. Lower central series extensions of finitely generated free groups and their profinite completions are classified by Massey products.

To apply Proposition 2.2.6, we compute ω_n when π_1 is the profinite completion of a finitely generated free group. (The notation ω_n is as in Proposition 2.2.6.)

Let $F = \langle x_1, \dots, x_r \rangle$ be the free group on the generators $x_1, \dots, x_r$, and let ϖ denote either F or the profinite completion of F, denoted $\hat{F}$. Let $\omega_n \in H^2(\varpi/[\varpi]_n, [\varpi]_n/[\varpi]_{n+1})$ be the element of cohomology classifying the extension

$$1 \to [\varpi]_n/[\varpi]_{n+1} \to \varpi/[\varpi]_{n+1} \to \varpi/[\varpi]_n \to 1.$$

as in 2.0.5.

The homomorphisms $x_j^* : F \to \mathbb{Z}$ defined by

$$x_j^*(x_i) = \delta_{ij} \tag{16}$$

extend to morphisms $\hat{F} \to \hat{\mathbb{Z}}$ also denoted x_j^*. Here δ_{ij} is the Kronecker delta: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. Let $Z = \mathbb{Z}$ if $\varpi = F$, and $Z = \hat{\mathbb{Z}}$ if $\varpi = \hat{F}$. The x_j^* determine cohomology classes in $H^1(\varpi/[\varpi]_n, Z)$ for any n. These cohomology classes will also be denoted x_j^*.

Using results of Dwyer [Dwy75], we compute ω_n in terms of n^{th} order Massey products of the x_j^*'s:

2.3.1. For a commutative ring A with a G action, or a profinite commutative ring A with a continuous G action, $C^*(G, A)$ (defined in 2.0.4) is a differential graded A algebra with multiplication given by the cup product. The cup product is defined

$$(a \cup b)(g_1, \dots, g_{p+q}) = a(g_1, \dots, g_p)((g_1 \cdots g_p) b(g_{p+1}, \dots, g_{p+1q})),$$

for $a \in C^p(G, A)$ and $b \in C^q(G, A)$.

2.3.2. *Massey product sign conventions.* Although Massey products can be defined in much greater generality [May69], here we limit ourselves to Massey products of elements of $H^1(G, A)$. We use the sign convention of [Dwy75], [Mor04], which is different from [Kra66] [May69]:

2.3.3. *Definition.* Let $t_1, \dots, t_n$ be elements of $H^1(G, A)$. The n^{th} order Massey product of the ordered n-tuple $(t_1, \dots, t_n)$ is defined if there exists a $(n+1) \times (n+1)$ strictly upper triangular matrix T with entries $T_{ij} \in C^1(G, A)$ such that

- $T_{i,i+1}$ represents t_i.
- $DT_{ij} = \sum_{p=i+1}^{j-1} T_{ip} \cup T_{pj}$ for $i + 1 < j$

T is called a *defining system*. The *Massey product relative to* T is defined by

$$\langle t_1, \dots t_{n-1} \rangle_T = \sum_{p=2}^{n} T_{1p} \cup T_{p,n+1}.$$

Let U_n denote the multiplicative group of $n \times n$ upper triangular matrices with coefficients in A whose diagonal entries are 1. ("U" stands for unipotent.) We have an inclusion $A \to U_n$ given by sending $a \in A$ to the matrix $e^a_{1,n}$, where $e^a_{1,n}$ is defined as the matrix with a as the $(1, n)$-entry, and with all other off diagonal matrix entries 0. This inclusion gives rise to a central extension

$$1 \to A \to U_n \to \overline{U}_n \to 1. \tag{17}$$

Note that the function $a_{ij} : U_n \to A$ taking a matrix to its (i, j) entry descends to a function on the quotient group $\overline{U}_n$ for $(i, j) \neq (1, n)$. We repeat an observation of Dwyer: for $g_1, g_2 \in \overline{U}_n$,

$$a_{ij}(g_1 g_2) = \sum_{p=1}^{n} a_{ip}(g_1) a_{pj}(g_2),$$

whence

$$a_{ij}(g_1 g_2) = a_{ij}(g_2) + \sum_{p=i+1}^{j-1} a_{ip}(g_1) a_{pj}(g_2) + a_{ij}(g_1).$$

Thus the matrix M with (i, j) entry given by $-a_{ij}$ is a defining system for $(-a_{12}, \dots, -a_{n-1,n})$. The section $\overline{U}_n \to U_n$ whose image lies in those matrices with vanishing $(1, n)$ entry gives rise to a 2-cocycle classifying (17), as in 2.0.5. This 2-cocycle is $\langle -a_{12}, \dots, -a_{n-1,n} \rangle_M$. Thus:

2.3.4. *Proposition.* — [Dwy75, Rmk p. 182]*The central extension (17) is classified by*

$$\langle -a_{12}, \dots, -a_{n-1,n} \rangle_M$$

It is well-known that U_n is n-nilpotent and $\overline{U}_n$ is $(n-1)$-nilpotent. (See 2.0.1 for the definition of n-nilpotent.) Although $F/[F]_n$ has an initial universal property among n-nilpotent groups that $\overline{U}_{n+1}$ does not, we show that ω_n is also given by Massey products.

Take $A = Z$. Choose $J : \{1, \dots, n\} \to \{1, \dots, r\}$. Let $\varphi_J : \varpi \to U_{n+1}$ be the homomorphism determined by

$$a_{i,i+1} \varphi_J(x_k) = x^*_{J(i)}(x_k),$$

$$a_{i,j}\varphi_J(x_k) = 0,$$

for $i = 1,\ldots,n$, $j > i+1$, $k = 1,\ldots,r$. Since $\overline{U}_{n+1}$ is n-nilpotent, φ_J descends to a map $\varphi_J(\varpi/[\varpi]_n) : \varpi/[\varpi]_n \to \overline{U}_{n+1}$. It follows that $\varphi_J(\varpi/[\varpi]_n)^*(M)$ is a defining system for $(x^*_{J(1)},\ldots x^*_{J(n)})$, so the n^{th} order Massey product $\langle x^*_{J(1)},\ldots,x^*_{J(n)}\rangle$ is defined in $H^2(\varpi/[\varpi]_n,\mathbb{Z})$

Since U_{n+1} is $n+1$-nilpotent, and $\overline{U}_{n+1}$ is n-nilpotent, φ_J gives rise to the morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & U_{n+1} & \longrightarrow & \overline{U}_{n+1} & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \varphi_J([\varpi]_n/[\varpi]_{n+1})} & & \uparrow{\scriptstyle \varphi_J(\varpi/[\varpi]_{n+1})} & & \uparrow{\scriptstyle \varphi_J(\varpi/[\varpi]_n)} & & \\ 1 & \longrightarrow & [\varpi]_n/[\varpi]_{n+1} & \longrightarrow & \varpi/[\varpi]_{n+1} & \longrightarrow & \varpi/[\varpi]_n & \longrightarrow & 1 \end{array} \tag{18}$$

Applying Proposition 2.2.3 to the morphism 18, we have:

$$\varphi_J([\varpi]_n/[\varpi]_{n+1})_*(\omega_n) = \langle -x^*_{J(1)},\ldots,-x^*_{J(n)}\rangle_{\varphi(\varpi/[\varpi]_n)^*M} \tag{19}$$

The function $\varphi_J([\varpi]_n/[\varpi]_{n+1}) : [\varpi]_n/[\varpi]_{n+1} \to Z$ is in fact the Magnus coefficient associated to $x_{J(1)}\cdots x_{J(n)}$ as is implied by a result of Dwyer, which we now explain (we will also recall the definition of the Magnus coefficient):

2.3.5. *Magnus embedding.* Let $Z\langle\langle z_1,\ldots,z_r\rangle\rangle$ be the ring of associative power series in the non-commuting variables $z_1,\ldots,z_n$ with coefficients in Z. The *Magnus embedding* $F \to \mathbb{Z}\langle\langle z_1,\ldots,z_r\rangle\rangle^\times$ is an injection of the free group into the multiplicative group of units of $\mathbb{Z}\langle\langle z_1,\ldots,z_r\rangle\rangle$ given by $x_j \mapsto 1+z_j$ for all j. Since $\hat{\mathbb{Z}}\langle\langle z_1,\ldots,z_r\rangle\rangle$ is profinite, $F \to \mathbb{Z}\langle\langle z_1,\ldots,z_r\rangle\rangle^\times$ extends to a continuous homomorphism $\hat{F} \to \hat{\mathbb{Z}}\langle\langle z_1,\ldots,z_r\rangle\rangle^\times$. Thus we have $m : \varpi \to Z\langle\langle z_1,\ldots,z_r\rangle\rangle$. For a degree d monomial $z_{I(1)}\cdots z_{I(d)}$ of $Z\langle\langle z_1,\ldots,z_r\rangle\rangle$ ($I : \{1,\ldots,d\} \to \{1,\ldots,r\}$), the associated *Magnus coefficient* $\mu_I : \varpi \to Z$ is defined by taking $f \in \varpi$ to the coefficient of $z_{I(1)}\cdots z_{I(d)}$ in $m(f)$. It is well known that $\mu_I(f) = 0$ for $f \in [F]_n$ and $0 < d < n$ (see [MKS04, §5.5, Cor. 5.7, p. 312]), and it follows by continuity that $\mu_I(f) = 0$ for $f \in [\varpi]_n$ and $0 < d < n$. Thus, μ_I induces a (well-defined) function $\mu_I : [\varpi]_d/[\varpi]_{d+1} \to Z$.

The following result is implied by [Dwy75, Lemma 4.2]. As the proof was omitted in [Dwy75], we include one for completeness.

2.3.6. *Proposition [Dwyer].* — $\varphi_J([\varpi]_n/[\varpi]_{n+1}) = \mu_J$

Proof. The homomorphism φ_J admits a factorization through m:

$$\begin{array}{ccc} \varpi & \xrightarrow{m} & Z\langle\langle z_1, \dots, z_r \rangle\rangle^\times \\ & \searrow^{\varphi_J} & \downarrow \\ & & U_{n+1} \end{array}$$

More explicitly, the map $Z\langle\langle z_1, \dots, z_r\rangle\rangle^\times \to U_{n+1}$ is the restriction of a homomorphism of algebras $\theta_J : Z\langle\langle z_1, \dots, z_r\rangle\rangle \to T_{n+1}$, where T_{n+1} denotes the algebra of upper triangular $(n+1) \times (n+1)$ matrices with coefficients in Z. θ_J is given by

$$a_{i,i+1}\theta_J(z_l) = \delta_{J(i),l},$$

$$a_{i,j}\theta_J(z_l) = 0,$$

for $l = 1, \dots, r$, $i = 1, \dots, n$ and $j \neq i+1$. T_{n+1} admits a grading, where the homogeneous degree d matrices are those such that $a_{i,j} = 0$ for $j \neq i + d$. θ_J is a map of graded algebras. (The fact that θ_J respects the gradings and T_{n+1} has no homogeneous elements of degree greater than n was used implicitly to see that θ_J is well-defined).

Let $b_{i,j}$ be the matrix of T_{n+1} with 1 in the (i,j) entry and all other entries 0 ($i = 1, \dots, n+1$, $j \geq i$). The only non-zero product of n elements of $\{b_{1,2}, b_{2,3}, \dots, b_{n,n+1}\}$ is $b_{1,2}b_{2,3}\cdots b_{n,n+1}$ and this product is $b_{1,2}b_{2,3}\cdots b_{n,n+1} = b_{1,n+1}$. It follows that

$$a_{1,n+1}\theta_J(z_{J(1)}\cdots z_{J(n)}) = 1$$

and that

$$a_{1,n+1}\theta_J(z_{I(1)}\cdots z_{I(n)}) = 0$$

for $I : \{1, \dots, n\} \to \{1, \dots, r\}$ such that $I \neq J$. Since θ_J respects the gradings on our algebras, we have $a_{1,n+1}\theta_J(z_{I(1)}\cdots z_{I(d)}) = 0$ for any $I : \{1, \dots, d\} \to \{1, \dots, r\}$ with $d \neq n$. Thus $a_{1,n+1}\theta_J m = \mu_J$, whence $\varphi_J([\varpi]_n/[\varpi]_{n+1}) = \mu_J$.

□

Combining Proposition 2.3.6 and equation (19), we have:

2.3.7. *Proposition. — For any* $J : \{1, \dots, n\} \to \{1, \dots, r\}$

$$(\mu_J)_*\omega_n = \langle -x^*_{J(1)}, \dots - x^*_{J(n)}\rangle_{\varphi(\varpi/[\varpi]_n)^*M}$$

Combining well known results, we now show that ω_n is determined by the elements $(\mu_J)_*\omega_n \in H^2(\varpi/[\varpi]_n, Z)$, where J ranges over all functions $J : \{1, \dots, n\} \to \{1, \dots, r\}$; this is a consequence of the Lie basis theorem and the relationship between the lower central series of free groups and the Magnus embedding [MKS04, Ch. 5], as we now explain:

The Lie elements of $Z\langle\langle z_1, \dots, z_r\rangle\rangle$ are the elements in the image of the map from the free Lie algebra over Z on the r generators $\zeta_1, \dots, \zeta_r$ to $Z\langle\langle z_1, \dots, z_r\rangle\rangle$ given by $\zeta_j \mapsto z_j$. (The Lie bracket on $Z\langle\langle z_1, \dots, z_r\rangle\rangle$ is $[z, z'] = zz' - z'z$.) As previously commented, $\mu_J(f) = 0$ for $f \in [\varpi]_n$ and J a function $J : \{1, \dots, d\} \to \{1, \dots, r\}$ such that $0 < d < n$. (A reference is [MKS04, §5.5, Cor. 5.7, p. 312]). Thus for any $f \in [\varpi]_n$, $m(f)$ is of the form $1+\wp_n+\wp_{>n}$, where $\wp_n$ is homogeneous of degree n and $\wp_{>n}$ only contains monomials of degree greater than n. It is well known that $f \mapsto \wp_n$ induces an isomorphism from $[F]_n/[F]_{n+1}$ to the homogeneous degree n Lie elements of $\mathbb{Z}\langle\langle z_1,\ldots,z_r\rangle\rangle$ ([MKS04, §5.7, Cor. 5.12(i), p. 341]). The inverse takes a weight n bracket in the variables $\zeta_1,\ldots,\zeta_r$ to the corresponding bracket in $[F]_n/[F]_{n+1}$ with the variable x_j substituted for ζ_j, $j = 1,\ldots,r$. As both this isomorphism and its inverse extend to continuous maps between $[\varpi]_n/[\varpi]_{n+1}$ and the homogeneous degree n Lie elements of $\hat{\mathbb{Z}}\langle\langle z_1,\ldots,z_r\rangle\rangle$, we have that $f \mapsto \wp_n$ induces an isomorphism from $[\varpi]_n/[\varpi]_{n+1}$ to the homogeneous degree n Lie elements of $Z\langle\langle z_1,\ldots,z_r\rangle\rangle$. By the Lie basis theorem ([MKS04, §5.6, Thm. 5.8(ii), p. 323]), the inclusion of the Lie elements of degree n into all the degree n elements of $\mathbb{Z}\langle\langle z_1,\ldots,z_r\rangle\rangle$ is a direct summand. It follows by continuity arguments that the same statement holds for $\hat{\mathbb{Z}}\langle\langle z_1,\ldots,z_r\rangle\rangle$. We may therefore choose a left inverse L to the map

$$\bigoplus_{J:\{1,\ldots,n\}\to\{1,\ldots,r\}} \mu_J : [\varpi]_n/[\varpi]_{n+1} \to \bigoplus_{J:\{1,\ldots,n\}\to\{1,\ldots,r\}} Z. \tag{20}$$

Let $L_* : \bigoplus_{J:\{1,\ldots,n\}\to\{1,\ldots,r\}} H^2(\varpi/[\varpi]_n,\mathbb{Z}) \to H^2(\varpi/[\varpi]_n, [\varpi]_n/[\varpi]_{n+1})$ be the map induced by L. By Proposition 2.3.7, we have:

2.3.8. *Proposition.* —

$$\omega_n = L_* \bigoplus_{J:\{1,\ldots,n\}\to\{1,\ldots,r\}} \langle x^*_{J(1)},\ldots x^*_{J(n)}\rangle_{\varphi(\varpi/[\varpi]_n)^*M}$$

2.3.9. The above calculation of the element of H^2 classifying

$$1 \to [\pi]_2/[\pi]_3 \to \pi/[\pi]_3 \to \pi/[\pi]_2 \to 1$$

as a cup product combines with Proposition 2.2.5 to give another proof of Proposition 2.1.4. Furthermore, this second proof is straightforward in the sense that it does not requiring 'guessing' the cochain γ whose boundary is the difference of two given cocycles, as in the proof in section 2.1.

Written out in excruciatingly unnecessary detail, this second proof goes as follows:

Proof. Let β be the map $x \mapsto [-,-]_*(x \cup x)$.

Let $\varpi_G \in H^1(\pi/[\pi]_2 \rtimes G, \pi/[\pi]_2)$ be the cohomology class represented by the projection $\pi/[\pi]_2 \rtimes G \to \pi/[\pi]_2$. Let $B_G \in H^2(\pi/[\pi]_2 \rtimes G, [\pi]_2/[\pi]_3)$ be the push-forward of $\varpi_G \cup \varpi_G$ under the commutator. Then, $\beta = \wp_{B_G}$ where $\wp_{B_G}$ is defined as in 2.2.1.

Let $\varpi \in H^1(\pi/[\pi]_2), \pi/[\pi]_2)$ be the cohomology class represented by the identity map. Let $B \in H^2(\pi/[\pi]_2, [\pi]_2/[\pi]_3)$ be the push-forward of $\varpi \cup \varpi$ under the commutator. So, $\varpi = i^*\varpi_G$ and $B = i^*B_G$, where $i : \pi \to \pi \rtimes G$ is the canonical inclusion.

As above, let $F = \langle x_1, \ldots, x_r \rangle$ be the free group, and if $r = 2g$ for some g, let $R = F/\langle [x_1, x_2][x_3, x_4] \cdots [x_{g-1}, x_g] \rangle$ be the fundamental group of a closed Riemann surface of genus g. Ignoring the action of G on π, π is isomorphic to $F^\wedge$ or $R^\wedge$, where $F^\wedge$ and $R^\wedge$ denote the profinite completions of F and R respectively.

Suppose $\pi = F^\wedge$. The abelianization of π is a free $\hat{\mathbb{Z}}$ module with basis $\{x_1, \ldots, x_r\}$, so an arbitrary element of $\pi/[\pi]_2$ can be expressed as a proword $x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ with $a_j \in \hat{\mathbb{Z}}$. **[do we need to explain the "pro-word" notation? — Kirsten]** In $[F]_2/[F]_3$, we have the computation:

$$\begin{aligned}
[x_i^{a_i} x_j^{a_j}, x_i^{b_i} x_j^{b_j}] &= (x_i^{a_i} x_j^{a_j})(x_i^{b_i} x_j^{b_j})(x_i^{a_i} x_j^{a_j})^{-1}(x_i^{b_i} x_j^{b_j})^{-1} \\
&= [x_i, x_j]^{-a_j b_i} x_i^{a_i + b_i} x_j^{a_j + b_j} x_j^{-a_j} x_i^{-a_i} x_j^{-b_j} x_i^{-b_i} \\
&= [x_i, x_j]^{-a_j b_i} x_i^{a_i + b_i} x_j^{a_j + b_j} [x_i, x_j]^{a_i b_j} x_j^{-a_j - b_j} x_i^{-a_i - b_i} \\
&= [x_i, x_j]^{a_i b_j - a_j b_i}
\end{aligned}$$

It follows that

$$B(x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}, x_1^{b_1} x_2^{b_2} \cdots x_r^{b_r}) = \prod_{i<j} [x_i, x_j]^{a_i b_j - a_j b_i}, \tag{21}$$

whence pushing forward B by the Magnus coefficient for the function $J : \{1, 2\} \to \{1, 2, \ldots, r\}$ with $J(1) = i$, $J(2) = j$, we have

$$(\mu_J)_* B = x_i^* \cup x_j^* - x_j^* \cup x_i^* = 2x_i^* \cup x_j^*.$$

Use the notation that $\hat{\omega}_n$ is the element of $H^2(F^\wedge/[F^\wedge]_n, [F^\wedge]_n/[F^\wedge]_{n+1})$ classifying the extension $1 \to [F^\wedge]_n/[F^\wedge]_{n+1} \to F^\wedge/[F^\wedge]_{n+1} \to F^\wedge/[F^\wedge]_n \to 1$.

