

UNSTABLE MOTIVIC HOMOTOPY THEORY

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1. INTRODUCTION

Morel–Voevodsky’s \mathbb{A}^1 -homotopy theory transports tools from algebraic topology into arithmetic and algebraic geometry, allowing us to draw arithmetic conclusions from topological arguments. Comparison results between classical and \mathbb{A}^1 -homotopy theories can also be used in the reverse direction, allowing us to infer topological results from algebraic calculations. For example, see the article by Isaksen and Østvær on Motivic Stable Homotopy Groups [IØstvær18]. The present article will introduce unstable \mathbb{A}^1 -homotopy theory and give several applications.

Underlying all \mathbb{A}^1 -homotopy theories is some category of schemes. A special case of a scheme is that of an *affine scheme*, $\mathrm{Spec} R$, which is a topological space, the points of which are the prime ideals of a ring R and on which there is a sheaf of rings essentially provided by R itself. For example, when R is a finitely generated k -algebra, R can be written as $k[x_1, \dots, x_n]/\langle f_1, \dots, f_m \rangle$, and $\mathrm{Spec} R$ can be thought of as the common zero locus of the polynomials f_1, f_2, \dots, f_m , that is to say, $\{(x_1, \dots, x_n) : f_i(x_1, \dots, x_n) = 0 \text{ for } i = 1, \dots, m\}$. Indeed, for a k -algebra S , the set

$$(\mathrm{Spec} R)(S) := \{(x_1, \dots, x_n) \in S^n : f_i(x_1, \dots, x_n) = 0 \text{ for } i = 1, \dots, m\}$$

is the set of S -points of $\mathrm{Spec} R$, where an S -point is a map $\mathrm{Spec} S \rightarrow \mathrm{Spec} R$. We remind the reader that a scheme X is a locally ringed space that is locally isomorphic to affine schemes. So heuristically, a scheme is formed by gluing together pieces, each of which is the common zero locus of a set of polynomials. The topology on a scheme X is called the *Zariski topology*, with basis given by subsets U of affines $\mathrm{Spec} R$ of the form $U = \{\mathfrak{p} \in \mathrm{Spec} R : g \notin \mathfrak{p}\}$ for some g in R . It is too coarse to be of use for classical homotopy theory; for instance, the topological space of an irreducible scheme is contractible, having the generic point as a deformation retract. Standard references for the theory of schemes include [Har77], [Vak15], and for the geometric view that motivates the theory [EH00].

Both to avoid certain pathologies and to make use of technical theorems that can be proved under certain assumptions, \mathbb{A}^1 -homotopy theory restricts itself to considering subcategories of the category of all schemes. In the seminal [MV99] the restriction is already made to \mathbf{Sm}_S , the full subcategory of smooth schemes of finite type over a finite dimensional noetherian base scheme S . The smoothness condition, while technical, is geometrically intuitive. For example, when $S = \mathrm{Spec} \mathbb{C}$, the smooth S -schemes are precisely those whose \mathbb{C} -points form a manifold. The “finite-type” condition is most easily understood if $S = \mathrm{Spec} R$ is affine, in which case the finite-type S -schemes are those covered by finitely many affine schemes, each of which is determined by the vanishing of finitely many

polynomials in finitely many variables over the ring R . The most important case, and the best understood, is when $S = \text{Spec } k$ is the spectrum of a field. In this case \mathbf{Sm}_k is the category of smooth k -varieties.

In this article, we will use the notation S to indicate a base scheme, always assumed noetherian and finite dimensional. For the more sophisticated results in the sequel, it will be necessary to assume $S = \text{Spec } k$ for some field. We have not made efforts to be precise about the maximal generality of base scheme S for which a particular result is known to hold.