By Proposition 2.3.7, we have that $2(\mu_J)_* \hat{\omega}_2 = (\mu_J)_* B$. Since (20) is a split injection, we have $2\hat{\omega}_n = B$. By Proposition 2.2.5, $\beta - 2\delta_2$ is linear as claimed.

The case $\pi = R^\wedge$ is similar: the quotient map $q : F^\wedge \to R^\wedge$ induces an isomorphism on abelianizations, and equation 21 remains valid. Thus,

$$2q([F^\wedge]_2/[F^\wedge]_3)_* \hat{\omega}_2 = q(F^\wedge/[F^\wedge]_2)^* B, \tag{22}$$

where $q([F^\wedge]_2/[F^\wedge]_3) : [F^\wedge]_2/[F^\wedge]_3 \to [R^\wedge]_2/[R^\wedge]_3$ and $q(F^\wedge/[F^\wedge]_2) : F^\wedge/[F^\wedge]_2 \to R^\wedge/[R^\wedge]_2$ denote the maps induced by q. q induces a morphism of short exact sequences:

(23)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & [F^\wedge]_2/[F^\wedge]_3 & \longrightarrow & F^\wedge/[F^\wedge]_3 & \longrightarrow & F^\wedge/[F^\wedge]_2 & \longrightarrow & 1 \\
 & & \downarrow q & & \downarrow q & & \downarrow q \, \cong & & \\
1 & \longrightarrow & [R^\wedge]_2/[R^\wedge]_3 & \longrightarrow & R^\wedge/[R^\wedge]_3 & \longrightarrow & R^\wedge/[R^\wedge]_2 & \longrightarrow & 1
\end{array}$$

where $q([F^\wedge]_2/[F^\wedge]_3)$ and $q(F^\wedge/[F^\wedge]_2)$ have been abbreviated to q.

Let $\hat{\rho}_n \in H^2(R^\wedge/[R^\wedge]_n, [R^\wedge]_n/[R^\wedge]_{n+1})$ be the element of cohomology classifying the bottom short exact sequence of (23). By Proposition 2.2.3, $q_*\hat{\omega}_2 = q^*\hat{\rho}_n$.

By (22), $2q^*\hat{\rho}_n = q^*B$. Since $q(F^\wedge/[F^\wedge]_2)$ is an isomorphism, this gives $2\hat{\rho}_n = B$, which by Proposition 2.2.5 completes the proof. □

2.4. **Bracket Massey products.** 2.4.1. *Definition.* For a short exact sequence of groups $1 \to N \to M \to Q \to 1$ with N abelian, and a (equivariant) homomorphism $f : N \to N'$ of abelian groups equipped with an action of M, an N *additive* N' *coordinate* is a function $A : M \to N'$ such that $A(nm) = f(n)A(m)$ for all n in N and m in M. In the case where $N' = N$ and f is the identity, we will say *N coordinate* to abbreviate 'N additive N coordinate.'

2.4.2. *Proposition.* — The boundary of an N additive N' coordinate descends to a 2-cocycle in $C^2(Q, N)$.

Proof. Let $A : M \to N$ be an N additive N' coordinate for $1 \to N \to M \to Q \to 1$ and $f : N \to N'$. $D(A)$ in $C^2(M, N')$ is defined $D(A)(x, y) = A(x) + xA(y) - A(xy)$. Let n be an element of N. We have the following straight-forward calculations:

$D(A)(nx, y) = A(nx) + nxA(y) - A(nxy) = f(n) + A(x) + xA(y) - (f(n) + A(xy)) = D(A)(x, y)$.

$D(A)(x, ny) = A(x) + xA(ny) - A(xny) = A(x) + xf(n) + xA(y) - A((xnx^{-1})(xy)) = A(x) + xf(n) + xA(y) - (xf(n) + A(xy)) = D(A)(x, y)$

Thus $D(A)$ descends to a function $Q \times Q \to N$. Since $D(A)$ considered as an element of $C^2(M, N)$ is a cocycle, $D(A)$ considered as an element of $C^2(Q, N)$ is a cocycle as well. □

2.4.3. The set of N coordinates of $1 \to N \to M \to Q \to 1$ is in bijection with the set of set-theoretic sections of the quotient map $M \to Q$, via the following bijection:

To an N coordinate $A : M \to N$, we associate the set-theoretic section

$$q \mapsto A(\overline{q})^{-1}\overline{q},$$

where $\overline{q}$ is any element of M mapping to q. (Because $A(n\overline{q})^{-1}n\overline{q} = (nA(\overline{q}))^{-1}n\overline{q} = A(\overline{q})^{-1}\overline{q}$, this section is well-defined.)

The inverse bijection associates to a set theoretic section $s : Q \to M$, the N coordinate

$$m \mapsto ms(q)^{-1},$$

where q is the image of m in Q.

2.4.4. *Proposition.* — The extension $1 \to N \to M \to Q \to 1$ is classified by $-D(A)$ in $H^2(Q, N)$, where A is any N coordinate.

Proof. Let $s : Q \to M$ denote the set-theoretic section of $M \to Q$ corresponding to A defined in 2.4.3. Let q_1, q_2 be elements of Q, and let m_1, m_2 be elements of M mapping to q_1, q_2 respectively. The proposition follows from the straight-forward algebraic manipulation:

$$\begin{aligned} s(q_1)s(q_2)s(q_1q_2)^{-1} =& (A(m_1)^{-1}m_1)(A(m_2)^{-1}m_2)(A(m_1m_2)^{-1}m_1m_2)^{-1} \\ =& A(m_1)^{-1}(m_1A(m_2)^{-1}m_1^{-1})A(m_1m_2) \end{aligned}$$

□

2.4.5. *Proposition.* — Let $\mathcal{A}$ be the set of N coordinates of $1 \to N \to M \to Q \to 1$, and let $\omega \in H^2(Q, N)$ classify this extension. The subset of cocycles of $C^2(Q, N)$ cohomologous to ω equals $\{-D(A) : A \in \mathcal{A}\}$.

Proof. By 2.0.5, ω is represented by the cocycle corresponding to a set-theoretic section s of $M \to Q$. Changing s to $s + e$ for $e : Q \to N$ changes the corresponding cocycle by adding $D(e)$. The proposition then follows by 2.4.3 and the proof of Proposition 2.4.4. □

2.4.6. Let M be a graded associative algebra with unit of the form:

$$M = \prod_{n=0}^{\infty} M_n,$$

where M_n denotes the homogeneous degree n elements of M. Let $a_n : M \to M_n$ denote the projection. Addition and multiplication in M will be denoted by + and · respectively.

Let U be multiplicative subgroup of the units of M given by $U = \{1 + x | x \in \prod_{n=1}^{\infty} M_n\}$.

Let N be the subset of M given by $N = \{x | x \in \prod_{n=1}^{\infty} M_n\}$. N has the structure of a Lie algebra by defining $[n_1, n_2]_N = n_1n_2 - n_2n_1$ for any n_1, n_2 in N.

Suppose $\mathbb{Q}$ injects into M_0. Then, $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ and $\log(x) = \sum_{n=1}^{\infty}(-1)^{n+1}(x-1)^n/n$ define functions $\exp : N \to U$ and $\log : U \to N$, which are inverse bijections between N and U (considered as sets). These bijections allow N to inherit the group structure of U, namely N has a group structure $*$ defined: $n_1 * n_2 = \log(\exp(n_1)\exp(n_2))$ making $\exp$ and $\log$ inverse isomorphisms of groups. The Campbell-Hausdorff formula

$$n_1 * n_2 = n_1 + n_2 + \frac{1}{2}[n_1, n_2] + \frac{1}{12}[[n_1, n_2], n_2] + \frac{1}{12}[[n_2, n_1], n_1] + \dots$$

expresses $n_1 * n_2$ in terms of the Lie algebra structure on N, where in the above formula $[-,-]$ abbreviates $[-,-]_N$. All the higher terms are higher order nested Lie brackets. For use later, we state the Campbell-Hausdorff formula in its entirety:

(24)

$$n_1 * n_2 = \sum_{m \geq 1} \sum_{\substack{\{p_i, q_i \geq 0 | \\ i=1,\dots,m, \\ p_i+q_i>0\}}} \frac{(-1)^m}{m\sum(p_i+q_i)} \frac{1}{\prod p_i! q_i!} [\cdots[\cdots[[n_1, n_1], \dots, n_1], n_2, \dots, n_2], n_1, \dots, n_2]$$

The bracket $[\cdots[\cdots[[n_1, n_1], \dots, n_1], n_2, \dots, n_2], n_1, \dots, n_2]$ has p_1 n_1's, then q_1 n_2's, then p_2 n_1's, then q_2 n_2's etc. For $I : \{1, \dots, m\} \to \{1, 2\}$, it will be convenient to let C_I denote the coefficient of $[\cdots[[n_{I(1)}, n_{I(2)}], n_{I(3)}], \dots, n_{I(m)}]$ in (24). In particular, we can abbreviate (24):

$$n_1 * n_2 = \sum_{m=1}^{\infty} \sum_{I:\{1,\dots,m\}\to\{1,2\}} C_I[\cdots[[n_{I(1)}, n_{I(2)}], n_{I(3)}], \dots, n_{I(m)}] \tag{25}$$

U is filtered by normal subgroups $U_n = \{1 + x | x \in \prod_{j=n}^{\infty} M_j\}$:

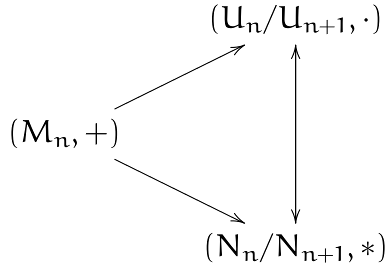
$$U = U_1 \supset U_2 \supset U_3 \dots$$

The corresponding filtration for the group $(N, *)$ is $N_n = \{x | x \in \prod_{j=n}^{\infty} M_j\}$:

$$N = N_1 \supset N_2 \supset N_3 \dots$$

These filtrations are additive with respect to the brackets $[u_1, u_2]_U = u_1 u_2 u_1^{-1} u_2^{-1}$ on U and $[n_1, n_2]_N = n_1 n_2 - n_2 n_1$ on N in the sense that $[U_j, U_k]_U \subset U_{j+k}$ and $[N_j, N_k]_N \subset N_{j+k}$.

2.4.7. We have a commutative diagram of group isomorphisms



where $m \mapsto 1 + m$ is the isomorphism $(M_n, +) \to (U_n/U_{n+1}, \cdot)$, $m \mapsto m$ is the isomorphism $(M_n, +) \to (N_n/N_{n+1}, *)$ and the vertical isomorphism is given by $\exp$ in one direction and $\log$ in the other.

2.4.8. *Proposition.* — $a_n \log : U/U_{n+1} \to U_n/U_{n+1}$ defines a U_n/U_{n+1} coordinate on U/U_{n+1}

Remark. $\log$ descends to a function $U/U_{n+1} \to N/N_{n+1}$. a_n descends to a function $N/N_{n+1} \to M_n$. We can therefore view $a_n \log$ as a function $U/U_{n+1} \to N/N_{n+1} \to M_n \to U_n/U_{n+1}$ via 2.4.7. This is what is meant by $a_n \log : U/U_{n+1} \to U_n/U_{n+1}$ in Proposition 2.4.8.

Proof. (of Proposition 2.4.8) Take x in U_n/U_{n+1} and y in U/U_{n+1}. $a_n \log(xy) = a_n(\log x * \log y)$. Since $\log x$ is in N_n/N_{n+1}, $[\log x, z]_N = 0$ for any z in N/N_{n+1}. Thus by the Campbell-Hausdorff formula, $\log x * \log y = \log x + \log y$, and the proposition follows. □

2.4.9. Proposition 2.4.2, Proposition 2.4.4, and Proposition 2.4.8 imply that $-D(a_n \log)$ determines a cocycle in $C^2(U/U_n, U_n/U_{n+1})$ which classifies the extension $U_n/U_{n+1} \to U/U_{n+1} \to U/U_n$. We calculate $-D(a_n \log)$ using the Campbell-Hausdorff formula:

To do this, we express $(u_1, u_2) \mapsto a_n \log(u_1 u_2)$ as a $\mathbb{Q}$ linear combination of cochains built from the $a_j \log$ for $j < n$. The cochains in this $\mathbb{Q}$ linear combination are indexed by pairs (I, P), where I is a function $I : \{1, \ldots, m\} \to \{1, 2\}$ for some integer m between 1 and n inclusive, i.e. $1 \leq m \leq n$, and where $P : \{1, \ldots, m\} \to \{1, \ldots, n-1\}$ is a function such that $\sum_{j=1}^{m} P(j) = n$, i.e. P is a partition of n into the sum of m ordered, strictly positive integers. For (I, P) as above, the associated cochain in $C^2(U/U_{n+1}, U_n/U_{n+1})$ is given by:

$$(26) \quad (u_1, u_2) \mapsto [\cdots[[a_{P(1)} \log(u_{I(1)}), a_{P(2)} \log(u_{I(2)})], \ldots, a_{P(3)} \log(u_{I(3)})], a_{P(m)} \log(u_{I(m)})]$$

Note that when $m > 1$, (26) defines a 2 cochain in $C^2(U/U_n, U_n/U_{n+1})$. By (25),

$$a_n \log(u_1 u_2) = \sum_{m=1}^{n} \sum_{(I,P)} C_I[\cdots[[a_{P(1)} \log(u_{I(1)}), a_{P(2)} \log(u_{I(2)})], \ldots, a_{P(3)} \log(u_{I(3)})], a_{P(m)} \log(u_{I(m)})]$$

It follows that

$$-D(a_n \log)(u_1,u_2) = \sum_{m=2}^{n}\sum_{(I,P)} C_I[\cdots[[a_{P(1)}\log(u_{I(1)}), a_{P(2)}\log(u_{I(2)})], \ldots, a_{P(3)}\log(u_{I(3)})], a_{P(m)}\log(u_{I(m)})].$$

2.4.10. Bracket Massey products. Recall that for a ring A equipped with an action of a profinite group G, one can define the n^{th} order Massey product $\langle t_1, t_2, \ldots, t_n\rangle$ of certain n-tuples $(t_1, t_2, \ldots, t_n)$ of elements of $H^1(G, A)$. (The definition was recalled in 2.3.3.) The definition of the Massey product uses the associativity of the cup product on $C^*(G, A)$. By replacing this associativity with the Jacobi relation, we define for a Lie algebra L with a G-action, an analogue of the n^{th} order Massey product $\langle t_1, t_1, \ldots, t_1\rangle$ of n copies of a 1-cocycle t_1 in $C^1(G, L)$. As in the case of the usual Massey product, t_1 must admit a defining system, and this defining system introduces indeterminacy.

2.4.11. *Bracket cup product.* Composing the usual cup product $H^1(G, L) \otimes H^1(G, L) \to H^2(G, L\otimes L)$ with the map $H^2(G, L\otimes L) \to H^2(G, L)$, induced from the Lie bracket $L\otimes L \to L$, gives the order 2 bracket Massey product, or bracket cup product. (See 2.1.2 as well. Also, note that although we claimed we would only define $\langle t_1, t_1\rangle$, here we define $\langle t_1, t_2\rangle$. For the higher order bracket Massey products, we will only give a definition of an analogue of $\langle t_1, t_1, \ldots, t_1\rangle$)

2.4.12. *Third order bracket Massey product.* Recall the following construction of the third order (usual) Massey product of three elements $\tau_i \in H^1(G, A)$ represented by cocycles $t_i \in C^1(G, A)$, $i = 1, 2, 3$. τ_1, τ_2, and τ_3 must admit a defining system (s_1, s_2), $s_1, s_2 \in C^1(G, A)$, i.e. we must have $Ds_1 = t_1 \cup t_2$ and $Ds_2 = t_2 \cup t_3$. We can then form

$$s_1 \cup t_3 + t_1 \cup s_2$$

and it follows from the associativity of the cup product that this expression is a cocycle.

We can replace this associativity with the Jacobi relation.

Let $[-,-] : L \otimes L \to L$ denote the Lie bracket. Suppose $\tau_1 \in H^1(G, L)$ is such that $[-,-]_*(\tau_1 \cup \tau_1) = 0$ in $H^2(G, L)$. Then we can form a *third order bracket Massey product*, denoted $\langle[\tau_1]\rangle_3$ as follows: let $t_1 \in C^1(G, L)$ be a cocycle representing τ_1. Since $[-,-]_*(\tau_1 \cup \tau_1) = 0$ in $H^2(G, L)$, we can find $t_2 \in C^1(G, L)$ such that $D(t_2) = [-,-]_*(t_1 \cup t_1)$.

The boundary of the cochain $[-,-]_*(t_2 \cup t_1)$ is $[-,-]_*(D(t_2) \cup t_1 - t_2 \cup D(t_1))$. Thus

$$D[-,-]_*(t_2 \cup t_1)(g_1, g_2, g_3) = [[t_1(g_1), g_1t_1(g_2)], g_1g_2t_1(g_3)]$$

Similarly

$$D[-,-]_*(t_1 \cup t_2)(g_1, g_2, g_3) = -[t_1(g_1), [g_1t_1(g_2), g_1g_2t_1(g_3)]] = [[g_1t_1(g_2), g_1g_2t_1(g_3)], t_1(g_1)]$$

Therefore, $[-,-]_*(t_2 \cup t_1) + [-,-]_*(t_2 \cup t_1)$ is not a cocycle, but because of the Jacobi relation

$$D([-,-]_*(t_2 \cup t_1) + [-,-]_*(t_2 \cup t_1))(g_1, g_2, g_3) = -[[g_1g_2t_1(g_3), t_1(g_1)], g_1t_1(g_2)]$$

We now form two additional cochains in $C^2(G, L)$, whose boundary evaluated at (g_1, g_2, g_3) can be expressed as brackets $[[-,-],-]$ evaluated at $t_1(g_1)$, $g_1t_1(g_2)$, and $g_1g_2t_1(g_3)$. We will add a $\mathbb{Q}$ linear combination of these cochains to $[-,-]_*(t_2 \cup t_1) + [-,-]_*(t_2 \cup t_1)$ to form a cocycle.

First, consider the cochain in $C^1(G, L \otimes L)$

$$\Delta(t_1, t_1)(g) = t_1(g) \otimes t_1(g)$$

Note that the boundary of $\Delta(t_1, t_1)$ is

$$\begin{aligned} D\Delta(t_1,t_1)(g_1,g_2) &= t_1(g_1) \otimes t_1(g_1) + g_1t_1(g_2) \otimes g_1t_1(g_2) - t_1(g_1g_2) \otimes t_1(g_1g_2) \\ &= t_1(g_1) \otimes t_1(g_1) + g_1t_1(g_2) \otimes g_1t_1(g_2) - (t_1(g_1) + g_1t_1(g_2)) \otimes (t_1(g_1) + g_1t_1(g_2)) \\ &= -(t_1(g_1) \otimes g_1t_1(g_2) + g_1t_1(g_2) \otimes t_1(g_1)) \end{aligned}$$

where the second equality follows because t_1 is a cocycle.

The two additional cochains in $C^2(G, L)$ mentioned above are $[[-,-]-]_*(t_1 \cup \Delta(t_1, t_1))$ and $[-,[-,-]]_*(\Delta(t_1,t_1) \cup t_1)$. We calculate the boundaries of these cochains:

$$D([[-,-]-]_*(t_1 \cup \Delta(t_1,t_1)))(g_1,g_2,g_3) = [[t_1(g_1), g_1t_1(g_2)]g_1g_2t_1(g_3)] + [[t_1(g_1), g_1g_2t_1(g_3)]g_1t_1(g_2)]$$

$$\begin{aligned} D[-,[-,-]]_*(\Delta(t_1,t_1) \cup t_1)(g_1,g_2,g_3) &= -[t_1(g_1), [g_1t_1(g_2), g_1g_2t_1(g_3)]] - [g_1t_1(g_2), [t_1(g_1), g_1g_2t_1(g_3)]] \\ &= [[g_1t_1(g_2), g_1g_2t_1(g_3)], t_1(g_1)] + [[t_1(g_1), g_1g_2t_1(g_3)], g_1t_1(g_2)] \end{aligned}$$

Thus,

$$\begin{aligned} &D([-,[-,-]]_*(\Delta(t_1,t_1) \cup t_1) + [[-,-]-]_*(t_1 \cup \Delta(t_1,t_1)))(g_1,g_2,g_3) \\ &= [[t_1(g_1), g_1t_1(g_2)]g_1g_2t_1(g_3)] + [[g_1t_1(g_2), g_1g_2t_1(g_3)], t_1(g_1)] + 2[[t_1(g_1), g_1g_2t_1(g_3)], g_1t_1(g_2)] \\ &= -[[g_1g_2t_1(g_3), t_1(g_1)], g_1t_1(g_2)] + 2[[t_1(g_1), g_1g_2t_1(g_3)], g_1t_1(g_2)] \\ &= 3[[t_1(g_1), g_1g_2t_1(g_3)], g_1t_1(g_2)] \end{aligned}$$

where the second equality follows from the Jacobi relation.

$D([-,-]_*(t_2 \cup t_1) + [-,-]_*(t_2 \cup t_1))(g_1,g_2,g_3)$, $D([[-,-]-]_*(t_1 \cup \Delta(t_1,t_1)))(g_1,g_2,g_3)$, and $D[-,[-,-]]_*(\Delta(t_1,t_1) \cup t_1)(g_1,g_2,g_3)$ are all in the substance of L spanned by

$$[[t_1(g_1), g_1g_2t_1(g_3)], g_1t_1(g_2)]$$

and

$$[[t_1(g_1), g_1t_1(g_2)], g_1g_2t_1(g_3)],$$

so they must satisfy a linear relation. From the above we see that this (unique up to scalar multiplication) linear relation is

$$\begin{aligned}0 = D([-,-]_*(t_2 \cup t_1) + [-,-]_*(t_2 \cup t_1))(g_1, g_2, g_3) \\ -\frac{1}{3}D([[-,-]-]_*(t_1 \cup \Delta(t_1, t_1)))(g_1, g_2, g_3) \\ -\frac{1}{3}D[-,[-,-]]_*(\Delta(t_1, t_1) \cup t_1)(g_1, g_2, g_3)\end{aligned}$$

We define $\langle[\tau_1]\rangle_3$ by

$$\langle[\tau_1]\rangle_3 = [-,-]_*(t_2 \cup t_1) + [-,-]_*(t_2 \cup t_1) - \frac{1}{3}[[-,-]-]_*(t_1 \cup \Delta(t_1, t_1)) - \frac{1}{3}[-,[-,-]]_*(\Delta(t_1, t_1) \cup t_1)$$

and by the above $\langle[\tau_1]\rangle_3$ is a cocycle.

2.4.13. n^{th} *order bracket Massey product.* Instead of searching by hand for linear combinations of brackets of cochains which determine cocycles by the Jacobi identity, the Campbell-Hausdorff formula produces such a linear combination automatically. We describe this linear combination in 2.4.14-2.4.19.

2.4.14. The cochain given in (26) generalizes to produce an operation $C^1(G, L)^m \to C^2(G, L)$ for each function $I : \{1, \ldots, m\} \to \{1, 2\}$. This operation takes $(b_1, \ldots, b_m) \in C^1(G, L)^m$ to the 2 cochain:

$$(g_1, g_2) \mapsto [\cdots[[l_1, l_2], l_3], \ldots, l_m],$$

where $l_i = b_i(g_1)$ if $I(i) = 1$, and $l_i = g_1 b_i(g_2)$ if $I(i) = 2$. We will denote this cochain by $\beta_I(b_1, \ldots, b_m)$.

2.4.15. The operations $\beta_I : C^1(G, L)^m \to C^2(G, L)$ of 2.4.14 can be expressed in terms of more familiar operations. We list these operations and give β_I as a composition of them.

To a pair $(b_1, b_2) \in C^1(G, \otimes^{m_1} L) \times C^1(G, \otimes^{m_2} L)$, associate the cochain $g \mapsto b_1(g) \otimes b_2(g)$ in $C^1(G, \otimes^{m_1+m_2} L)$. (The justification for calling this operation 'more familiar' than β_I is that if $\otimes^{m_1} L$, $\otimes^{m_2} L$, and $\otimes^{m_1+m_2} L$ are replaced by a ring R, the boundary of the analogous cochain shows that the cup product $H^1(G, R) \otimes H^1(G, R) \to H^2(G, R)$ is antisymmetric.) Repeating this process gives a map $\Delta : C^1(G, L)^m \to C^1(G, \otimes^m L)$ which takes $(b_1, \ldots, b_m)$ to the cochain

$$g \mapsto b_1(g) \otimes b_2(g) \otimes \ldots \otimes b_m(g) \tag{27}$$

Let $i_1, i_2, \ldots, i_{m_1}$ be the list of elements of $I^{-1}(1)$ given in increasing order, and let $i_{m_1+1}, i_{m_1+2}, \ldots, i_{m_1+m_2}$ be the list of elements of $I^{-1}(2)$ given in increasing order. Let σ be the permutation of $\{1, \ldots, m\}$ given by $\sigma(i_j) = j$. In other words, σ puts the list $i_1, \ldots i_m$ back into increasing order. σ induces $\sigma_* : C^*(G, \otimes^m L) \to C^*(G, \otimes^m L)$. The m^{th} order Lie bracket $[\cdots[[-,-],-],\dots,-] : \otimes^m L \to L$ induces $[\cdots[[-,-],-],\dots,-]_* : C^*(G,\otimes^m L) \to C^*(G,L)$. The operation β_I is a composition of these operations:

(28)
$$\beta_I(b_1,\dots,b_m) = [\cdots[[-,-],-],\dots,-]_*\sigma_*(\Delta(b_{i_1},b_{i_2},\dots,b_{i_{m_1}})\cup\Delta(b_{i_{m_1+1}},b_{i_{m_1+2}},\dots,b_{i_{m_1+m_2}}))$$

2.4.16. *Definition.* Let L be a Lie-algebra with an action of a profinite group G. Let τ_1 be an element of $H^1(G,L)$. A *defining system* for the order n bracket Massey product of τ_1 is $(t_1,\dots,t_{n-1}) \in C^1(G,L)^{n-1}$ such that:

(1) t_1 is a cocycle representing τ_1
(2) $D(t_j) = -\sum_{m=2}^{j}\sum_{(I,P)} C_I\beta_I(t_{P(1)},t_{P(2)},\dots,t_{P(m)})$ for $1 < j < n$

The sum in condition (2) runs over all pairs (I,P) where I is a function $I : \{1,\dots,m\} \to \{1,2\}$ and $P : \{1,\dots,m\} \to \{1,\dots,n-1\}$ is a function such that $\sum_{j=1}^{m} P(j) = n$. (See also 2.4.9.)

2.4.17. *Definition.* The *order* n *bracket Massey product* of τ_1 with respect to the defining system $(t_1,\dots,t_{n-1})$ is an element of $C^2(G,L)$ denoted by $[\langle\tau_1;t_1,\dots,t_{n-1}\rangle]_n$ and is defined

$$[\langle\tau_1;t_1,\dots,t_{n-1}\rangle]_n = \sum_{m=2}^{n}\sum_{(I,P)} C_I\beta_I(t_{P(1)},t_{P(2)},\dots,t_{P(m)}),$$

where the indexing of the sum is as in 2.4.16.

Although we have defined $[\langle\tau_1;t_1,\dots,t_{n-1}\rangle]_n$ as a cochain, we will show in Proposition 2.4.19 that $[\langle\tau_1;t_1,\dots,t_{n-1}\rangle]_n$ is a cocycle, and we will sometimes denote the corresponding cohomology class by $[\langle\tau_1;t_1,\dots,t_{n-1}\rangle]_n$ or by $[\langle\tau_1\rangle]_n$ if the defining system is understood.

Note that condition (2) of Definition 2.4.16 is equivalent to $D(t_j) = -[\langle\tau_1;t_1,\dots,t_{j-1}\rangle]_j$ for $1 < j < n$

2.4.18. *Example.* By construction, the extensions $U_n/U_{n+1} \to U/U_{n+1} \to U/U_n$ of 2.4.6 are classified by bracket Massey products: let $U = U_1 \supset U_2 \supset U_3 \cdots$ and the cochains $a_j \log \in C^1(U/U_n, U_j/U_{j+1})$, $j < n$, be as in 2.4.6. Let $G = U/U_n$ and $L = \oplus_{j=1}^{\infty} U_j/U_{j+1}$. G acts trivially on L. By 2.4.9 we have a cocycle $-D(a_n \log) \in C^2(G, U_n/U_{n+1})$ classifying $U_n/U_{n+1} \to U/U_{n+1} \to U/U_n$. Viewing $-D(a_n \log)$ as a cocycle in $C^2(G,L)$, note that

$$-D(a_n \log) = [\langle a_1 \log; a_1 \log, a_2 \log,\dots,a_{n-1}\log\rangle]_n$$

2.4.19. *Proposition. — For* $(t_1, \dots, t_{n-1}) \in C^1(G, L)^{n-1}$ *a defining system of* $\tau_1 \in H^1(G, L)$, $[\langle \tau_1; t_1, \dots, t_{n-1} \rangle]_n \in C^2(G, L)$ *is a cocycle.*

Proof. Abbreviate the cochain $[\langle \tau_1; t_1, \dots, t_{n-1} \rangle]_n \in C^2(G, L)$ by $[\langle \tau_1 \rangle]_n$. Take $(g_1, g_2, g_3) \in G^3$. We show $D[\langle \tau_1 \rangle]_n(g_1, g_2, g_3) = 0$. Let E be the subset of L given by

$$E = \{t_1(g_1), ..., t_n(g_1), g_1 t_1(g_2), ..., g_1 t_n(g_2), g_1 g_2 t_1(g_3), ..., g_1 g_2 t_n(g_3)\} \subset L.$$

By example 2.4.18, it suffices to show that $D[\langle \tau_1 \rangle]_n(g_1, g_2, g_3)$ can be expressed as a $\mathbb{Q}$ linear combination of Lie brackets of any weight and arrangement in the elements of E.

By 2.4.15, it suffices to show that

$$D(\Delta(b_{i_1}, b_{i_2}, \dots, b_{i_{m_1}}) \cup \Delta(b_{i_{m_1+1}}, b_{i_{m_1+2}}, \dots, b_{i_{m_1+m_2}}))(g_1, g_2, g_3)$$

is a $\mathbb{Q}$ linear combination of simple tensors in Lie brackets of any weight and arrangement in the elements of E.

Let x abbreviate $\Delta(b_{i_1}, b_{i_2}, \dots, b_{i_{m_1}})$ and let y abbreviate $\Delta(b_{i_{m_1+1}}, b_{i_{m_1+2}}, \dots, b_{i_{m_1+m_2}})$.

$$D(x \cup y)(g_1, g_2, g_3) = Dx(g_1, g_2) \otimes g_1 g_2 y(g_3) - x(g_1) \otimes g_1 Dy(g_2, g_3)$$

Since $x(g_1)$ and $g_1 g_2 y(g_3)$ are simple tensors in the elements of E, it suffices to show that $Dx(g_1, g_2)$ and $g_1 Dy(g_2, g_3)$ are $\mathbb{Q}$ linear combinations of simple tensors in Lie brackets of any weight and arrangement in the elements of E. Since $(t_1, \dots, t_{n-1})$ is a defining system, $Dt_j(g_1, g_2)$ and $g_1 Dt_j(g_2, g_3)$ are $\mathbb{Q}$ linear combinations of simple tensors in Lie brackets of any weight and arrangement in the elements of E. The proposition follows from the calculation that for any $(b_1, b_2, \dots, b_{m'}) \in C^1(G, L)^{m'}$, we have

$$\begin{aligned} D(\Delta(b_1, b_2, \dots, b_{m'}))(g_1, g_2) &= \otimes_{i=1}^{m'} b_i(g_1) + \otimes_{i=1}^{m'} g_1 b_i(g_2) - \otimes_{i=1}^{m'} b_i(g_1 g_2) \\ &= \otimes_{i=1}^{m'} b_i(g_1) + \otimes_{i=1}^{m'} g_1 b_i(g_2) - \otimes_{i=1}^{m'} (b_i(g_1) + g_1 b_i(g_2) - Db_i(g_1, g_2)) \end{aligned}$$

(Here it is understood that the tensor product over the index i respects order, e.g. $\otimes_{i=1}^{k} b_i(g_1) = b_1(g_1) \otimes b_2(g_1) \otimes \ldots \otimes b_m(g_1)$.)

□

2.4.20. Obtaining defining systems.

We describe a situation in which we can form an order n bracket Massey product. Let G and ϖ be profinite groups, with G acting continuously on ϖ. Let $\varpi > \varpi_2 > \varpi_3 \ldots$ be a G invariant filtration of ϖ by normal subgroups such that $[\varpi_i, \varpi_j] \subset \varpi_{i+j}$. For instance, we could take the lower central series filtration $\varpi_n = [\varpi]_n$. Let $\tau_1 \in H^1(\varpi/\varpi_2 \rtimes G, \varpi/\varpi_2)$ be the cohomology class represented by the projection $t_1 : \varpi/\varpi_2 \rtimes G \to \varpi/\varpi_2$, and by abuse of notation, τ_1 and t_1 will also denote the pullbacks to $\varpi/\varpi_n \rtimes G$ for $n > 2$.

Form the Lie algebra $L = \oplus_{j=1}^{\infty} \varpi_j/\varpi_{j+1}$. G acts on L and therefore so does $\varpi/\varpi_n \rtimes G$ for any n, by letting ϖ/ϖ_n act trivially. We describe a situation in which we can form the order n bracket Massey product of τ_1 for the group $\varpi/\varpi_n \rtimes G$ (or $\varpi/\varpi_m \rtimes G$ for $m \geq n$) acting on L. The resulting bracket Massey product from the below procedure will be the push-forward of a cocycle in $C^2(\varpi/\varpi_n \rtimes G, \varpi_n/\varpi_{n+1})$.

(t_1) is a defining system for the order 2 bracket Massey product as above.

Suppose inductively that $(t_1, \ldots, t_{n-2})$ is a defining system for the order $n-1$ bracket Massey product on the group $\varpi/\varpi_{n-1} \rtimes G$. Thus, we can form $[\langle \tau_1; t_1, \ldots, t_{n-2}\rangle]_{n-1}$ in $H^2(\varpi/\varpi_{n-1}, L)$. Under the hypothesis that $[\langle \tau_1; t_1, \ldots, t_{n-2}\rangle]_{n-1}$ is the pushforward to L of the element of cohomology classifying

$$1 \to \varpi_{n-1}/\varpi_n \to \varpi/\varpi_n \rtimes G \to \varpi/\varpi_{n-1} \rtimes G \to 1,$$

we can form the order n bracket Massey product on $\varpi/\varpi_n \rtimes G$ as follows:

By Proposition 2.4.5, we have a ϖ_{n-1}/ϖ_n coordinate

$$t_{n-1} : \varpi/\varpi_n \rtimes G \to \varpi_{n-1}/\varpi_n$$

such that $D(t_{n-1}) = -[\langle \tau_1; t_1, \ldots, t_{n-2}\rangle]_{n-1}$. (For this equality, view $D(t_{n-1})$ as a cocycle in $C^2(\varpi/\varpi_{n-1} \rtimes G, L)$ via $\varpi_{n-1}/\varpi_n \to L$. c.f. Proposition 2.4.2.)

Pulling back $(t_1, \ldots, t_{n-2})$ to $\varpi/\varpi_n \rtimes G$, we have the defining system $(t_1, \ldots, t_{n-2}, t_{n-1})$ for the order n bracket Massey product of τ_1, making it possible to repeat this process.

2.4.21. Let G, $\varpi > \varpi_2 > \varpi_3 \ldots$, τ_1, and t_1 be as in 2.4.20. As above, suppose that for each $j < n$, we have a ϖ_j/ϖ_{j+1} coordinate t_j for

$$1 \to \varpi_j/\varpi_{j+1} \to \varpi/\varpi_{j+1} \rtimes G \to \varpi/\varpi_j \rtimes G \to 1$$

such that $D(t_j) = -[\langle \tau_1; t_1, \ldots, t_{j-1}\rangle]_j$ in $C^2(\varpi/\varpi_j \rtimes G, \varpi_j/\varpi_{j+1})$.

In particular, $(t_1, \ldots, t_{n-1})$ is a defining system for the order n bracket Massey product of τ_1, and this defining system pulls back along any group homomorphism $G \to \varpi/\varpi_n \rtimes G$ to a defining system of the pull-back of τ_1. This gives an alternate description of the map

$$\wp_\eta : H^1(G, \varpi/[\varpi]_n) \to H^2(G, [\varpi]_n/[\varpi]_{n+1})$$

for

$$\eta = [\langle \tau_1; t_1, \ldots, t_{n-1}\rangle]_n$$

defined in Notation 2.2.1: for every x in $H^1(G, \varpi/[\varpi]_n)$, we have a homomorphism $G \to \varpi/[\varpi]_n \rtimes G$ defined up to conjugation by an element of $\varpi/[\varpi]_n$. (This is also described in Notation 2.2.1.) The pullback of $(t_1, \ldots, t_{n-1})$ is a defining system for the order n bracket Massey product of x, and the resulting bracket Massey product $[\langle x\rangle]_n \in H^2(G, [\varpi]_n/[\varpi]_{n+1})$ is independent of the choice of homomorphism $G \to \varpi/[\varpi]_n \rtimes G$. Furthermore, it is immediate from the definitions that

$$\wp_\eta(x) = [\langle x\rangle]_n$$

where the defining system on the right hand side is understood to be the pull-back of $(t_1, \ldots, t_{n-1})$.

2.4.22. Let G, $\varpi > \varpi_2 > \varpi_3 \ldots$, τ_1, and t_1 be as in 2.4.20. Recall from 2.4.20 that under the hypothesis that for each $j < n$, we have a ϖ_j/ϖ_{j+1} coordinate t_j for $\varpi/\varpi_{j+1} \rtimes G$ such that $D(t_j) = -[\langle\tau_1; t_1, \ldots, t_{j-1}\rangle]_j$, we can form $[\langle\tau_1; t_1, \ldots, t_{n-1}\rangle]_n$ in $C^2(\varpi/\varpi_n \rtimes G, \varpi_n/\varpi_{n+1})$. We can repeat this process, i.e. we can find a ϖ_n/ϖ_{n+1} coordinate t_n for $\varpi/\varpi_{n+1} \rtimes G$ such that $D(t_n) = -[\langle\tau_1; t_1, \ldots, t_{n-1}\rangle]_n$ and form $[\langle\tau_1; t_1, \ldots, t_n\rangle]_{n+1}$, if

$$[\langle\tau_1\rangle]_n \in H^2(\varpi/\varpi_n \rtimes G, \varpi_n/\varpi_{n+1})$$

classifies

$$1 \to \varpi_n/\varpi_{n+1} \to \varpi/\varpi_{n+1} \rtimes G \to \varpi/\varpi_j \rtimes G \to 1 \tag{29}$$

($[\langle\tau_1\rangle]_n$ denotes the cohomology class of the cocycle $[\langle\tau_1; t_1, \ldots, t_{n-1}\rangle]_n$).

The next proposition shows that when ϖ is a finitely generated free group filtered by its lower central series, the pullback of $[\langle\tau_1\rangle]_n$ along $i : \varpi_n/\varpi_{n+1} \to \varpi_n/\varpi_{n+1} \rtimes G$ always classifies the pullback of (29).