To see the applicability of \mathbb{A}^1 -homotopy theory, consider the following example. The topological Brouwer degree map

$$\text{deg} : [S^n, S^n] \rightarrow \mathbb{Z}$$

from pointed homotopy classes of maps of the n -sphere to itself can be evaluated on a smooth map

$$f : S^n \rightarrow S^n$$

in the following manner. Choose a regular value p in S^n and consider its finitely many preimages $f^{-1}(p) = \{q_1, \dots, q_m\}$. At each point q_i , choose local coordinates compatible with a fixed orientation on S^n . The induced map on tangent spaces can then be viewed as an \mathbb{R} -linear isomorphism $T_{q_i} f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with an associated Jacobian determinant $\text{Jac } f(q_i) = \det\left(\frac{\partial(T_{q_i} f)_j}{x_k}\right)_{j,k}$. The assumption that p is a regular value implies that $\text{Jac } f(q_i) \neq 0$, and therefore there is a local degree $\text{deg}_{q_i} f$ of f at q_i such that

$$\text{deg}_{q_i} f = \begin{cases} +1 & \text{if } \text{Jac } f(q_i) > 0, \\ -1 & \text{if } \text{Jac } f(q_i) < 0. \end{cases}$$

The degree $\text{deg } f$ of f is then given

$$\text{deg } f = \sum_{q \in f^{-1}(p)} \text{deg}_q f,$$

as the appropriate sum of $+1$'s and -1 's.

Lannes and Morel suggested the following modification of this formula to give a degree for an algebraic function $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ valued in nondegenerate symmetric bilinear forms. Namely, let k be a field and let $\text{GW}(k)$ denote the Grothendieck–Witt group, whose elements are formal differences of k -valued, non-degenerate, symmetric, bilinear forms on finite dimensional k -vector spaces. We will say more about this group in Section 4. For a in $k^\times / (k^\times)^2$, denote by $\langle a \rangle$ the element of $\text{GW}(k)$ determined by the isomorphism class of the bilinear form $(x, y) \mapsto axy$. For simplicity, assume that p is a k -point of \mathbb{P}^1 , i.e., p is an element of k or ∞ , and that the points q of $f^{-1}(p)$ are also k -points such that $\text{Jac } f(q) \neq 0$. Then the \mathbb{A}^1 -degree of f in $\text{GW}(k)$ is given by

$$\text{deg } f = \sum_{q \in f^{-1}(p)} \langle \text{Jac } f(q) \rangle.$$

In other words, the \mathbb{A}^1 -degree counts the points of the inverse image weighted by their Jacobians, instead of weighting only by the signs of their Jacobians.

Morel shows that this definition extends to an \mathbb{A}^1 -degree homomorphism

$$(1) \quad \text{deg} : [\mathbb{P}_k^n / \mathbb{P}_k^{n-1}, \mathbb{P}_k^n / \mathbb{P}_k^{n-1}] \rightarrow \text{GW}(k)$$

from \mathbb{A}^1 -homotopy classes of endomorphisms of the quotient $\mathbb{P}_k^n / \mathbb{P}_k^{n-1}$ to the Grothendieck–Witt group. We will use this degree to enrich results in classical enumerative geometry over \mathbb{C} to equalities in $\text{GW}(k)$ in section 4.5. For (1) to make sense, we must define the quotient $\mathbb{P}_k^n / \mathbb{P}_k^{n-1}$. It is not a scheme; it is a space in the sense of Morel–Voevodsky. In section 2, we sketch the construction of the homotopy theory of spaces, including Thom spaces and the Purity Theorem. In section 3, we discuss realization functors to topological spaces, which allow us to see how \mathbb{A}^1 -homotopy theory combines phenomena associated to the real and complex points of a variety. An example of this is the degree, discussed in the follow section, along with Euler classes. The Milnor–Witt K -theory groups are also introduced in section 4. These are the global sections of certain unstable homotopy sheaves of spheres. The following section, section 5, discusses homotopy sheaves of spaces, characterizing important properties. It also states the unstable connectivity theorem. The last section describes some beautiful applications to the study of algebraic vector bundles.

Some things we do not do in this article include Voevodsky’s groundbreaking work on the Bloch–Kato and Milnor conjectures. A superb overview of this is given [Mor98]. We also do not deal with any stable results, which have their own article in this handbook [IØstvæ18].