2.4.23. *Proposition. — Suppose that ϖ is a finitely generated free group with an action of G such that the lower central series filtration $\varpi > [\varpi]_2 > [\varpi]_3 \ldots$ satisfies the hypothesis that for each $j < n$ we have a $[\varpi]_j/[\varpi]_{j+1}$ coordinate t_j for $\varpi/[\varpi]_{j+1} \rtimes G$ such that $D(t_j) = -[\langle\tau_1; t_1, \ldots, t_{j-1}\rangle]_j$ in $C^2(\varpi/[\varpi]_j \rtimes G, \varpi_j/[\varpi]_{j+1})$. Let $i : \varpi/[\varpi]_n \to \varpi/[\varpi]_n \rtimes G$ be the inclusion. Then*

$$i^*[\langle\tau_1; t_1, \ldots, t_{n-1}\rangle]_n$$

classifies

$$1 \to [\varpi]_n/[\varpi]_{n+1} \to \varpi/[\varpi]_{n+1} \to \varpi/[\varpi]_n \to 1$$

To prove Proposition 2.4.23 we will use the following lemma and corollary to the lemma:

2.4.24. *Lemma. — Let ϖ be a finitely generated free group. The pullback $H^2(\varpi/[\varpi]_n, \mathbb{Z}) \to H^2(\varpi/[\varpi]_{n+1}, \mathbb{Z})$ is the zero map for all n.*

Remark. The hypothesis that ϖ be free is necessary: if ϖ is the fundamental group of a compact surface of genus $g > 1$, $H^2(\varpi/[\varpi]_2, \mathbb{Z}) \to H^2(\varpi/[\varpi]_3, \mathbb{Z})$ is non-zero. (The kernel is the image of the boundary map $H^1([\varpi]_2/[\varpi]_3, \mathbb{Z}) \to H^2(\varpi/[\varpi]_2, \mathbb{Z})$ of the E_2 page of the Serre spectral sequence $H^i(\varpi/[\varpi]_2, H^j([\varpi]_2/[\varpi]_3, \mathbb{Z})) \Rightarrow H^2(\varpi/[\varpi]_3, \mathbb{Z})$, which can not be surjective as $H^1([\varpi]_2/[\varpi]_3, \mathbb{Z})$ is free of rank $\binom{g}{2} - 1$ and $H^2(\varpi/[\varpi]_2, \mathbb{Z})$ is free of rank $\binom{g}{2}$

Proof. By the universal coefficient theorem, $H^2(\varpi/[\varpi]_n, \mathbb{Z})$ is dual to $H_2(\varpi/[\varpi]_n, \mathbb{Z})$, so it suffices to show that $H_2(\varpi/[\varpi]_{n+1}, \mathbb{Z}) \to H_2(\varpi/[\varpi]_n, \mathbb{Z})$ is the zero map. This follows by the natural isomorphisms $H_2(\varpi/[\varpi]_n, \mathbb{Z}) \cong [\varpi]_n/[\varpi]_{n+1}$ given by Hopf's theorem (see for instance [Bro94, II Thm. 5.3]). ☐

Lemma 2.4.24 implies the following uniqueness result for bracket Massey products in free groups.

2.4.25. *Corollary. — Let ϖ be a finitely generated free group. Form the Lie algebra $\oplus[\varpi]_n/[\varpi]_{n+1}$. Suppose $\oplus[\varpi]_n/[\varpi]_{n+1} \to L$ is a morphism of Lie algebras, and view L and $\oplus[\varpi]_n/[\varpi]_{n+1}$ as equipped with the trivial action of ϖ. Suppose $(t_1, \ldots, t_{n-1})$ is a defining system for the order n bracket Massey product of $\tau_1 \in H^1(\varpi/[\varpi]_n, L)$ such that t_j is a $[\varpi]_j/[\varpi]_{j+1}$ additive L coordinate for $\varpi/[\varpi]_{j+1}$. Suppose $(s_1, \ldots, s_{n-1})$ is another such defining system. Then in $H^2(\varpi/[\varpi]_n, L)$, we have the equality*

$$[\langle \tau_1; t_1, \ldots, t_{n-1}\rangle]_n = [\langle \tau_1; s_1, \ldots, s_{n-1}\rangle]_n$$

Proof. By Lemma 2.4.24, $H^2(\varpi/[\varpi]_{n-1}, L) \to H^2(\varpi/[\varpi]_n, L)$ is the 0 map. Thus, it suffices to show that $[\langle \tau_1; t_1, \ldots, t_{n-1}\rangle]_n - [\langle \tau_1; s_1, \ldots, s_{n-1}\rangle]_n$ descends to a cochain (which is therefore automatically a cocycle) in $C^2(\varpi/[\varpi]_{n-1}, L)$.The only summands of $[\langle \tau_1; t_1, \ldots, t_{n-1}\rangle]_n$ given in 2.4.17 which do not descend to $\varpi/[\varpi]_{n-1}$ are $(x, y) \mapsto [t_1(x), t_{n-1}(y)]$ and $(x, y) \mapsto [t_{n-1}(x), t_1(y)]$. Because t_{n-1} and s_{n-1} are $[\varpi]_{n-1}/[\varpi]_n$ additive L coordinates for $\varpi/[\varpi]_n$, their difference $t_{n-1} - s_{n-1}$ descends to an element of $C^1(\varpi/[\varpi]_{n-1}, L)$. Therefore $(x, y) \mapsto [t_1(x), (t_{n-1} - s_{n-1})(y)]$ and $(x, y) \mapsto [(t_{n-1} - s_{n-1})(x), t_1(y)]$ descend to $C^2(\varpi/[\varpi]_{n-1}, L)$. □

Proof. (of Proposition 2.4.23)Let $s_j = i^* t_j$ for $j = 1, \ldots, n-1$. Notice that $s_j \in C^2(\varpi/[\varpi]_j, [\varpi]_j/[\varpi]_{j+1})$ is a $[\varpi]_j/[\varpi]_{j+1}$ coordinate for $1 \to [\varpi]_j/[\varpi]_{j+1} \to \varpi/[\varpi]_{j+1} \to \varpi/[\varpi]_j \to 1$, and that $D(s_j) = -[\langle i^*\tau_1; s_1, \ldots, s_{j-1}\rangle]_j$ (where this equality is an equality of cocycles in $C^2(\varpi/[\varpi]_j, [\varpi]_j/[\varpi]_{j+1})$). We need to show that $[\langle i^*\tau_1; s_1, \ldots, s_{n-1}\rangle]_n$ classifies $1 \to [\varpi]_n/[\varpi]_{n+1} \to \varpi/[\varpi]_{n+1} \to \varpi/[\varpi]_n \to 1$.

As in 2.3.5, let $m : \varpi \to \mathbb{Z}\langle\langle z_1, \ldots, z_r\rangle\rangle^\times$ be the Magnus embedding. Let U and U_j ($j = 1, 2, \ldots$) be as in 2.4.6 for the graded associative algebra $M = \mathbb{Z}\langle\langle z_1, \ldots, z_r\rangle\rangle$.

Let $\omega_j \in H^2(\varpi/[\varpi]_j, [\varpi]_j/[\varpi]_{j+1})$ denote the cohomology class classifying

$$1 \to [\varpi]_j/[\varpi]_{j+1} \to \varpi/[\varpi]_{j+1} \to \varpi/[\varpi]_j \to 1 \tag{30}$$

(cf. Proposition 2.3.8). Let $\nu_j \in H^2(U/U_j, U_j/U_{j+1})$ denote the cohomology class classifying

$$1 \to U_j/U_{j+1} \to U/U_{j+1} \to U/U_j \to 1 \tag{31}$$

m induces a map from the central extension (30) to the central extension (31). By 2.2.3, $m_*\omega_n = m^*\nu_n$.

Because $m : [\varpi]_n/[\varpi]_{n+1} \to U_n/U_{n+1}$ is a split injection (see equation 20), it suffices to show that $m_*\omega_n = m_*[\langle i^*\tau_1; s_1, \ldots, s_{n-1}\rangle]_n$ in $H^2(\varpi/\varpi_n, U_n/U_{n+1})$.

Let a_j and $\log$ be as in 2.4.6. By Example 2.4.18, $(a_1, a_2 \log, \ldots, a_{n-1} \log)$ is a defining system for the order n bracket Massey product of a_1, and $[\langle a_1; a_1, a_2 \log, \ldots, a_{n-1} \log\rangle]_n = \nu_n$ in $H^2(U/U_n, U_n/U_{n+1})$.

We are therefore reduced to showing that

$$m_*[\langle i^*\tau_1; s_1, \ldots, s_{n-1}\rangle]_n = m^*[\langle a_1; a_1, a_2 \log, \ldots, a_{n-1} \log\rangle]_n$$

This is equivalent to showing

$$[\langle m_* i^*\tau_1; m_* s_1, \ldots, m_* s_{n-1}\rangle]_n = [\langle m^* a_1; m^* a_1, m^* a_2 \log, \ldots, m^* a_{n-1} \log\rangle]_n$$

This last equality follows by Corollary 2.4.25 □

2.4.26. Let G, ϖ be profinite groups with G acting on ϖ. Let $\delta_n : H^1(G, \varpi/[\varpi]_n) \to H^2(G, [\varpi]_n/[\varpi]_{n+1})$ be as in 2.0.3.

Let ϵ_n denote the element of $H^2(\varpi/[\varpi]_n \rtimes G, [\varpi]_n/[\varpi]_{n+1})$ classifying

$$1 \to [\varpi]_n/[\varpi]_{n+1} \to \varpi/[\varpi]_{n+1} \rtimes G \to \varpi/[\varpi]_n \rtimes G \to 1$$

So, $\delta_n = \wp_{\epsilon_n}$ (see 2.2.2 and 2.2.1).

Suppose additionally that ϖ is a finitely generated free group. Combining Proposition 2.2.5 and Proposition 2.4.23 gives the following recursive understanding of the structure of δ_n for small n (or "until the linear term comes from a non vanishing cohomology class"): $\delta_2(x) = [\langle x\rangle]_2 + L_2(x)$, where L_2 is linear. L_2 is the linear map associated to $\ell_2 \in H^2(\varpi/[\varpi]_2 \rtimes G, [\varpi]_2/[\varpi]_3)$. If ℓ_2 is 0, then $L_2 = 0$ and $\delta_3(x) = [\langle x\rangle]_3 + L_3(x)$, where L_3 is linear in the sense of Proposition 2.2.5. L_3 is the linear map associated to $\ell_3 \in H^2(\varpi/[\varpi]_3 \rtimes G, [\varpi]_3/[\varpi]_4)$. If ℓ_3 is 0, then $L_3 = 0$ and $\delta_4(x) = [\langle x\rangle]_4 + L_4(x)$, where L_4 is linear in the sense of Proposition 2.2.5 etc. ($[\langle x\rangle]_n$ is as in 2.4.21. The map associated to a cohomology class is given in 2.2.1. For δ_2, see also Proposition 2.1.4.) Formally:

2.4.27. *Definition.* Let G, $\varpi > \varpi_2 > \varpi_3 > \ldots$, τ_1, and t_1 be as in 2.4.20. Consider ϖ as a filtered group with a G action. Say that ϖ has *property* $\mathcal{P}_n$ if there is a defining system $(t_1, \ldots, t_{n-1})$ for the order n bracket Massey product of τ_1, satisfying the hypothesis that t_j is a ϖ_j/ϖ_{j+1} coordinate for $\varpi/[\varpi]_{j+1} \rtimes G$. (The order n bracket Massey product of τ_1 is for the group $\varpi/\varpi_n \rtimes G$ acting on the Lie algebra $\oplus\varpi_j/\varpi_{j+1}$.)

2.4.28. *Theorem. — Suppose G is a profinite group acting continuously on ϖ, where ϖ is a finitely generated free group or the profinite completion of such a group. If ϖ filtered by its lower central series has property $\mathcal{P}_n$, then*

- $\delta_m(x) = [\langle x\rangle]_m$ *for $m < n$, where $[\langle x\rangle]_m$ denotes the order m bracket Massey product as in 2.4.21.*
- $\delta_n(x) = [\langle x\rangle]_n + L_n(x)$, *where $L_n : H^1(G, \varpi/[\varpi]_n) \to H^2(G, [\varpi]_n/[\varpi]_{n+1})$ is linear in the sense of Proposition 2.2.5.*

- $L_n = \wp_{\ell_n}$, *where* $\ell_n \in H^2(\varpi/[\varpi]_n \rtimes G, [\varpi]_n/[\varpi]_{n+1})$ *is given by* $\ell_n = \epsilon_n - [\langle \tau_1 \rangle]_n$ *with* $[\langle \tau_1 \rangle]_n$ *as in 2.4.17 and 2.4.27,* ϵ_n *as in 2.4.26, and* $\wp_{\ell_n}$ *as in 2.2.1.*

Furthermore, if ℓ_n *is* 0*, then* ϖ *has property* $\mathcal{P}_{n+1}$.

Proof. Since ϖ has property $\mathcal{P}_n$, we have for $m < n$, a ϖ_m/ϖ_{m+1} coordinate t_m for $\varpi/[\varpi]_{m+1} \rtimes G$ such that $-D(t_m) = [\langle \tau_1; t_1, \ldots, t_{m-1} \rangle]_m$ in $C^2(\varpi/[\varpi]_m \rtimes G, \varpi_m/[\varpi]_{m+1})$, where τ_1 and t_1 are as in 2.4.20 for the lower central series filtration of ϖ. By Proposition 2.4.4, it follows that for $m < n$

$$1 \to [\varpi]_m/[\varpi]_{m+1} \to \varpi/[\varpi]_{m+1} \rtimes G \to \varpi/[\varpi]_m \rtimes G \to 1$$

is classified by the cohomology class $[\langle \tau_1 \rangle]_m$ of the cocycle $[\langle \tau_1; t_1, \ldots, t_{m-1} \rangle]_m$. By 2.2.2, we therefore have $\delta_m = \wp_\eta$ for $\eta = [\langle \tau_1 \rangle]_m$ for $m < n$. Thus $\delta_m(x) = [\langle x \rangle]_m$, as in 2.4.21 for $m < n$.

Since ϖ has property $\mathcal{P}_n$, we can form the cocycle $[\langle \tau_1; t_1, \ldots, t_{n-1} \rangle]_n$ and its corresponding cohomology class $[\langle \tau_1 \rangle]_n \in H^2(\varpi/[\varpi]_n \rtimes G, [\varpi]_n/[\varpi]_{n+1})$. By 2.2.2 and 2.4.21, $\delta_n(x) - [\langle x \rangle]_n = \wp_{\ell_n}(x)$ (for any $x \in H^1(G, \varpi/[\varpi]_n)$). Let $i : \varpi/[\varpi]_n \to \varpi/[\varpi]_n \rtimes G$ denote the inclusion. By Proposition 2.4.23, $i^*[\langle \tau_1 \rangle]_n$ classifies

$$1 \to [\varpi]_n/[\varpi]_{n+1} \to \varpi/[\varpi]_{n+1} \to \varpi/[\varpi]_n \to 1 \tag{32}$$

By 2.2.3, $i^*\epsilon_n$ classifies (32). Since both $i^*[\langle \tau_1 \rangle]_n$ and $i^*\epsilon_n$ classify (32), $i^*\ell_n = 0$. By 2.2.5, $\wp_{\ell_n}$ is linear in the sense of Proposition 2.2.5, i.e. $\wp_{\ell_n}(x_1 + x_2) = \wp_{\ell_n}(x_1) + \wp_{\ell_n}(x_2)$ for any x_1, x_2 in $H^1(G, \varpi/[\varpi]_n)$ such that we have $x_1 + x_2$ in $H^1(G, \varpi/[\varpi]_n)$. So, we may define $L_n = \wp_{\ell_n}$, and we have $\delta_n(x) = [\langle x \rangle]_n + L_n(x)$ with L_n linear in the sense of Proposition 2.2.5.

Now suppose $\ell = 0$. Then $[\langle \tau_1 \rangle]_n$ classifies

$$1 \to [\varpi]_n/[\varpi]_{n+1} \to \varpi/[\varpi]_{n+1} \rtimes G \to \varpi/[\varpi]_n \rtimes G \to 1$$

By 2.4.20, ϖ has property $\mathcal{P}_{n+1}$. □

2.4.29. Theorem 2.4.28 applies to the obstructions δ_n for any non-proper, smooth, geometrically irreducible, algebraic curve X over a subfield k of $\mathbb{C}$, as in 1.1.3. For instance, π_1 filtered by its lower central series and equipped with its action of G_k (with π_1 as in 1.1.3), has property $\mathcal{P}_2$ and we recursively apply Theorem 2.4.28 as above to describe δ_n for small n. (We may only get a description of δ_2.)

2.4.30. By 2.4.20 and 2.4.17, $[\langle \tau_1 \rangle]_n$ is independent of the lift of the action of G on $\varpi/[\varpi]_n$ to the action of G on $\varpi/[\varpi]_{n+1}$. On the other hand, ϵ_n depends on this lift. The bracket Massey term in the decomposition of δ_n of Theorem 2.4.28 therefore requires less understanding of the G action than is required to compute δ_n. It is only the term subject to linearity conditions that depends on G's action on all of $\varpi/[\varpi]_{n+1}$.

2.4.31. Dependence of the linear term of δ_2 on the base point.

Let b_1, b_2 be rational base points or tangential base points of $X \otimes \overline{k}$, and let π_{b_1},π_{b_1} denote the etale fundamental group of $X \otimes \overline{k}$ based at b_1, b_2 respectively.

π_{b_1} and π_{b_2} are isomorphic as groups, but not (necessarily) as G groups.

Let $\wp$ be a path from b_1 to b_2. $\wp$ determines a group isomorphism $i_\wp : \pi_{b_1} \to \pi_{b_2}$, given by $\gamma \mapsto \wp\gamma\wp^{-1}$. The failure of $i_\wp$ to be G equivariant is measured by the cocycle $g \mapsto \wp^{-1}(g\wp)$ in $C^1(G, \pi_{b_2})$ corresponding to $(b_1, \wp)$. (This correspondence refers to the second map from rational points to homotopy sections given in 1.1.3.)

$$gi_\wp(\gamma) = i_\wp((\wp^{-1}(g\wp))(g\gamma)(\wp^{-1}(g\wp))^{-1})$$

Note that $i_\wp$ induces a G equivariant isomorphism $\pi_{b_2}/[\pi_{b_2}]_2 \to \pi_{b_1}/[\pi_{b_1}]_2$.

Furthermore, $i_\wp$ induces a G equivariant isomorphism $[\pi_{b_2}]_2/[\pi_{b_2}]_3 \to [\pi_{b_1}]_2/[\pi_{b_1}]_3$, because for any $\gamma \in [\pi_{b_2}]_2$, we have $(\wp^{-1}(g\wp))(g\gamma)(\wp^{-1}(g\wp))^{-1}(g\gamma)^{-1} \in [\pi_{b_2}]_3$.

We therefore denote both $\pi_{b_1}/[\pi_{b_1}]_2$ and $\pi_{b_2}/[\pi_{b_2}]_2$ by $\pi/[\pi]_2$. Similarly, $[\pi]_2/[\pi]_3$ denotes both $[\pi_{b_1}]_2/[\pi_{b_1}]_3$ and $[\pi_{b_2}]_2/[\pi_{b_2}]_3$.

Let δ_{2,b_1} and δ_{2,b_2} denote the obstructions $H^1(G_k, \pi/[\pi]_2) \to H^2(G_k, [\pi]_2/[\pi]_3)$ corresponding to b_1 and b_2 respectively. Both δ_{2,b_1} and δ_{2,b_2} are the sum of the bracket cup product $H^1(G_k, \pi/[\pi]_2) \to H^2(G_k, [\pi]_2/[\pi]_3)$ and a linear term. (The bracket cup product is given in 2.1.2 or 2.4.11. It is the same map for either base point because $i_\wp$ identifies the two maps $\pi_{b_i}/[\pi_{b_i}]_2 \otimes \pi_{b_i}/[\pi_{b_i}]_2 \to [\pi_{b_i}]_2/[\pi_{b_i}]_3$ for $i = 1, 2$, in the sense that the obvious diagram commutes.) Denote the linear term of δ_{2,b_1} by L_{b_1}, and denote the linear term of δ_{2,b_2} by L_{b_2}.

$L_{b_1} - L_{b_2}$ is the obstruction $H^1(G_k, \pi/[\pi]_2) \to H^2(G_k, [\pi]_2/[\pi]_3)$ corresponding to $\epsilon_1 - \epsilon_2$, where $\epsilon_i \in H^2(\pi/[\pi]_2 \rtimes G_k, [\pi]_2/[\pi]_3)$ is the element of cohomology classifying

$$1 \to [\pi]_2/[\pi]_3 \to \pi_{b_i}/[\pi_{b_i}]_3 \rtimes G_k \to \pi/[\pi]_2 \rtimes G_k$$

for $i = 1, 2$.

Let a_2 be a $[\pi]_2/[\pi]_3$ additive $[\pi]_2/[\pi]_3$ coordinate for

$$[\pi]_2/[\pi]_3 \to \pi_{b_1}/[\pi_{b_1}]_3 \to \pi/[\pi]_2$$

Then $a_2 i_\wp$ is a $[\pi]_2/[\pi]_3$ additive $[\pi]_2/[\pi]_3$ coordinate for

$$[\pi]_2/[\pi]_3 \to \pi_{b_2}/[\pi_{b_2}]_3 \to \pi/[\pi]_2$$

It follows that the map $\pi_{b_1}/[\pi_{b_1}]_3 \rtimes G_k \to [\pi]_2/[\pi]_3$, also denoted a_2, defined by

$$\gamma \rtimes g \mapsto a_2(\gamma)$$

is a $[\pi]_2/[\pi]_3$ coordinate for

$$[\pi]_2/[\pi]_3 \to \pi_{b_1}/[\pi_{b_2}]_3 \rtimes G_k \to \pi/[\pi]_2 \rtimes G_k$$

In the same way, $a_2 i_\wp$ is a $[\pi]_2/[\pi]_3$ coordinate for

$$[\pi]_2/[\pi]_3 \to \pi_{b_2}/[\pi_{b_2}]_3 \rtimes G_k \to \pi/[\pi]_2 \rtimes G_k$$

By Proposition 2.4.4, $\epsilon_1 - \epsilon_2 = D(a_2 i_\wp) - D(a_2)$.

$(D(a_2) - D(a_2 i_\wp))(\gamma_1 \rtimes g_1, \gamma_2 \rtimes g_2) = a_2(\gamma_1 \rtimes g_1 \cdot_{\pi_{b_1}} \gamma_2 \rtimes g_2) - a_2 i_\wp(\gamma_1 \rtimes g_1 \cdot_{\pi_{b_2}} \gamma_2 \rtimes g_2)$

$i_\wp(\gamma_1 \rtimes g_1 \cdot_{\pi_{b_2}} \gamma_2 \rtimes g_2) = i_\wp(\gamma_1 g_1 \gamma_2) \rtimes g_1 g_2 = \gamma_1 i_\wp(g_1 \gamma_2) \rtimes g_1 g_2 = \gamma_1(\wp^{-1}(g_1 \wp))^{-1} g_1 \gamma_2 (\wp^{-1}(g_1 \wp)) \rtimes g_1 g_2$.

Thus $(D(a_2) - D(a_2 i_\wp))(\gamma_1 \rtimes g_1, \gamma_2 \rtimes g_2) = a_2(\gamma_1 g_1 \gamma_2) - a_2(\gamma_1(\wp^{-1}(g_1\wp))^{-1} g_1 \gamma_2 (\wp^{-1}(g_1 \wp)))$.

$\gamma_1(\wp^{-1}(g_1\wp))^{-1} g_1 \gamma_2 (\wp^{-1}(g_1 \wp)) = \gamma_1 g_1 \gamma_2 [-g_1\gamma_2, -\wp^{-1}(g_1\wp)] = [g_1 \gamma_2, \wp^{-1}(g_1 \wp)] \gamma_1 g_1 \gamma_2$

Thus $(D(a_2) - D(a_2 i_\wp))(\gamma_1 \rtimes g_1, \gamma_2 \rtimes g_2) = [\wp^{-1}(g_1\wp), g_1\gamma_2]$. This is the bracket cup product of the point b_1 with the 'identity' i.e. this is the bracket cup product of the cocycle corresponding to $(b_1, \wp)$ with the twisted homomorphism $\pi/[\pi]_2 \rtimes G_k \to \pi/[\pi]_2$ given by $\gamma \rtimes g \mapsto \gamma$.

Thus, $(L_1 - L_2)(x) = b_1 \cup x$, where the cup product here denotes the bracket cup product.

3. Examples and computing δ_n

3.1. δ_n **over** $\mathbb{R}$**.** For geometrically connected, finite type $\mathbb{R}$ schemes, the $G_\mathbb{R}$ action on π_1^{et} comes from the $G_\mathbb{R}$ action on π_1^{top} via the canonical isomorphism $(\pi_1^{top})^\wedge \cong \pi_1^{et}$ [SGAI, Exp. XII Cor. 5.2]. To study δ_n over $\mathbb{R}$, we first establish results allowing us to replace π_1^{et} by π_1^{top}. (See Proposition 3.1.19.) The analogue of the injection $\operatorname{Jac} X(k) \to H^1(G_k, \pi_1/[\pi_1]_2)$ for number fields (discussed in 1.1.3) does not hold over $\mathbb{R}$. Instead homotopy sections can only record connected components of real points (3.1.2). In particular, Ellenberg's δ_n are obstructions to connected components of real points of the Jacobian coming from the curve itself. In 3.2 , we show that for proper, smooth, geometrically connected curves over $\mathbb{R}$, the δ_n succeed in determining the connected components of $\mathbb{R}$-points of the curve, and in fact δ_2 itself is sufficient for this (Proposition 3.2.1).

For the above, we use a 'section conjecture for π_1^{top},' under certain hypotheses (Corollaries 3.1.8 and 3.1.9). This 'section conjecture' is an immediate corollary of a theorem of Gunnar Carlsson (reproduced below as Theorem 3.1.4). It implies the usual section conjecture over $\mathbb{R}$ for schemes such that $H^1(G_\mathbb{R}, \pi_1^{top}) \to H^1(G_\mathbb{R}, (\pi_1^{top})^\wedge)$ is an isomorphism. To establish Proposition 3.1.19, we show $H^1(G_\mathbb{R}, \pi_1^{top}) \to H^1(G_\mathbb{R}, (\pi_1^{top})^\wedge)$ is an isomorphism for smooth proper (geometrically connected) curves and abelian varieties with an $\mathbb{R}$-point (Propositions 3.1.14 and 3.1.17), thereby also giving another proof of the usual section conjecture for these schemes (Remark 3.1.20). See [Pál] for a nice discussion and a topological proof of the section conjecture over $\mathbb{R}$ in full generality.

3.1.1. *Homotopy sections over* $\mathbb{R}$.

For $k = \mathbb{R}$, the map from sections of $X \to \operatorname{Spec} k$ to homotopy sections (1.1.3) passes through $\pi_0(X(\mathbb{R}))$. (In particular, for curves and their Jacobians with $X(\mathbb{R}) \neq \emptyset$, $X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$ is not injective.)

3.1.2. *Proposition. — Let* $X \to \operatorname{Spec} \mathbb{R}$ *be finite type and geometrically connected, and suppose that* $X(\mathbb{R}) \neq \emptyset$. *Then the map* $X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$ *defined in 1.1.3 admits a factorization*

$$X(\mathbb{R}) \to \pi_0(X(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$$

(where $\pi_0(X(\mathbb{R}))$ *denotes the connected components of the real points of the complex analytic space corresponding to* $X_{\mathbb{C}}$*).*

Proof. Let b denote the chosen basepoint. Let $x_1, x_2 \in X(\mathbb{R})$ be two points with the same image in $\pi_0(X(\mathbb{R}))$. Let $X(\mathbb{C})$ denote the complex analytic space corresponding to $X_{\mathbb{C}}$, by a slight abuse of notation. Choose a path $\gamma_1 : [0,1] \to X(\mathbb{C})$ from b to x_1, and a path $\eta : [0,1] \to X(\mathbb{C})^{G_{\mathbb{R}}}$ from x_1 to x_2 whose image lies entirely inside the real points of $X(\mathbb{C})$. In particular, $\gamma_2 = \gamma_1\eta$ is a path from b to x_2. (Composition in $\pi_1^{top}(X(\mathbb{C}))$ is denoted left to right here.) Let $\chi_i \in C^1(G_{\mathbb{R}}, \pi_1^{top}(X(\mathbb{C})))$ be the cocycle $g \mapsto \gamma_i g(\gamma_i^{-1})$ for $i = 1, 2$. Let $\iota : H^1(G_{\mathbb{R}}, \pi_1^{top}(X(\mathbb{C}))) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$ be the map induced by the canonical isomorphism $\pi_1^{top}(X(\mathbb{C}))^{\wedge} \cong \pi_1^{et}(X_{\mathbb{C}})$. x_1 and x_2 are sent to $\iota(\chi_1)$ and $\iota(\chi_2)$ respectively in $H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$. Since $g(\eta) = \eta$,

$$\chi_2(g) = \gamma_1\eta g(\gamma_1\eta)^{-1} = \gamma_1\eta g(\eta)^{-1} g(\gamma_1^{-1}) = \chi_1(g),$$

proving the proposition. □

3.1.3. 'Section conjecture for π_1^{top}' over $\mathbb{R}$.

Let G be a finite group. By a G *CW complex* we mean a CW complex X with an action of G by cellular maps such that for each $g \in G$, $\{x \in X | g(x) = x\}$ is a subcomplex of X (see [Bre67]). By a *finite* G *complex* we mean a finite dimensional G CW complex X with only finitely many cells in each dimension. For a topological space T, let $T_p^{\wedge}$ denote the Bousfield-Kan mod p completion.

3.1.4. *Theorem [Carlsson]. — Let* G *be a finite* p*-group. Let* X *be a finite based* G *CW complex, and let*

$$X^G = \coprod_{\alpha \in \pi_0(X^G)} (X^G)_\alpha$$

be the decomposition of X^G *into its connected components. Let* K_α *be the kernel of*

$$\pi_1((X^G)_\alpha) \to \pi_1(X)$$

and let $\widetilde{(X^G)}_\alpha \to (X^G)_\alpha$ *be the connected covering space corresponding to* K_α. *Let* $L_\alpha = \pi_1((X^G)_\alpha)/K_\alpha$. *Let* $\overline{X^G_\alpha} = EL_\alpha \times_{L_\alpha} (\widetilde{(X^G)}_\alpha)_p^{\wedge}$.

There is a natural map $F(EG, X)^G \to \coprod_{\alpha \in \pi_0(X^G)} \overline{X^G_\alpha}$ *which induces an isomorphism on mod-*p *homology.*

This is Theorem B(a) of [Car91].

A *weak* G*-homotopy equivalence* is a G equivariant map $f : X \to Y$ such that for any subgroup H of G, the induced map on fixed points $f^H : X^H \to Y^H$ is a weak equivalence.

3.1.5. *Corollary. — Let G be a finite p-group and let X be weakly G homotopy equivalent to a finite based G CW complex. Then there is a natural bijection $\pi_0(F(EG, X)^G) \cong \pi_0(X^G)$.*

Proof. The spaces $\overline{X^G_\alpha}$ are connected. □

When X is a $K(\pi, 1)$, there is a natural bijection $\pi_0(F(EG, X)^G) = H^1(G, \pi)$ by a standard representability result for group cohomology with twisted coefficients. For the convenience of the reader, we include a proof.

3.1.6. *Proposition. — Let G be a group and let T be a topological space with a G action such that $\pi_i(T) = 0$ for $i \neq 1$ and $\pi_1(T) \cong \varpi$, i.e. T is a $K(\varpi, 1)$. Assume there is a G invariant simply connected set B (e.g. B could be a fixed point), and fix an isomorphism $\pi_1(T, B) \cong \varpi$, so ϖ inherits a G action from the G action on $\pi_1(T, B)$. Then, there is a canonical isomorphism of pointed sets*

$$\pi_0(F(EG, T)^G) \cong H^1(G, \varpi)$$

Proof. Composition in $\pi_1(T, B)$ is written left to right, so $\gamma_1\gamma_2$ is the loop obtained by 'first following the loop γ_1 and then following the loop γ_2. Choose a base point p of EG. To $f \in F(EG, T)^G$ and a choice of path γ from B to $f(p)$, we associate a cochain $\sigma \in C^1(G, \varpi)$, defined as follows: let γ_g be a path in EG from p to gp. Define

$$\sigma(g) = \gamma f_*(\gamma_g) g(\gamma^{-1}) \in \pi_1(T, B) = \varpi$$

Since EG is contractible, $\sigma(g)$ does not depend on the choice of γ_g. Choosing $\gamma_{g_1 g_2}$ to be $\gamma_{g_1} g_1(\gamma_{g_2})$ shows that σ is a cocycle. Choosing a different path from B to $f(p)$ produces a cocycle of the form

$$g \mapsto \eta^{-1}\sigma(g)g\eta$$

for $\eta \in \pi_1(T, B)$ and therefore does not change the cohomology class of σ. Choosing a different base point p of EG does not change the cohomology class of σ either. We therefore have a map $F(EG, T)^G \to H^1(G, \varpi)$. It is not hard to see that this map descends to a map $\pi_0(F(EG, T)^G) \to H^1(G, \varpi)$.

It suffices to show that this map is an isomorphism for the model of EG which is the classifying space of the category whose objects are G and which has a unique morphism between any two objects. We show surjectivity. Choose $b \in B$. Let σ be a cocycle in $C^1(G, \varpi)$. The 0-skeleton, EG^0, of EG is in G equivariant bijection with G equipped with the G action of left translation. Map EG^0 to T by sending g to gb. The 1-skeleton, EG^1, of EG is in G equivariant bijection with $G \times G$ equipped with the G action of left translation on each factor of G. Choose a path representing $\sigma(g)$ starting at b and ending at gb. Map the 1-simplex $1 \times g$ in EG^1 to this path. This determines a G-equivariant map $EG^1 \to T$. Since σ is a cocycle, we can extend this map to a G-equivariant map $EG^2 \to T$. Since the higher homotopy groups of T are trivial. We can extend this map to a G-equivariant map $EG \to T$, showing surjectivity. Injectivity is shown similarly, by constructing a G equivariant map $EG \times [0, 1] \to T$ whose restriction to $EG \times \{0\}$ and $EG \times \{1\}$ are two given maps determining the same element of $H^1(G, \varpi)$. (Here, $[0, 1]$ has the trivial G-action.) □

3.1.7. *Corollary. — Let* G *be a finite* p*-group and let* X *be a* $K(\pi_1(X), 1)$ *such that* X *is weakly* G*-homotopy equivalent to a finite based* G *CW complex. Then there is a natural bijection* $\pi_0(X^G) \cong H^1(G, \pi_1(X))$.

This follows immediately from Corollary 3.1.5 and Proposition 3.1.6.

For a smooth, geometrically connected curve over $\mathbb{R}$, it is not hard to see that the associated complex analytic space is $G_{\mathbb{R}}$-homotopy equivalent to a finite $G_{\mathbb{R}}$ CW complex. By Corollary 3.1.7, we have:

3.1.8. *Corollary. — Let* $X \to \operatorname{Spec} \mathbb{R}$ *be a geometrically connected, smooth curve such that the associated complex analytic space has Euler characteristic* ≤ 0*. Assume that* $X(\mathbb{R}) \neq \emptyset$*. Then there is a natural bijection* $\pi_0(X(\mathbb{R})) \cong H^1(G_{\mathbb{R}}, \pi_1^{top}(X(\mathbb{C})))$

It is tautological that the natural map $X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \pi_1^{top}(X(\mathbb{C})))$ from rational points to homotopy sections described in 1.1.3 induces the bijection of Corollary 3.1.8.

A smooth compact $\mathbb{Z}/2$ manifold is triangulable as a finite $\mathbb{Z}/2$-CW complex by a theorem of Illman [LMSM86, I.1 Thm 1.2]. By Corollary 3.1.7, we have:

3.1.9. *Corollary. — Let* $X \to \operatorname{Spec} \mathbb{R}$ *be a geometrically connected, smooth, and proper scheme over* $\mathbb{R}$ *such that the associated complex analytic space is a* $K(\pi, 1)$*. Assume that* $X(\mathbb{R}) \neq \emptyset$*. (In particular,* X *could be an abelian variety over* $\mathbb{R}$ *such that* $X(\mathbb{R}) \neq \emptyset$*.) Then there is a natural bijection* $\pi_0(X(\mathbb{R})) \cong H^1(G_{\mathbb{R}}, \pi_1^{top}(X(\mathbb{C})))$.

This bijection is again induced by the map $X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \pi_1^{top}(X(\mathbb{C})))$ described in 1.1.3.

Corollaries 3.1.8 and 3.1.9 can be viewed as 'section conjectures for π_1^{top},' as in the heading of this subsection. We now explain what is meant by the name 'section conjecture for π_1^{top},' but this explanation is not necessary to understand δ_n over $\mathbb{R}$ (as in Proposition 3.1.19 and Proposition 3.2.1).

For a smooth, proper, geometrically connected curve X over a number field k, Grothendieck's *Section Conjecture* is that the map from sections of $X \to \operatorname{Spec} k$ to homotopy sections is a bijection, in the terminology of 1.1.3. When $X(k) \neq \emptyset$, this is equivalent to the natural map $X(k) \to H^1(G_k, \pi_1^{et}(X_{\overline{k}}))$ of 1.1.3 being a bijection.

As the map $X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$ descends to a map $\pi_0(X(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$ (3.1.2), it is this later map that the section conjecture over $\mathbb{R}$ says is a bijection. Explicitly, the *section conjecture over* $\mathbb{R}$ for a smooth, geometrically connected curve X such that

$X(\mathbb{R}) \neq \emptyset$ is that the natural map $\pi_0(X(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi_1^{et}(X_{\mathbb{C}}))$ is a bijection. This map factors through the map $\pi_0(X(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi_1^{top}(X_{\mathbb{C}}))$ By the *section conjecture for* π_1^{top}, we mean that the map $\pi_0(X(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi_1^{top}(X_{\mathbb{C}}))$ is a bijection. Thus Corollaries 3.1.8 and 3.1.9 are section conjectures for π_1^{top}.

The section conjecture over $\mathbb{R}$ has been proven by Mochizuki, Stix, and Pál (Pál also mentions relevant work of Cox and Scheiderer); see [Pál] for more information.

3.1.10. δ_n^{top} determines δ_n^{et}.

Let $G = \langle \tau | \tau^2 = 1 \rangle \cong \mathbb{Z}/2$ and let ϖ be a group with a G action. The G action on ϖ extends uniquely to a continuous G action on $\varpi^\wedge$ by the universal property of profinite completion.

Let I denote the set of normal finite index subgroups N of ϖ which are stable under G (i.e. $\tau N \subset N$). For each $N \in I$, the G action on ϖ determines a G action on ϖ/N. I is cofinal in the set of all normal finite index subgroups (because for a normal finite index subgroup N, $N \cap \tau N$ is in I). Thus we have a canonical G isomorphism $\varpi^\wedge = \varprojlim_{N \in I} \varpi/N$.

A group ϖ is *residually finite* if for any $x \in \varpi$ such that $x \neq 1$, there exists a normal subgroup N of finite index such that $x \notin N$.

3.1.11. *Lemma. — Let* G *and* ϖ *be as above. Suppose that* ϖ *is residually finite, and that* $H^1(G, \varpi)$ *is finite. The projection maps* $\varpi \to \varpi/N$ *for* $N \in I$ *induce a surjection of pointed sets*

$$H^1(G, \varpi) \to \varprojlim_{N \in I} H^1(G, \varpi/N)$$

Proof. Take $N \in I$. Any element of $H^1(G, \varpi/N)$ is represented by $x \in \varpi$ such that $\tau x x \in N$. If $\tau x x = 1$, then [x] is in the image of the map $H^1(G, \varpi) \to H^1(G, \varpi/N)$. Otherwise, we can find $M \in I$ such that $\tau x x \notin M$, since ϖ is residually finite. Replacing M by $M \cap N$ allows us to assume that $M \subset N$. For any $y \in \varpi$, $\tau y x y^{-1}$ does not determine a cocycle in $C^1(G, \varpi/M)$ because

$$\tau(\tau y x y^{-1})(\tau y x y^{-1}) = y \tau x x y^{-1}$$

is not in M. Thus [x] is not in the image of the map $H^1(G, \varpi/M) \to H^1(G, \varpi/N)$. Thus $H^1(G, \varpi)$ surjects onto the image of the projection $\varprojlim_{N \in I} H^1(G, \varpi/N) \to H^1(G, \varpi/N)$. Let α be an element of $\varprojlim_{N \in I} H^1(G, \varpi/N)$, and α_N the image of α in $H^1(G, \varpi/N)$. Let O_N be the subset of $H^1(G, \varpi)$ of elements mapping to α_N. Note that $O_N \cap O_M \supset O_{N \cap M}$ for $N, M \in I$, and that by the above, $O_N \neq \emptyset$. Since $H^1(G, \varpi)$ is finite, it follows that $\cap_{N \in I} O_N \neq \emptyset$. This shows surjectivity. □

3.1.12. *Lemma. — Let* G *and* ϖ *be as above. Suppose that* ϖ *is residually finite. The map*

$$H^1(G, \varpi^\wedge) \to \varprojlim_{N \in I} H^1(G, \varpi/N)$$

induced by the projection maps $\varprojlim_{N\in I}\varpi/N \to \varpi/N$ *is an isomorphism of pointed sets.*

Proof. Let $x, y \in \varpi^\wedge$ represent cohomology classes $[x], [y] \in H^1(G, \varpi^\wedge)$ with the same image in $\varprojlim_{N\in I} H^1(G, \varpi/N)$. Thus, for each $N \in I$, there is $z \in \varpi^\wedge$ such that $x = (\tau z)yz^{-1} \mod N$. Equivalently, $zy^{-1}(\tau z)^{-1}x \in N$. For each N, let $O_N = \{z \in \varpi^\wedge | zy^{-1}(\tau z)^{-1}x \in N\}$. By the previous, $O_N \neq \emptyset$. Note that $O_N \cap O_M \supset O_{N\cap M}$. Thus $\{O_N | N \in I\}$ has the finite intersection property. Since $\varpi^\wedge$ is compact, $\cap_{N\in I} O_N \neq \emptyset$. Thus there exists $z \in \varpi^\wedge$ such that $zy^{-1}(\tau z)^{-1}x \in \cap_{N\in I} N$. Since ϖ is residually finite, $\cap_{N\in I} N = 1$. Thus $[x] = [y]$ in $H^1(G, \varpi^\wedge)$, showing injectivity.