In our presentation, we concentrate on those aspects and applications of the theory that relate to the calculation of the unstable homotopy sheaves of spheres. The most notable of the applications, at present, is the formation of an \mathbb{A}^1 obstruction theory of BGL_n , and from there, the proof of strong results about the existence and classification of vector bundles on smooth affine varieties. These can be found in Section 6.

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2. OVERVIEW OF THE CONSTRUCTION OF UNSTABLE \mathbb{A}^1 -HOMOTOPY THEORY

2.1. Homotopy theory of Spaces. Morel and Voevodsky [MV99] constructed a “homotopy theory of schemes,” in a category sufficiently general to include both schemes and simplicial sets. By convention, the objects in such a category may be called a “motivic space” or an “ \mathbb{A}^1 -space” or, most commonly, simply a “space”. The underlying category of spaces is a category of simplicial presheaves on a category of schemes, where schemes themselves are embedded by means of a Yoneda functor. Two localizations are then performed: a Nisnevich localization and an \mathbb{A}^1 -localization. Before describing these, we clarify the notion of “a homotopy theory” that we use. Two standard choices are that a homotopy theory is a simplicial model category or a homotopy theory is an ∞ -category, a.k.a., a quasi-category. Background on simplicial model categories can be found in [Hir03] and [Rie14]. Lurie’s *Higher Topos Theory* [Lur09] contains many tools used

for doing \mathbb{A}^1 -homotopy theory with ∞ -categories. In the present case, either the model-categorical or the ∞ -categorical approach can be chosen, and once this choice has been made, presheaves of simplicial sets and the resulting localizations take on two different, although compatible, meanings, either of which produces a homotopy theory of schemes as described below.

To fix ideas, we use simplicial model categories. Let S be a noetherian scheme of finite dimension, and let \mathbf{Sm}_S denote the category of smooth schemes of finite type over S , defined as a full subcategory of schemes over S . Let $\mathbf{sPre}(\mathbf{Sm}_S)$ denote the category of functors from the opposite category of \mathbf{Sm}_S to simplicial sets, i.e., $\mathbf{sPre}(\mathbf{Sm}_S) = \text{Fun}(\mathbf{Sm}_S^{\text{op}}, \mathbf{sSet})$. Objects of $\mathbf{sPre}(\mathbf{Sm}_S)$ are called *simplicial presheaves*. The properties of $\mathbf{sPre}(\mathbf{Sm}_S)$ that make it desirable for homotopy theoretic purposes are the following:

- There is a *Yoneda embedding* $\eta : \mathbf{Sm}_S \rightarrow \mathbf{sPre}(\mathbf{Sm}_S)$, sending a scheme X to the simplicial presheaf $U \mapsto \mathbf{Sm}_S(U, X)$, where the set of maps is understood as a 0-dimensional simplicial set.
- The category $\mathbf{sPre}(\mathbf{Sm}_S)$ has all small limits and colimits. In this it is different from the category \mathbf{Sm}_S itself, which is far from containing all the colimits one might wish for in doing homotopy theory. The Yoneda embedding is continuous, but not cocontinuous.
- The category $\mathbf{sPre}(\mathbf{Sm}_S)$ is simplicial.

The Yoneda embedding will be tacitly used throughout to identify a smooth scheme X with the presheaf it represents, a 0-dimensional object of $\mathbf{sPre}(\mathbf{Sm}_S)$. We remark in passing that S itself represents a terminal object of $\mathbf{sPre}(\mathbf{Sm}_S)$.

One can put several, Quillen equivalent, model structures on $\mathbf{sPre}(\mathbf{Sm}_S)$ such that weak equivalences are detected objectwise, meaning that a map $X \rightarrow Y$ of simplicial presheaves is a weak equivalence if and only if the map $X(U) \rightarrow Y(U)$ of simplicial sets is a weak equivalence for all U in \mathbf{Sm}_S . Some standard choices for such a model structure are the injective [Jar87], projective [Bla01] and flasque [Isa05] [DHI04]. We then carry out two left Bousfield localizations. One can learn about Bousfield localization in [Hir03].

The first localization is analogous to the passage from presheaves to sheaves and depends on a choice of Grothendieck topology on \mathbf{Sm}_S . We remind the reader that a Grothendieck topology is not a topology in the point–set sense, rather it is sufficient data to allow one to speak meaningfully about locality and sheaves. We refer to [MLM92] for generalities on sheaves.

The standard choice of Grothendieck topology for \mathbb{A}^1 -homotopy theory is the *Nisnevich* topology, although the étale topology is also used, producing a different homotopy theory of spaces. If one wishes to study possibly nonsmooth schemes, then different topologies, for instance the cdh topology, can be used. The theory in the nonsmooth case is less well developed, but [Voe10b] establishes that the theory for all schemes and the cdh topology is a localization of the theory for smooth schemes and the Nisnevich topology, so that there is a localization functor L from one homotopy category to the other. Strikingly, the same paper proves that that for a map $f : X \rightarrow Y$

in the smooth, Nisnevich theory, Lf is a weak equivalence if and only if Σf , the suspension, is a weak equivalence.

In order to describe the Nisnevich topology, it is first necessary to describe étale maps: A map $Y \rightarrow X$ in \mathbf{Sm}_S is *étale* if for every point y of Y , the induced map on tangent spaces is an isomorphism [BLR90, 2.2 Proposition 8 and Corollary 10]. Many other characterizations of étale morphisms can be found in [Gro67, §17]. A finite collection of maps $\{U_i \rightarrow X\}$ is a *Nisnevich cover* if the maps $U_i \rightarrow X$ are étale and for every point $x \in X$, there is a point u in some U_i mapping to x such that the induced map on residue fields $\kappa(x) \rightarrow \kappa(u)$ is an isomorphism. The topology generated by this pretopology is the Nisnevich topology on \mathbf{Sm}_S . We remark that the additional constraint imposed by the Nisnevich condition is not vacuous even when $S = \text{Spec } \bar{k}$ is the spectrum of an algebraically closed field. For instance, when $|n| > 1$, the n -th power map $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$ —or more correctly $\mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}$ —is an étale cover but not a Nisnevich cover because the map on the residue fields of the generic points is the homomorphism $\mathbb{C}(x) \rightarrow \mathbb{C}(x)$ sending $x \mapsto x^n$, which is not an isomorphism. One sees that given Nisnevich cover $f : \coprod U_i \rightarrow X$, there must be a dense open subset V_0 of X on which f admits a section, and a dense open subset V_1 of the complement of V_0 on which f admits a (possibly different) section and so on, so that X has a stratification such that f admits sections on the open complements of the strata; this definition is used in [Del01, Section 3.1].

A hypercover is a generalization of the Čech cover

$$\dots \rightrightarrows \coprod_i (U_i \cap U_j) \rightrightarrows \coprod_i U_i \rightarrow X$$

associated to the cover $\{U_i \rightarrow X\}$. See, for example [Fri82, Chapter 3] or [DHI04, §4]. One may therefore define hypercovers for the Nisnevich topology. The first localization we carry out forces the maps $\text{hocolim}_n U_n \rightarrow X$ to be weak equivalences for every hypercover $U \rightarrow X$. This localization is called *Nisnevich localization* when the corresponding topology is the Nisnevich topology. The resulting simplicial model structures are referred to as the local model structures, and they are all Quillen equivalent via identity functors. The local injective model structure on presheaves was originally constructed by Jardine in [Jar87, Theorem 2.3], in a slight generalization of a construction due to Joyal [Joy83] who considered only sheaves rather than presheaves. What we call the “local injective” model structure is notably referred to the “simplicial model structure” in [MV99].