As in the proof of Lemma 3.1.11, $H^1(G, \varpi^\wedge)$ surjects onto the image of the projection $\varprojlim_{N\in I} H^1(G, \varpi/N) \to H^1(G, \varpi/N)$. Let α be an element of $\varprojlim_{N\in I} H^1(G, \varpi/N)$, and α_N the image of α in $H^1(G, \varpi/N)$. Let O_N be the subset of $\varpi^\wedge$ of elements mapping to α_N. Note that $O_N \cap O_M \supset O_{N\cap M}$ for $N, M \in I$, and that by the above, $O_N \neq \emptyset$. Since $\varpi^\wedge$ is compact, it follows that $\cap_{N\in I} O_N \neq \emptyset$. This shows surjectivity.

□

3.1.13. *Lemma. — Let* G *and* ϖ *be as above. Suppose that* ϖ *is a finitely generated abelian group. Then for all* $i \geq 1$ *the map* G *map* $\varpi \to \varpi^\wedge$ *induces an isomorphism of finite abelian groups*

$$H^i(G, \varpi) \cong H^i(G, \varpi^\wedge)$$

Proof. The periodic resolution

$$\cdots \mathbb{Z}G_{\mathbb{R}} \xrightarrow{\tau-1} \mathbb{Z}G_{\mathbb{R}} \xrightarrow{\tau+1} \mathbb{Z}G_{\mathbb{R}} \xrightarrow{\tau-1} \mathbb{Z}G_{\mathbb{R}} \longrightarrow \mathbb{Z}$$

shows that the cohomology groups $H^i(G, \varpi)$ and $H^i(G, \varpi^\wedge)$ are the cohomology groups of the lower and upper complexes of the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \varpi' & \xleftarrow[\tau-1]{} & \varpi' & \xleftarrow[\tau+1]{} & \varpi' & \xleftarrow[\tau-1]{} & \varpi' & \longleftarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots & \varpi & \xleftarrow[\tau-1]{} & \varpi & \xleftarrow[\tau+1]{} & \varpi & \xleftarrow[\tau-1]{} & \varpi & \longleftarrow 0 \end{array}$$

Since ϖ is a finitely generated abelian group, the map $\varpi \to \varpi^\wedge$ is $\varpi \to \varpi \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong \varpi^\wedge$. Since $\hat{\mathbb{Z}}$ is a flat $\mathbb{Z}$ module, $H^i(G, \varpi) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \to H^i(G, \varpi^\wedge)$ is an isomorphism.

$H^i(G, \varpi)$ is a finitely generated abelian group, because ϖ is finitely generated. Additionally, $H^i(G, \varpi)$ is 2-torsion for $i \geq 1$. Thus $H^i(G, \varpi)$ is finite, whence $H^i(G, \varpi) = H^i(G, \varpi) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, proving the lemma. □

Applying Lemma 3.1.13 to the topological fundamental group of an abelian variety over $\mathbb{R}$ implies:

3.1.14. *Proposition. — Let* X *be an abelian variety over* $\mathbb{R}$ *such that* $X(\mathbb{R}) \neq \emptyset$. *Let* π_1^{top} *denote the fundamental group of the associated complex analytic space to* X *based at a real point. Then the natural map* $\pi_1^{\mathrm{top}} \to (\pi_1^{\mathrm{top}})^\wedge$ *induces an isomorphism*

$$H^1(G_\mathbb{R}, \pi_1^{\mathrm{top}}) \to H^1(G_\mathbb{R}, (\pi_1^{\mathrm{top}})^\wedge)$$

3.1.15. *Lemma. — Let* G *be as above. Let* ϖ *and* ϖ' *be groups with* G *actions such that* ϖ *is residually finite, and* ϖ' *is a finitely generated abelian group. Suppose that there is a morphism* $\varpi \to \varpi'$ *inducing an injection of sets* $H^1(G, \varpi) \to H^1(G, \varpi')$. *Then the* G *morphism* $\varpi \to \varpi^\wedge$ *induces an isomorphism of pointed sets*

$$H^1(G, \varpi) \cong H^1(G, \varpi^\wedge)$$

Proof. Since ϖ' is a finitely generated abelian group, $H^1(G, \varpi')$ is finite. Thus $H^1(G, \varpi)$ is finite. By lemmas 3.1.12 and 3.1.11, $H^1(G, \varpi) \to H^1(G, \varpi^\wedge)$ is surjective. By Lemma 3.1.13, $H^1(G, \varpi') \to H^1(G, \varpi'^\wedge)$ is injective. By hypothesis, $H^1(G, \varpi) \to H^1(G, \varpi')$ is injective. Injectivity of $H^1(G, \varpi) \to H^1(G, \varpi^\wedge)$ follows from the commutative diagram

$$\begin{array}{ccc} \varpi^\wedge & \longrightarrow & \varpi'^\wedge \\ \uparrow & & \uparrow \\ \varpi & \longrightarrow & \varpi' \end{array}$$

□

We will apply Lemma 3.1.15 with $\varpi \to \varpi'$ equal to the map on π_1^{et} induced from the Abel-Jacobi map $X \to \operatorname{Jac} X$. To do this, we will use the following well known lemma. This lemma follows directly from [GH81], but we include a proof for completeness.

3.1.16. *Lemma. — Let* X *be a geometrically connected, smooth, proper, curve of genus* $g \geq 1$ *over* $\mathbb{R}$ *such that* $X(\mathbb{R}) \neq \emptyset$. *Let* $b \in X(\mathbb{R})$ *be a real point and let* $X \to \operatorname{Jac}(X)$ *be the associated Abel-Jacobi map. Then, the induced map on connected components*

$$\pi_0(X(\mathbb{R})) \to \pi_0(\operatorname{Jac}(X)(\mathbb{R}))$$

is an injection.

Proof. By [GH81, Prop. 2.2a], every point of $\operatorname{Jac}(X)(\mathbb{R})$ can be represented by a $G_\mathbb{R}$ invariant divisor of $X(\mathbb{C})$. Let $\mathbb{C}(X)$ denote the rational functions on $X_\mathbb{C}$, $\mathbb{R}(X)$ the rational functions on X, and let P denote the principal divisors of $X_\mathbb{C}$. The exact sequence of multiplicative $G_\mathbb{R}$ modules

$$1 \to \mathbb{C}^* \to \mathbb{C}(X)^* \to P \to 1$$

gives the exact sequence in cohomology $\mathbb{R}(X)^* \to P^{G_\mathbb{R}} \to H^1(G_\mathbb{R}, \mathbb{C}^*) = 1$. Thus $\operatorname{Jac}(X)(\mathbb{R})$ is the quotient of the $G_\mathbb{R}$ invariant divisors of $X(\mathbb{C})$ by $\{\operatorname{div} f | f \in \mathbb{R}(X)^*\}$. By [GH81, Lem. 4.1], for any $f \in \mathbb{R}(X)^*$, $\operatorname{div} f$ has an even number of points on each component of $X(\mathbb{R})$, whence there is a map $\operatorname{Jac}(X)(\mathbb{R}) \to (\mathbb{Z}/2)^{\pi_0(X(\mathbb{R}))}$, sending a divisor to the number of points it contains on each component of $X(\mathbb{R})$ mod 2. Since the image of two real points of X in different connected components of $X(\mathbb{R})$ under the composite morphism $X(\mathbb{R}) \to \mathrm{Jac}(X)(\mathbb{R}) \to (\mathbb{Z}/2)^{\pi_0(X(\mathbb{R}))}$ are different, $\pi_0(X(\mathbb{R})) \to \pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ is an injection. $\square$

3.1.17. *Proposition. — Let* X *be a geometrically connected, smooth, proper, curve of genus* $g \geq 1$ *over* $\mathbb{R}$ *such that* $X(\mathbb{R}) \neq \emptyset$. *Let* π_1^{top} *denote the fundamental group of the associated complex analytic space to* X *based at a real point. Then the natural map* $\pi_1^{top} \to (\pi_1^{top})^\wedge$ *induces an isomorphism*

$$H^1(G_{\mathbb{R}}, \pi_1^{top}) \to H^1(G_{\mathbb{R}}, (\pi_1^{top})^\wedge)$$

Proof. π_1^{top} is the fundamental group of a surface of genus g, and in particular, π_1^{top} is finitely generated. π_1^{top} is residually finite by [Lop94, Thm.A] (originally shown by Peter Scott [Sco78]) By 3.1.9, we have natural isomorphisms $H^1(G_{\mathbb{R}}, \pi_1^{top}) \cong \pi_0(X(\mathbb{R}))$ and $H^1(G_{\mathbb{R}}, \pi_1^{top}/[\pi_1^{top}]_2) \cong \pi_0(\mathrm{Jac}\, X(\mathbb{R}))$. The Abel-Jacobi map $X \to \mathrm{Jac}\, X$ (for the base point of π_1^{top}) induces an injection $\pi_0(X(\mathbb{R})) \to \pi_0(\mathrm{Jac}\, X(\mathbb{R}))$ by Lemma 3.1.16. Thus $H^1(G_{\mathbb{R}}, \pi_1^{top}) \to H^1(G_{\mathbb{R}}, \pi_1^{top}/[\pi_1^{top}]_2)$ is an injection. The Proposition follows by Lemma 3.1.15. $\square$

3.1.18. *Definition* δ_n^{top} *and* δ_n^{et}. Let $X \to \mathrm{Spec}\, \mathbb{R}$ be a geometrically connected curve such that $X(\mathbb{R}) \neq \emptyset$ and let $\delta_n^{et} = \delta_n$ be Ellenberg's obstructions as in 1.1.3 with respect to some real base point b. Let $X(\mathbb{C})$ denote the complex analytic space corresponding to X. $G_{\mathbb{R}}$ acts on $\pi_1^{top}(X_{\mathbb{C}}) = \pi_1^{top}(X(\mathbb{C}), b)$. Define $\delta_n^{top} : H^1(G_{\mathbb{R}}, \pi_1^{top}(X_{\mathbb{C}})/[\pi_1^{top}(X_{\mathbb{C}})]_n) \to H^2(G_{\mathbb{R}}, [\pi_1^{top}(X_{\mathbb{C}})]_n/[\pi_1^{top}(X_{\mathbb{C}})]_{n+1})$ to be the boundary map associated to

$$1 \to [\pi_1^{top}(X_{\mathbb{C}})]_n/[\pi_1^{top}(X_{\mathbb{C}})]_{n+1} \to \pi_1^{top}(X_{\mathbb{C}})/[\pi_1^{top}(X_{\mathbb{C}})]_{n+1} \to \pi_1^{top}(X_{\mathbb{C}})/[\pi_1^{top}(X_{\mathbb{C}})]_n \to 1$$

For smooth curves over $\mathbb{R}$, δ_n^{top} determines δ_n^{et} for any n, and δ_2^{top} is equivalent to δ_2^{et}:

3.1.19. *Proposition. — Let* X *be a smooth, geometrically connected curve over* $\mathbb{R}$ *such that* $X(\mathbb{R}) \neq \emptyset$. *Let* δ_n^{top} *and* δ_n^{et} *be as above. The diagram*

$$\begin{array}{ccc}
H^2(G_{\mathbb{R}}, [\pi_1^{top}]_n/[\pi_1^{top}]_{n+1}) & \xrightarrow{\cong} & H^2(G_{\mathbb{R}}, [\pi_1^{et}]_n/[\pi_1^{et}]_{n+1}) \\
\uparrow{\scriptstyle \delta_n^{top}} & & \uparrow{\scriptstyle \delta_n^{et}} \\
H^1(G_{\mathbb{R}}, \pi_1^{top}/[\pi_1^{top}]_n) & \longrightarrow & H^1(G_{\mathbb{R}}, \pi_1^{et}/[\pi_1^{et}]_n) \\
\uparrow & & \uparrow \\
H^1(G_{\mathbb{R}}, \pi_1^{top}) & \longrightarrow & H^1(G_{\mathbb{R}}, \pi_1^{et})
\end{array}$$

commutes. (The horizontal arrows are induced by the natural isomorphism $(\pi_1^{top})^\wedge \cong \pi_1^{et}$. π_1^{top} *and* π_1^{et} *abbreviate* $\pi_1^{top}(X(\mathbb{C}), b)$ *and* $\pi_1^{et}(X_{\mathbb{C}}, b)$ *respectively.)*

The top horizontal arrow is an isomorphism. The bottom two are (at least) surjections. For $n = 2$, *the middle arrow is an isomorphism. For* X *proper, the bottom arrow is an isomorphism.*

Proof. The natural morphism of $G_{\mathbb{R}}$ equivariant short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & [\pi_1^{et}]_n/[\pi_1^{et}]_{n+1} & \longrightarrow & \pi_1^{et}/[\pi_1^{et}]_{n+1} & \longrightarrow & \pi_1^{et}/[\pi_1^{et}]_n & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & [\pi_1^{top}]_n/[\pi_1^{top}]_{n+1} & \longrightarrow & \pi_1^{top}/[\pi_1^{top}]_{n+1} & \longrightarrow & \pi_1^{top}/[\pi_1^{top}]_n & \longrightarrow & 1
\end{array}$$

gives the commutativity of the top square in the diagram. Commutativity of the bottom square is immediate.

By [SGAI, Exp. XII Cor. 5.2], there is a canonical isomorphism $\pi_1^{et} = (\pi_1^{top})^\wedge$. By the universal properties of profinite completion and taking the quotient by subgroups of the lower central series, the profinite completion of $[\pi_1^{top}]_n/[\pi_1^{top}]_{n+1}$ is canonically $[\pi_1^{et}]_n/[\pi_1^{et}]_{n+1}$. Thus, the top horizontal arrows are isomorphisms by Lemma 3.1.13. Similarly, the profinite completion of $\pi_1^{top}/[\pi_1^{top}]_n$ is canonically $\pi_1^{et}/[\pi_1^{et}]_n$. For $n = 2$, the middle arrow is an isomorphism by Lemma 3.1.13. π_1^{top} is either a free group or the fundamental group of a surface of genus g, and in particular, π_1^{top} is finitely generated and residually finite [MKS04, 2.4 ex.24] [Lop94, Thm.A] [Sco78]. Finitely generated nilpotent groups are residually finite [MKS04, 6.5]. $H^1(G_{\mathbb{R}}, \pi_1^{top})$ is finite by Proposition 3.1.8. By induction and [Ser02, I.5.7 Prop. 42], $H^1(G, \pi^{top}/[\pi^{top}]_n)$ is finite for all n. Therefore, Lemmas 3.1.11 and 3.1.12 imply that the bottom two arrows are surjections. For X proper, the bottom horizontal arrow is an isomorphism by Proposition 3.1.17. □

I see no reason why $H^1(G_{\mathbb{R}}, \pi_1^{top}) \to H^1(G_{\mathbb{R}}, \pi_1^{et})$ should not be an isomorphism for all curves. Similarly, δ_n^{top} may equal δ_n^{et} for all n. (However, we will show below that for X proper, the kernel of δ_2 is precisely $\pi_0(X(\mathbb{R}))$, so Ellenberg's first obstruction already succeeds at determining the connected components of $\mathbb{R}$-rational points of the curve from those of the Jacobian, making the higher obstructions less interesting.)

3.1.20. *Remark.* Propositions 3.1.17 and 3.1.14 turn the topological section conjectures of Corollaries 3.1.9 and 3.1.8 into the usual section conjecture over $\mathbb{R}$ for π_1^{et}.

3.2. δ_2 for proper smooth curves over $\mathbb{R}$.

Following a suggestion of Jordan Ellenberg, we show that δ_2 over $\mathbb{R}$ determines the connected components of $X(\mathbb{R})$ from those of $\operatorname{Jac} X(\mathbb{R})$ for X a proper curve.

Let $X \to \operatorname{Spec} \mathbb{R}$ be a proper smooth geometrically connected curve. Let g be the genus of X, and suppose that $g > 0$. Let $\operatorname{Jac}(X) \to \operatorname{Spec} \mathbb{R}$ denote the Jacobian of X. Assume that $X(\mathbb{R})$ is non-empty and choose b in $X(\mathbb{R})$.

As above, let $G_{\mathbb{R}}$ denote the absolute Galois group of $\mathbb{R}$, let $\pi = \pi_1^{top}(X(\mathbb{C}), b)$, and let δ_2 be δ_2^{top} as in 3.1.18. By Proposition 3.1.19, δ_2 is Ellenberg's obstruction of 1.1.3. Let $\mathcal{H}$ denote the $G_{\mathbb{R}}$ module $H_1(X(\mathbb{C}), \mathbb{Z}) = \pi^{ab} = \pi/[\pi]_2 = \pi_1(\operatorname{Jac}(X)(\mathbb{C}))$. Let $\tau \in G_{\mathbb{R}}$ denote complex conjugation.

The abelian group structure on $\mathrm{Jac}(X)(\mathbb{R})$ gives $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ the structure of an abelian group. In fact, $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ is a $\mathbb{Z}/2$ vector space and is isomorphic to the Tate cohomology group $\hat{H}^0(G_{\mathbb{R}}, \mathrm{Jac}(X)(\mathbb{C}))$ by, for instance, [GH81, Prop 1.1]. (For the reader's convenience, here is the proof in [GH81]: the norm map $\mathbb{N} : \mathrm{Jac}(X)(\mathbb{C}) \to \mathrm{Jac}(X)(\mathbb{R})$, defined by sending x in $\mathrm{Jac}(X)(\mathbb{C})$ to $x + \tau x$, is a continuous homomorphism from a compact connected group. The image of $\mathbb{N}$ is therefore a closed connected subgroup. The image of $\mathbb{N}$ also contains $2\,\mathrm{Jac}(X)(\mathbb{R})$, and is therefore finite index, whence open. Thus the image is the connected component of the identity of $\mathrm{Jac}(X)(\mathbb{R})$, whence $\hat{H}^0(G_{\mathbb{R}}, \mathrm{Jac}(X)(\mathbb{C})) = \pi_0(\mathrm{Jac}(X)(\mathbb{R}))$.)

We have the commutative diagram

$$\begin{array}{ccc} H^1(G_{\mathbb{R}}, \pi) & \longrightarrow & H^1(G_{\mathbb{R}}, \mathcal{H}) \\ \cong\uparrow & & \cong\uparrow \\ \pi_0(X(\mathbb{R})) & \longrightarrow & \pi_0(\mathrm{Jac}(X)(\mathbb{R})) \end{array}$$

where the vertical arrows are isomorphisms by Corollaries 3.1.9 and 3.1.8. The horizontal arrows are injections by Lemma 3.1.16.

The obstruction $\delta_2 : H^1(G_{\mathbb{R}}, \mathcal{H}) \to H^1(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ can therefore be viewed as a map with domain $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$. By Proposition 2.1.3, δ_2 is a quadratic form, and by construction, δ_2 vanishes on $\pi_0(X(\mathbb{R}))$. In fact, the kernel of δ_2 is precisely $\pi_0(X(\mathbb{R}))$ (and therefore none of the higher obstructions eliminate further elements of $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$). In other words, δ_2 is a quadratic form on the $\mathbb{Z}/2$ vector space $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ which vanishes precisely on $\pi_0(X(\mathbb{R}))$.

3.2.1. *Proposition. — The kernel of* δ_2 *is* $\pi_0(X(\mathbb{R}))$

Proof. Recall that as a complex manifold, $\mathrm{Jac}(X)(\mathbb{C})$ is $\Omega(X(\mathbb{C}))^*/\mathcal{H}$, where $\Omega(X(\mathbb{C}))$ denotes the g dimensional complex vector space of holomorphic one-forms on $X(\mathbb{C})$, and $\Omega(X(\mathbb{C}))^*$ denotes its dual. Choosing a basis for $\Omega(X(\mathbb{C}))^*$, we have that $\mathrm{Jac}(X)(\mathbb{C})$ is isomorphic to $\mathbb{C}^g/\mathcal{H}$. Since the connected component of the identity of $\mathrm{Jac}(X)(\mathbb{R})$ is a connected, compact, abelian, real Lie group of dimension g, it is isomorphic to $(\mathbb{R}/\mathbb{Z})^g$. Thus $\mathcal{H} \cap \mathbb{R}^g \cong \mathbb{Z}^g$. View $\mathbb{Z}^g$ as a $G_{\mathbb{R}}$ module with τ acting by the identity. We therefore have an injection of $G_{\mathbb{R}}$ modules $\mathbb{Z}^g \hookrightarrow \mathcal{H}$. For any $v \in \mathcal{H}$, $\tau v + v$ is an element of $\mathcal{H} \cap \mathbb{R}^g$. The cokernel of the injection $\mathbb{Z}^g \hookrightarrow \mathcal{H}$ is therefore $\mathcal{I}$, where $\mathcal{I}$ denotes the $G_{\mathbb{R}}$ module which is $\mathbb{Z}^g$ as an abelian group and where τ acts by multiplication by -1. So we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}^g \xrightarrow{\ r\ } \mathcal{H} \xrightarrow{\ \iota\ } \mathcal{I} \longrightarrow 0$$

Since $H^1(G_{\mathbb{R}}, \mathbb{Z}^g) = 0$, it follows that $H^1(G_{\mathbb{R}}, \mathcal{H}) \to H^1(G_{\mathbb{R}}, \mathcal{I})$ is an injection.

For any $G_{\mathbb{R}}$ module M, we have the cup product

$$\cup : H^1(G_{\mathbb{R}}, M) \otimes H^1(G_{\mathbb{R}}, M) \to H^2(G_{\mathbb{R}}, M \otimes M).$$

We also have the quotient map $p_M : M \otimes M \to M \wedge M$, where $M \wedge M := M \otimes M / \langle m \otimes m | m \in M \rangle$. Combining the two, we have a symmetric pairing

$$H^1(G_{\mathbb{R}}, M) \otimes H^1(G_{\mathbb{R}}, M) \to H^2(G_{\mathbb{R}}, M \wedge M),$$

which is natural in M.

The periodic resolution

$$\cdots \mathbb{Z}G_{\mathbb{R}} \xrightarrow{\tau-1} \mathbb{Z}G_{\mathbb{R}} \xrightarrow{\tau+1} \mathbb{Z}G_{\mathbb{R}} \xrightarrow{\tau-1} \mathbb{Z}G_{\mathbb{R}} \longrightarrow \mathbb{Z}$$

gives rise to isomorphisms $H^1(G_{\mathbb{R}}, M) \cong \mathrm{Ker}(\tau+1)/\mathrm{Image}(\tau-1)$, and $H^2(G_{\mathbb{R}}, M) \cong \mathrm{Ker}(\tau-1)/\mathrm{Image}(\tau+1)$, where $\tau+1$ and $\tau-1$ are viewed as endomorphisms of M. It is straightforward to see that in terms of these isomorphisms, the above symmetric pairing is

$$m_1 \otimes m_2 \mapsto m_1 \wedge \tau m_2$$

Thus for the $G_{\mathbb{R}}$ module $\mathcal{I}$, the symmetric pairing $H^1(G_{\mathbb{R}}, \mathcal{I}) \otimes H^1(G_{\mathbb{R}}, \mathcal{I}) \to H^2(G_{\mathbb{R}}, \mathcal{I} \wedge \mathcal{I})$ is isomorphic to $p_{(\mathbb{Z}/2)^g} : (\mathbb{Z}/2)^g \otimes (\mathbb{Z}/2)^g \to (\mathbb{Z}/2)^g \wedge (\mathbb{Z}/2)^g$.

The map $\iota : \mathcal{H} \to \mathcal{I}$ gives rise to the commutative diagram:

$$\begin{array}{ccc} H^1(G_{\mathbb{R}}, \mathcal{H}) \otimes H^1(G_{\mathbb{R}}, \mathcal{H}) & \longrightarrow & H^2(G_{\mathbb{R}}, \mathcal{H} \wedge \mathcal{H}) \\ \downarrow & & \downarrow \\ (\mathbb{Z}/2)^g \otimes (\mathbb{Z}/2)^g & \xrightarrow{p_{(\mathbb{Z}/2)^g}} & (\mathbb{Z}/2)^g \wedge (\mathbb{Z}/2)^g \end{array} \tag{33}$$

The commutator $v \otimes w \mapsto vwv^{-1}w^{-1}$ defines a map $q : \mathcal{H} \otimes \mathcal{H} \to [\pi]_2/[\pi]_3$. We show that the induced map $q_* : H^2(G_{\mathbb{R}}, \mathcal{H} \wedge \mathcal{H}) \to H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ is injective:

Let $E : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}$ be the intersection pairing on $H_1(X(\mathbb{C}), Z)$. Recall that E is antisymmetric. Since τ induces an orientation reversing homeomorphism of $X(\mathbb{C})$, $E(\tau v, \tau w) = -E(v, w)$ for all $v, w \in \mathcal{H}$. From the construction of the genus g surface which consists of gluing the sides of a 4g-gon following the pattern

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1},$$

we see that the kernel of $\mathcal{H} \otimes \mathcal{H} \to [\pi]_2/[\pi]_3$ is generated as a $\mathbb{Z}$ module by an element x of the form

$$x = a_1 \wedge b_1 + \ldots + a_g \wedge b_g,$$

such that $E(a_i, b_i) = 1$ for all i. Let K denote this kernel. Since $E(x) = g$, we have that $E(\tau x) = -g$. Thus, x is not fixed by τ. Because $G_{\mathbb{R}}$ acts on K, it follows that $\tau x = -x$. Thus $H^2(G_{\mathbb{R}}, K) = 0$, so the map $H^2(G_{\mathbb{R}}, \mathcal{H} \wedge \mathcal{H}) \to H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ is injective.

Let [b] denote the element of $\pi_0(X(\mathbb{R}))$ containing b. By [GH81] Prop 2.2, Lemma 4.1, and Prop 4.2, the image of $\pi_0(X(\mathbb{R})) - \{[b]\}$ in $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ is a basis of $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ as a $\mathbb{Z}/2$ vector space. (For the convenience of the reader, we include a proof below. See Lemma 3.2.2.) Choose one point of each connected component of $X(\mathbb{R})$ other then

[b], and denote these points by $b_1, b_2, \ldots, b_n$. So, $\pi_0(X(\mathbb{R}) = \{[b], [b_1], [b_2], \ldots, [b_n]\}$, and $\{[b_1], [b_2], \ldots, [b_n]\}$ is a basis of $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$.

By Proposition 2.1.3, $\delta_2(v+w) = \delta_2(v)+\delta_2(w)+q_*(p_{\mathcal{H}})_*(v \cup w)$. Since $\{[b_1], [b_2], \ldots, [b_n]\}$ is a $\mathbb{Z}/2$ basis of $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$, an element of $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ can be expressed uniquely in the form $b_{i_1} + b_{i_2} + \ldots + b_{i_m}$ with $i_j < i_{j+1}$. Since $\delta_2([b_i]) = 0$ for all i, $\delta_2(b_{i_1} + b_{i_2} + \ldots + b_{i_m}) = \sum_{1 \le j < k \le m} q_*(p_{\mathcal{H}})_*(b_{i_j} \cup b_{i_k})$.

Since $H^1(G_{\mathbb{R}}, \mathcal{H})$ injects into $H^1(G_{\mathbb{R}}, \mathcal{I})$, $\sum_{1 \le j < k \le m}(p_{\mathcal{H}})_*(b_{i_j} \cup b_{i_k}) \neq 0$ for $m > 1$ by equation 33. Since q_* is injective, we have that $\delta_2(b_{i_1} + b_{i_2} + \ldots + b_{i_m}) \neq 0$ for $m > 1$, proving Proposition 3.2.1. □

As above, b is the basepoint of π, and $X \to \mathrm{Jac}(X)$ is the Abel-Jacobi map corresponding to b. As promised above, we give a more detailed proof of the following:

3.2.2. *Lemma. — Let* $[b]$ *denote the connected component of* $X(\mathbb{R})$ *containing* b*. The image of* $\pi_0(X(\mathbb{R})) - \{[b]\}$ *in* $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ *is a basis of* $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ *as a* $\mathbb{Z}/2$ *vector space.*

Proof. As described in the proof of Lemma 3.1.16, [GH81] shows there is a continuous homomorphism $c : \mathrm{Jac}(\mathbb{R}) \to (\mathbb{Z}/2)^{\pi_0(X(\mathbb{R}))}$, which sends a divisor to the number of points it contains on each connected component of $X(\mathbb{R})$ mod 2. In particular, c is a surjective homomorphism, and $X(\mathbb{R}) \longrightarrow \mathrm{Jac}\, X(\mathbb{R}) \xrightarrow{c} (\mathbb{Z}/2)^{\pi_0(X(\mathbb{R}))}$ determines a well defined map on $\pi_0(X(\mathbb{R})) - \{[b]\}$ whose image is the standard basis of $(\mathbb{Z}/2)^{\pi_0(X(\mathbb{R}))}$. [GH81, Prop. 4.2] identifies the kernel of c as $2\,\mathrm{Jac}\, X(\mathbb{R})$. As commented above, $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ is a $\mathbb{Z}/2$ vector space, whence $2\,\mathrm{Jac}\, X(\mathbb{R})$ is contained in the connected component of the identity of $\mathrm{Jac}\, X(\mathbb{R})$. As commented above, the connected component of the identity of $\mathrm{Jac}\, X(\mathbb{R})$ is isomorphic to $(\mathbb{R}/\mathbb{Z})^g$ and is therefore divisible. Thus, $2\,\mathrm{Jac}\, X(\mathbb{R})$ equals the connected component of the identity of $\mathrm{Jac}\, X(\mathbb{R})$. Thus, c induces an isomorphism $\pi_0(\mathrm{Jac}(X)(\mathbb{R})) \to (\mathbb{Z}/2)^{\pi_0(X(\mathbb{R}))}$, and under this isomorphism, $\pi_0(X(\mathbb{R})) - \{[b]\}$ is sent to the standard basis. □

Remark: For the above, we used Carlsson's Theorem 3.1.4 to identify $\pi_0(X(\mathbb{R}))$ and $\pi_0(\mathrm{Jac}(X)(\mathbb{R}))$ with $H^1(G_{\mathbb{R}}, \pi)$ and $H^1(G_{\mathbb{R}}, \mathcal{H})$ respectively. One can give an alternate proof that $\pi_0(\mathrm{Jac}(X)(\mathbb{R})) = H^1(G_{\mathbb{R}}, \mathcal{H})$ using specific information about Jacobians, as follows:

3.2.3. *Proposition. — The natural map* $\pi_0(\mathrm{Jac}(X)(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \mathcal{H})$ *is an isomorphism of* $\mathbb{Z}/2$ *vector spaces.*

Proof. As above, we have that $\mathrm{Jac}(X)(\mathbb{C})$ is isomorphic to $\mathbb{C}^g/\mathcal{H}$. We therefore have the exact sequence of $G_{\mathbb{R}}$ modules

$$0 \to \mathcal{H} \to \mathbb{C}^g \to \mathrm{Jac}(X)(\mathbb{C}) \to 0,$$

and the resulting exact sequence of Tate cohomology groups

$$\ldots \hat{H}^0(G_{\mathbb{R}}, \mathbb{C}^g) \to \hat{H}^0(G_{\mathbb{R}}, \mathrm{Jac}(X)(\mathbb{C})) \to \hat{H}^1(G_{\mathbb{R}}, \mathcal{H}) \to \hat{H}^1(G_{\mathbb{R}}, \mathbb{C}^g) \ldots$$

Since $\hat{H}^0(G_{\mathbb{R}}, \mathbb{C}^g)$ and $\hat{H}^1(G_{\mathbb{R}}, \mathbb{C}^g)$ are both 0, and $\hat{H}^0(G_{\mathbb{R}}, \mathrm{Jac}(X)(\mathbb{C})) = \pi_0(\mathrm{Jac}(X)(\mathbb{R}))$, we have that $\pi_0(\mathrm{Jac}(X)(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \mathcal{H})$ is an isomorphism as claimed. □

3.3. $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$**: lower order terms of** δ_2 **and** δ_3**.** 3.3.1. Note that the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ for α, β, γ such that $\alpha \cup \beta = 0$ and $\beta \cup \gamma = 0$ is described by choosing cochains A, B such that $DA = \alpha \cup \beta$ and $DB = \beta \cup \gamma$, and setting $\langle \alpha, \beta, \gamma \rangle = A \cup \gamma + \alpha \cup B$. The indeterminacy is the ideal generated by α and γ.

Let k be a number field, and $X = \mathbb{P}^1_k - \{0, 1, \infty\}$. Base $\pi = \pi_1(\mathbb{P}^1_{\bar{k}} - \{0, 1, \infty\})$ at the tangential base point at 0 pointing towards 1 along the real line. Then $\pi = \langle x, y \rangle^\wedge$, where x is a loop around 0 and y is a loop around 1. The map $\pi \to \pi/[\pi]_2$ is the fundamental group functor applied to the Abel-Jacobi map $X_{\bar{k}} \to \mathrm{Jac}(X_{\bar{k}}) = \mathbb{G}_{m,\bar{k}} \times \mathbb{G}_{m,\bar{k}}$ associated to the same base point. Thus, the action of G_k on $\pi/[\pi]_2$ is given by:

$$\sigma(x) = x^{\chi(\sigma)}$$

$$\sigma(y) = y^{\chi(\sigma)},$$

where $\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^*$ denotes the cyclotomic character. It follows that $[\pi]_n/[\pi]_{n+1}$ is a free $\hat{\mathbb{Z}}(n)$ module of rank 2^{n-2}. ($\hat{\mathbb{Z}}(n)$ denotes the n^{th} Tate twist of $\hat{\mathbb{Z}}$, which is $\hat{\mathbb{Z}}$ equipped with the G_k action given by χ^n.)

$\{[x, y]\}$ is a basis for $[\pi]_2/[\pi]_3$ as a $\hat{\mathbb{Z}}(2)$ module.

$\{[[x, y], x], [[x, y], y]\}$ is a basis for $[\pi]_3/[\pi]_4$ as a $\hat{\mathbb{Z}}(3)$ module. This basis gives an isomorphism $H^2(G_k, [\pi]_3/[\pi]_4) \cong H^2(G_k, \hat{\mathbb{Z}}(3)) \oplus H^2(G_k, \hat{\mathbb{Z}}(3))$, decomposing δ_3 into a direct sum of two obstructions

$$\delta_{3,[[x,y],x]}, \delta_{3,[[x,y],y]} : H^1(G_k, \pi/[\pi]_3) \to H^2(G_k, \hat{\mathbb{Z}}(3)).$$

More specifically, taking the image of ϵ_3 (see 2.4.26 for the definition of ϵ_3) under the isomorphism

$$b : H^2(\pi/[\pi]_3 \rtimes G_k, [\pi]_3/[\pi]_4) \cong H^2(\pi/[\pi]_3 \rtimes G_k, \hat{\mathbb{Z}}(3)) \oplus H^2(\pi/[\pi]_3 \rtimes G_k, \hat{\mathbb{Z}}(3))$$

given by the basis $\{[[x, y], x], [[x, y], y]\}$, we obtain two classes $\epsilon_{3,[[x,y],x]}, \epsilon_{3,[[x,y],y]} \in H^2(\pi/[\pi]_3 \rtimes G_k, \hat{\mathbb{Z}}(3))$ defined by

$$b(\epsilon_3) = \epsilon_{3,[[x,y],x]} \oplus \epsilon_{3,[[x,y],y]}.$$

3.3.2. *Notation.* Let $\delta_{3,[[x,y],x]} = \wp_{\epsilon_{3,[[x,y],x]}}$ and $\delta_{3,[[x,y],y]} = \wp_{\epsilon_{3,[[x,y],y]}}$, where $\wp$ is as in 2.2.1.

Many properties of the Galois action on $\pi_1(\mathbb{P}^1_{\bar{k}} - \{0, 1, \infty\})$ are encoded in Drinfeld's Grothendieck-Teichmüller group [Dri90] . One such property is that the action of G_k on π is of the form

$$\sigma(x) = x^{\chi(\sigma)} \tag{34}$$

$$\sigma(y) = \mathfrak{f}(\sigma)^{-1} y^{\chi(\sigma)} \mathfrak{f}(\sigma),$$

where $\mathfrak{f} : G_{\mathbb{Q}} \to [\pi]_2$ is a certain cocyle. (Here, σ is any element of G_k; x and y are as above; and as above, χ is the cyclotomic character.) This property of the action follows from symmetry arguments, the fact that Galois actions on fundamental groups preserve inertia, and the identification of $\mathrm{Jac}(X_{\bar{k}})$ with $\mathbb{G}_{m,\bar{k}} \times \mathbb{G}_{m,\bar{k}}$ (see [Iha94]). Also see [Iha94] for a discussion of $\mathfrak{f}$ and further properties of the action as described by Drinfeld's Grothendieck-Techmüller group.

3.3.3. *Notation.* We will use the following notation in this subsection: the basis $\{x, y\}$ for $\pi/[\pi]_2$ as a $\hat{\mathbb{Z}}(1)$ module gives rise to homomorphisms $x^*, y^* : H^1(G_k, \pi/[\pi]_3) \to H^1(G_k, \hat{\mathbb{Z}}(1))$ defined by pushing forward by the G_k homomorphisms $\pi/[\pi]_3 \to \pi/[\pi]_2 \to \mathbb{Z}\hat{(1)}$, where the maps $\pi/[\pi]_2 \to \hat{\mathbb{Z}}(1)$ are determined by the basis $\{x, y\}$ in the obvious manner. Note that this definition of x^* and y^* is different from that which would be suggested by (16).

3.3.4. *Theorem. — Let $X = \mathbb{P}^1_k - \{0, 1, \infty\}$, and base the fundamental group of $\mathbb{P}^1_{\bar{k}} - \{0, 1, \infty\}$ at the tangential base point at 0 pointing towards 1 along the real line. Then:*

$$\begin{aligned} \delta_{3,[[x,y],x]} &= \langle y^*, x^*, x^* \rangle + \langle (\chi - 1)/2, x^*, y^* \rangle \\ \delta_{3,[[x,y],y]} &= -\langle x^*, y^*, y^* \rangle + \langle (\chi - 1)/2, x^*, y^* \rangle - \mathfrak{f} \cup y^* \end{aligned}$$

where $\delta_{3,[[x,y],x]}$ and $\delta_{3,[[x,y],y]}$ are as in (3.3.2), x^* and y^* are as in 3.3.3, $\langle \cdot, \cdot, \cdot \rangle$ denotes the triple Massey product, $\chi : G_k \to \hat{\mathbb{Z}}^*$ denotes the cyclotomic character, and $\mathfrak{f} : G_k \to [\pi]_2/[\pi]_3$ is as in (34).

Furthermore, the indeterminacy of the Massey products encodes the indeterminacy of lifting an element of $H^1(G_k, \pi/[\pi]_2)$ to an element of $H^1(G_k, \pi/[\pi]_3)$, as is made precise in the following remark.

3.3.5. *Remark.* Note that x^* and y^* factor through $H^1(G_k, \pi/[\pi]_3) \to H^1(G_k, \pi/[\pi]_2)$. The condition on an element of $H^1(G_k, \pi/[\pi]_2)$ to lift to an element of $H^1(G_k, \pi/[\pi]_3)$ and the indeterminacy of the choice of lift are precisely the conditions required for the existence of the Massey products and the indeterminacy of the Massey products when they are defined, respectively; namely, given an element p of $H^1(G_k, \pi/[\pi]_3)$, we have a distinguished choice of cochain whose boundary is $x^*(p) \cup y^*(p)$, and making that choice in the construction of the Massey product of 3.3.1 (along with making certain standard choices for the cochain whose boundary is $x^*(p) \cup x^*(p)$, $(\chi - 1)/2 \cup x^*(p)$ etc.), the above equations for $\delta_3(p)$ are equalities of cohomology classes (i.e. without any indeterminacy). Starting from the right hand side of the se equations, if we have a choice of cohain whose boundary is $x^*(p) \cup y^*(q)$, we have a distinguished element of $H^1(G_k, \pi/[\pi]_3)$, and the equations of Theorem 3.3.4 are again equalities of cohomology classes.

Proof. (of Theorem 3.3.4)

For $a \in \hat{\mathbb{Z}}$ and $n \in \mathbb{Z}$, let $\binom{a}{n} = (a(a-1)\cdots(a-n+1))/(n!) \in \hat{\mathbb{Z}}$ denote the binomial coefficient. We have the following equalities in $\pi/[\pi]_4$:

A straightforward computation shows:

$$x^b y^a = y^a x^b [x,y]^{ab} [[x,y],y]^{b\binom{a+1}{2}} [[x,y],x]^{a\binom{b+1}{2}} \tag{35}$$

Using equation (35), one obtains the further computation:

$$[x^a, y^a] = [x,y]^{a^2} [[x,y],x]^{-a\binom{a}{2}} [[x,y],y]^{-a\binom{a}{2}} \tag{36}$$

Let $f : G_k \to \hat{\mathbb{Z}}$ be defined by

$$\mathfrak{f}(\sigma) = [x,y]^{f(\sigma)} \tag{37}$$

f is a cocycle in $C^1(G_k, \hat{\mathbb{Z}}(2))$.

For any $g \in G_k$, a straightforward computation using (34), (36) and (37) shows that:

$$\begin{gathered} g(y^a x^b [x,y]^c) = \\ y^{\chi(g)a} x^{\chi(g)b} [x,y]^{\chi(g)^2 c} [[x,y],x]^{-\frac{\chi(g)-1}{2}\chi(g)^2 c} [[x,y],y]^{-\frac{\chi(g)-1}{2}\chi(g)^2 c} [[x,y],y]^{-f(g)\chi(g)a} \end{gathered} \tag{38}$$

An arbitrary element of $\pi/[\pi]_3$ can be written uniquely in the form $y^a x^b [x,y]^c$ for $a, b, c \in \hat{\mathbb{Z}}$. Sending $y^a x^b [x,y]^c \in \pi/[\pi]_3$ to $y^a x^b [x,y]^c \in \pi/[\pi]_4$ determines a section of the quotient map $\pi/[\pi]_4 \to \pi/[\pi]_3$, which gives rise to a cocycle $e \in C^2(\pi/[\pi]_3 \rtimes G_k, [\pi]_3/[\pi]_4)$ representing ϵ_3 (see 2.0.5), as well as cocycles $e_{[[x,y],x]}, e_{[[x,y],y]} \in C^2(\pi/[\pi]_3 \rtimes G_k, \hat{\mathbb{Z}}(3))$ representing $\epsilon_{3,[[x,y],x]}, \epsilon_{3,[[x,y],y]}$ respectively.

Combining (35) and (38):

$$\begin{gathered} e_{[[x,y],x]}(y^{a_1} x^{b_1} [x,y]^{c_1} \rtimes g_1, y^{a_2} x^{b_2} [x,y]^{c_2} \rtimes g_2) = \\ c_1 \chi(g_1) b_2 + \binom{b_1+1}{2} \chi(g_1) a_2 \\ + b_1 \chi(g_1)^2 a_2 b_2 - \frac{\chi(g_1)-1}{2} \chi(g_1)^2 c_2 \end{gathered} \tag{39}$$

$$\begin{gathered} e_{[[x,y],y]}(y^{a_1} x^{b_1} [x,y]^{c_1} \rtimes g_1, y^{a_2} x^{b_2} [x,y]^{c_2} \rtimes g_2) = \\ c_1 \chi(g_1) a_2 + b_1 \binom{\chi(g_1) a_2 + 1}{2} - c_2 \chi(g_1) \binom{\chi(g_1)}{2} - f(g_1) \chi(g_1) a_2 \end{gathered} \tag{40}$$

We now show that (39) and (40) are the claimed Massey products with respect to defining systems as in Remark 3.3.5.

Let $a, b \in C^1(\pi/[\pi]_3 \rtimes G_k, \hat{Z}(1))$ denote the cocycles determined by $y^a x^b [x,y]^c \mapsto a$ and $y^a x^b [x,y]^c \mapsto b$ respectively. Let $c \in C^1(\pi/[\pi]_3 \rtimes G_k, \hat{Z}(2))$ denote the cochain $y^a x^b [x,y]^c \mapsto c$. $Dc = -b \cup a$. Note that $g \mapsto (\chi(g)-1)/2$ determines a cocycle in $C^1(G_k, \hat{\mathbb{Z}}(1))$. Recall that $f \in C^1(G_k, \hat{\mathbb{Z}}(2))$ is a cocycle.

The "standard choices" discussed in Remark 3.3.5 for the cochains (in $C^1(G_k, \hat{\mathbb{Z}}(2))$) whose boundaries are $a \cup a$, $b \cup b$, or $(\chi-1)/2 \cup b$ are as follows:

We invert 2.

Define $B \in C^1(G_k, \hat{\mathbb{Z}}(2))$ by $B(y^a x^b [x,y]^c) = b$. (So B is not b because the former is in $C^1(G_k, \hat{\mathbb{Z}}(2))$ and the latter is in $C^1(G_k, \hat{\mathbb{Z}}(1))$.) Note that

$$B(y^{a_1} x^{b_1} [x,y]^{c_1} \rtimes g_1, y^{a_2} x^{b_2} [x,y]^{c_2} \rtimes g_2) =$$
$$b_1 + \chi(g_1)^2 b_2 - (b_1 + \chi(g_1) b_2) = (\chi(g_1)-1)\chi(g_1) b_2$$

Thus, $DB = (\chi-1) \cup b$. We take $B/2$ as the "standard choice" of cochain in $C^1(G_k, \hat{\mathbb{Z}}(2))$ whose boundary is $(\chi-1)/2 \cup b$.

Define $a^2 \in C^1(G_k, \hat{\mathbb{Z}}(2))$ by $a^2(y^a x^b [x,y]^c) = a^2$. Then $Da^2 = -2a \cup a$. We take $-a^2/2$ as the "standard choice" of cochain in $C^1(G_k, \hat{\mathbb{Z}}(2))$ whose boundary is $a \cup a$. We do the same for b.

Consider $e_{[[x,y],y]}$. Note that the cochain $g \mapsto b_1 \binom{\chi(g_1)a_2+1}{2}$ equals $b \cup a^2/2 + B/2 \cup a$. By equation (40), it follows that

$$e_{[[x,y],y]} = c \cup a + b \cup a^2/2 + B/2 \cup a - (\chi-1)/2 \cup c - f \cup a.$$

Since $c \cup a + b \cup a^2/2 = \langle x^*, -y^*, y^* \rangle$, and $B/2 \cup a - (\chi-1)/2 \cup c = \langle (\chi-1)/2, x^*, y^* \rangle$, we have established the claimed decomposition of $\delta_{3,[[x,y],y]}$ into a Massey product and the lower order term $\langle (\chi-1)/2, x^*, y^* \rangle - f \cup y^*$. Furthermore, the defining systems implicit in the Massey products are as in Remark 3.3.5.

Consider $e_{[[x,y],x]}$. By decomposing the cochain $g \mapsto \binom{b_1+1}{2}\chi(g_1)a_2$ into $b^2/2 \cup a + B/2 \cup a$, we see that equation (39) becomes

$$e_{[[x,y],x]} = c \cup b + b^2/2 \cup a + B/2 \cup a + b \cup (ab) - (\chi-1)/2 \cup c,$$

where $(ab) \in C^1(G_k, \hat{\mathbb{Z}}(2))$ is the cochain $g \mapsto ab$. A short calculation shows that $D(ab) = -a \cup b - b \cup a$. In particular $D(c - ab) = a \cup b$. Thus $(c-ab) \cup b + a \cup (-B/2) = \langle y^*, x^*, x^* \rangle$. The expression for $e_{[[x,y],x]}$ follows. □

3.4. $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$: **evaluating** δ_2 **and** δ_3**.** We wish to use the results of §3.3 to determine whether a given homotopy section of one of the bottom two spaces of the nilpotent tower of $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ lifts to the next. This is equivalent to determining whether certain elements of $H^2(G_{\mathbb{Q}}, \hat{\mathbb{Z}}(2))$ and $H^2(G_{\mathbb{Q}}, \hat{\mathbb{Z}}(3))$ vanish. Using Bloch-Kato, questions about the vanishing of elements of $H^2(G_{\mathbb{Q}}, \hat{\mathbb{Z}}(2))$ can be converted into questions about the Milnor K-group $K_2\mathbb{Q}$. Determining whether the cocycles of Theorem 3.3.4 vanish in $H^2(G_{\mathbb{Q}}, \hat{\mathbb{Z}}(3))$, however, seems difficult. Instead, we replace the lower central series with the lower exponent 2 central series, and perform calculations in the Brauer group.

Filter π by the lower exponent 2 central series, $\pi > [\pi]^2_2 > [\pi]^2_3 > \dots$, and let δ^2_n denote the corresponding obstructions. Theorem 3.3.4 remains valid if we replace δ_3 by δ^2_3. For simplicity, take $k = \mathbb{Q}$.

By Theorem 3.3.4, to explicitly evaluate $\delta^2_{3,[[x,y],y]}$, we need further information about $\mathfrak{f}$ (where $\mathfrak{f} : G_k \to [\pi]_2$ is as in 34).

An explicit formula for the Magnus coefficients of the projection of $\mathfrak{f}$ onto the nilpotent completion of π in terms of the cyclotomic elements of Soulé and Deligne is known due to contributions of Anderson, Coleman, Deligne, Ihara, Kaneko and Yukinari (see for instance [Iha91, 6.3 p.115]; this result is also in [And89], [Col89], and [IKY87]). Furthermore, the projection of $\mathfrak{f}$ to a function $\mathfrak{f} : G_k \to [\pi]_2/[\pi]_3$ is independent of the cyclotomic elements. From this formula, we obtain:

3.4.1. *Proposition. — Let* $[\mathfrak{f}]^2_3 \in H^1(G_{\mathbb{Q}}, [\pi]^2_2/[\pi]^2_3)$ *denote the cohomology class represented by* $\mathfrak{f}$. *The basis* $\{[x, y]\}$ *for* $[\pi]^2_2/[\pi]^2_3$ *as a* $\mathbb{Z}/2$*-module gives an isomorphism* $H^1(G_{\mathbb{Q}}, [\pi]^2_2/[\pi]^2_3) \cong H^1(G_{\mathbb{Q}}, \mathbb{Z}/2)$. *View* $[\mathfrak{f}]^2_3$ *as an element of* $H^1(G_{\mathbb{Q}}, \mathbb{Z}/2)$. *Then,* $[\mathfrak{f}]^2_3$ *equals the image of* -1 *under the Kummer map* $\mathbb{Q} \to H^1(G_{\mathbb{Q}}, \mathbb{Z}/2)$.

3.4.2. *Convention.* We will choose specific cocycles representing elements of cohomology in the image of Kummer maps. This is equivalent to choosing an n^{th} root for every n and every element of $\mathbb{Q}$. We make this choice by declaring the argument of a positive rational number to be 0, and the argument of a negative rational number to be π. By abuse of notation, we will let a rational number $a \in \mathbb{Q}$ also denote the corresponding cocycle in $C^1(G_{\mathbb{Q}}, \mu_\infty)$. Choosing an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, and choosing the primitive n^{th} roots of unity $e^{(2\pi i)/n}$, we identify μ_∞ and $\hat{\mathbb{Z}}(1)$, and let $a \in \mathbb{Q}$ also denote the corresponding cocycle in $C^1(G_{\mathbb{Q}}, \hat{\mathbb{Z}}(1))$.

3.4.3. Let p be an odd prime. Recall the following well-known computation of the cup product map

$$H^1(\mathbb{Q}_p, \mathbb{Z}/2) \otimes H^1(\mathbb{Q}_p, \mathbb{Z}/2) \to H^2(\mathbb{Q}_p, \mathbb{Z}/2):$$

Let K be a finite extension of $\mathbb{Q}_p$. The Kummer exact sequence $1 \to \mathbb{Z}/2 \to \overline{K}^* \to \overline{K}^* \to 1$ and Hilbert 90 give an isomorphism $K^*/(K^*)^2 \cong H^1(G_K, \mathbb{Z}/2)$. Let $\mathfrak{p}$ be a uniformizer of the ring of integers $\mathcal{O}_K$ of K, and let $u \in \mathcal{O}_K$ be a unit which reduces to an element of the residue field with is not a square, i.e. u is not a quatratic residue. $K^*/(K^*)^2$ is a free $\mathbb{Z}/2$ module with basis $\{u, \mathfrak{p}\}$. $H^2(G_K, \mathbb{Z}/2) \cong \mathbb{Q}/\mathbb{Z}[2]$ is the 2 torsion of the Brauer group. The computation of $\mathfrak{p} \cup \mathfrak{p}$ depends on if -1 is a quadratic residue. The following table gives the cup product, with the case where -1 is a quadratic residue given before the comma, and the case where -1 is not a quadratic residue given after the comma:

$\cup$	$\mathfrak{u}$	$\mathfrak{p}$	$\mathfrak{u}+\mathfrak{p}$
$\mathfrak{u}$	0	$1/2$	$1/2$
$\mathfrak{p}$	$1/2$	$0\,,1/2$	$1/2\,,0$
$\mathfrak{u}+\mathfrak{p}$	$1/2$	$1/2\,,0$	$0\,,1/2$

3.4.4. *Lemma. — Let* p *be a prime and let* $a, b \in Z$ *be such that* p *does not divide* b *and* p *divides* a *exactly once. Suppose we have* $u, v, w \in \mathbb{Q}$ *not all* 0 *such that* $u^2 = bv^2 + aw^2$. *Then* b *is a square mod* p.

3.4.5. *Lemma. — Take* b *in* $\mathbb{Q}$. *The cochain in* $C^1(G_{\mathbb{Q}}, \mathbb{Z}/2)$

$$\sigma \mapsto \binom{b(\sigma)+1}{2}$$

equals $\sqrt{b}$ *when restricted to an element of* $C^1(G_{\mathbb{Q}(\sqrt{b})}, \mathbb{Z}/2)$. *In particular, the restriction is a cocycle.*

Proof. By the choices of Convention (3.4.2), $b(\sigma) \in \mathbb{Z}/4$ is determined by

$$\sigma\sqrt[4]{b} = e^{2\pi i b(\sigma)/4}\sqrt[4]{b}.$$

Note that the choices of Convention (3.4.2) are such that $(\sqrt[4]{b})^2 = \sqrt{b}$. Thus $\sigma\sqrt{b} = e^{2\pi i 2b(\sigma)/4}\sqrt{b}$ and $b(\sigma) = 0$ or 2 for $\sigma \in G_{\mathbb{Q}(\sqrt{b})}$.

If $b(\sigma) = 0$, then $\binom{b(\sigma)+1}{2} = 0$ in $\mathbb{Z}/2$.

If $b(\sigma) = 2$, then $\binom{b(\sigma)+1}{2} = 1$ in $\mathbb{Z}/2$.

This proves the Lemma. □

3.4.6. *Lemma. — Let* p *be an odd prime and let* $a, b \in \mathbb{Q}$. *Suppose that* b *is a square in* $\mathbb{Q}_p$ *and let* $\sqrt{b}$ *be a chosen square root. If* $\sqrt{b}$ *is a square in* $\mathbb{Q}_p$, *then* $(0, 0)$ *is contained in the subset* Δ *of* $H^2(G_{\mathbb{Q}_p}, \mathbb{Z}/2) \oplus H^2(G_{\mathbb{Q}_p}, \mathbb{Z}/2)$:

$$\Delta = \{(\tilde{c} \cup a + (-1) \cup \tilde{c} + (-1) \cup a, \sqrt{b} \cup a + (-1) \cup \tilde{c}) : \tilde{c} \in H^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)\}$$

Proof. Since $\sqrt{b}$ is a square in $\mathbb{Q}_p$, $\sqrt{b} \cup a = 0$.

Case 1: $p \equiv 1 \mod 4$. Then -1 vanishes in $H^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)$. Thus,

$$\Delta = \{(\tilde{c} \cup a, 0) : \tilde{c} \in H^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)\}$$

It follows from 3.4.3 that we can choose $\tilde{c}$ such that $\tilde{c} \cup a = 0$.

Case 2: $p \equiv 3 \mod 4$. Since -1 is not a square mod p and has valuation 0, we have that $-1 = u \in H^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)$ in the notation of 3.4.3. For $\tilde{c} = u$, we therefore have that $(\tilde{c} \cup a + (-1) \cup \tilde{c} + (-1) \cup a, \sqrt{b} \cup a + (-1) \cup \tilde{c}) = (2u \cup a, 0) = (0, 0)$. □

3.4.7. *Proposition. — Let* p *be an odd prime and let* $a, b \in \mathbb{Z}$ *be such that* p *does not divide* b *and* p *divides* a *exactly once. Suppose further that* $\delta_2^2(b \times a) = 0$. *Then there exists a lift* $(b \times a)_c \in H^1(G_{\mathbb{Q}_p}, \pi/[\pi]_3^2)$ *of* $(b \times a) \in H^1(G_{\mathbb{Q}_p}, \pi/[\pi]_2^2)$ *such that* $\delta_3^2(b \times a)_c = 0$ *if and only if* b *is a fourth power mod* p *or* $p \equiv 1$ *mod 4*

Proof. (39) and (40) imply:

$$\delta^2_{3,[[x,y],x]}(b \times a)_c(\sigma, \tau) = c(\sigma)b(\tau) + \binom{b(\sigma)+1}{2} a(\tau) + b(\sigma)a(\tau)b(\tau) - \frac{\chi(\sigma)-1}{2} c(\tau)$$

$$\delta^2_{3,[[x,y],y]}(b \times a)_c(\sigma, \tau) = c(\sigma)a(\tau) + b(\sigma)\binom{\chi(\sigma)a(\tau)+1}{2} - \frac{\chi(\sigma)-1}{2} c(\tau) - \frac{\chi(\sigma)-1}{2} a(\tau)$$

Since $\delta_2^2(b \times a) = b \cup a$, we have that $b \cup a = 0$ in $H^2(G_{\mathbb{Q}}, \mathbb{Z}/2)$. Thus the corresponding Brauer-Severi variety is trivial. It is well known that $b \cup a$ corresponds to $u^2 = bv^2 + aw^2$. Thus we have $u, v, w \in \mathbb{Q}$ not all 0 such that $u^2 = bv^2 + aw^2$. By 3.4.4, we have that b is a square in $\mathbb{Q}_p$, and therefore that $\delta_3^2(b \times a)_c$ in $H^2(G_{\mathbb{Q}_p}, [\pi]_3^2/[\pi]_4^2)$ factors through the restriction $G_{\mathbb{Q}} \to G_{\mathbb{Q}(\sqrt{b})}$

Thus in $H^2(G_{\mathbb{Q}_p}, \mathbb{Z}/2)$:

$$\delta^2_{3,[[x,y],x]}(\mathbb{Q}_p)(b \times a)_c(\sigma, \tau) = \binom{b(\sigma)+1}{2} a(\tau) - \frac{\chi(\sigma)-1}{2} c(\tau)$$

$$\delta^2_{3,[[x,y],y]}(\mathbb{Q}_p)(b \times a)_c(\sigma, \tau) = c(\sigma)a(\tau) - \frac{\chi(\sigma)-1}{2} c(\tau) - \frac{\chi(\sigma)-1}{2} a(\tau)$$

Because $Dc = b \cup a$, c restricts to a cocycle in $C^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)$. Since an arbitrary lift of $(b \times a) \in H^1(G_{\mathbb{Q}_p}, \pi/[\pi]_2^2)$ to $H^1(G_{\mathbb{Q}_p}, \pi/[\pi]_3^2)$ differs from a given one by changing c by any cocycle, we have that c varies over all elements of $H^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)$.

Let $\Delta_3^2(\mathbb{Q}_p)$ denote the set of the images under $\delta_3^2(\mathbb{Q}_p) = (\delta^2_{3,[[x,y],x]}(\mathbb{Q}_p), \delta^2_{3,[[x,y],y]}(\mathbb{Q}_p))$ of lifts of $(b \times a) \in H^1(G_{\mathbb{Q}_p}, \pi/[\pi]_2^2)$:

$$\Delta_3^2(\mathbb{Q}_p) = \{(\delta^2_{3,[[x,y],x]}(\mathbb{Q}_p)(b \times a)_c, \delta^2_{3,[[x,y],y]}(\mathbb{Q}_p))(b \times a)_c : (b \times a)_c \text{ lifts } b \times a\}.$$

Then,

$$\Delta_3^2(\mathbb{Q}_p) = \{(\sqrt{b} \cup a + (-1) \cup \tilde{c}, \tilde{c} \cup a + (-1) \cup \tilde{c} + (-1) \cup a) : \tilde{c} \in H^1(G_{\mathbb{Q}_p}, \mathbb{Z}/2)\}$$

By Lemma 3.4.6, if b is a fourth power, we can choose a lift such that δ_3^2 vanishes.

If b is not a fourth power and -1 is a quadratic residue, $\sqrt{b} \cup a + (-1) \cup \tilde{c}$ will never vanish.

If b is not a fourth power and -1 is not a quadratic residue, letting $\tilde{c} = u + \mathfrak{p}$ in the notation of 3.4.3 shows that δ_3^2 vanishes on such a lift. □

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