The theory constructed by Joyal and Jardine is very general, applying to all sites. We refer to [JSS15] for more about local homotopy theory *per se*. In the cases called for by \mathbb{A}^1 -homotopy, there is a more elementary approach to the theory. Building on ideas of Brown & Gersten [BG73], one defines a *distinguished square* of schemes to be a diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

of schemes where i is an open immersion, p is an étale morphism and p restricts to an isomorphism on the closed complements $p^{-1}(X - U) \rightarrow X - U$ (given the reduced induced subscheme structures). Then a functor $\mathcal{F} : \mathbf{Sm}_S^{\text{op}} \rightarrow \mathbf{Set}$ is a Nisnevich sheaf if \mathcal{F} takes distinguished squares to cartesian squares of sets, and a functor $\mathcal{X} : \mathbf{Sm}_S^{\text{op}} \rightarrow \mathbf{sSet}$ is, loosely speaking, suitable for \mathbb{A}^1 -homotopy theory if it takes distinguished squares to homotopy cartesian diagrams of simplicial sets. The following result, [Bla01, Lemma 4.1] is the simplest in a family of many such results.

Proposition 2.1. *A simplicial presheaf $\mathcal{X} : \mathbf{Sm}_S^{\text{op}} \rightarrow \mathbf{sSet}$ is fibrant in the projective local model structure if $\mathcal{X}(\cdot)$ takes values in Kan complexes and \mathcal{X} takes distinguished squares to homotopy cartesian squares.*

Variations on this idea are considered in [Isa05, Section 4.2], [AHW17a, Section 3.2] and [Voe10a], where we have listed the references in order from least to most general.

The second localization makes the projection maps $X \times \mathbb{A}^1 \rightarrow X$ into weak equivalences for all smooth schemes X in \mathbf{Sm}_S . Any resulting simplicial model structure is an \mathbb{A}^1 -model structure. A pleasant overview of the relationships between the various model categories appearing in this story can be found in [Isa05].

Definition 2.2. Let $R_{\mathbb{A}^1}$ denote a fibrant replacement functor in the injective \mathbb{A}^1 model structure. We will say that an object \mathcal{X} is \mathbb{A}^1 -local if the map $\mathcal{X} \rightarrow R_{\mathbb{A}^1}\mathcal{X}$ is a weak equivalence in a local model structure.

That is \mathbb{A}^1 -local objects are the objects for which the local homotopy type already recovers the \mathbb{A}^1 -homotopy type.

These simplicial model categories of spaces allows us to carry out homotopy theory on schemes. For example, we can form limits and colimits, in particular smash products of schemes or of schemes and simplicial sets, and give meaning to the spaces in (1). Many results from the classical homotopy theory of simplicial sets carry over. For example, we have excision: suppose that $Z \rightarrow X$ is a closed immersion in \mathbf{Sm}_k and $Y \rightarrow X$ is an open subset of X . Then there is a pushout square of schemes

$$(2) \quad \begin{array}{ccc} Y - Z & \longrightarrow & X - Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

corresponding to a Nisnevich cover of X . The Nisnevich localization procedure guarantees that Nisnevich covers give rise to pushouts in spaces, causing (2) to be a homotopy pushout, giving the excision weak-equivalence

$$X/(X - Z) \simeq Y/(Y - Z).$$

We also can take any vector bundle $p : E \rightarrow X$, where X is a smooth scheme, and decompose X as an open cover of subschemes U_i such that the induced maps $p^{-1}(U_i) \rightarrow U_i$ is isomorphic to the projection $\mathbb{A}^n \times U_i \rightarrow U_i$, which is a weak equivalence. A colimiting argument then shows that $p : E \rightarrow X$ is itself an \mathbb{A}^1 -weak equivalence. This works for any sort of Nisnevich-locally-trivial

fibration with \mathbb{A}^1 -contractible fibres, and in particular, for quasiprojective varieties over a field k , the Jouanolou trick [Wei89, Proposition 4.3] produces a map $p : \text{Spec } R \rightarrow X$ which is an \mathbb{A}^1 -equivalence having affine source. Any quasiprojective smooth k -variety is therefore \mathbb{A}^1 -equivalent to an affine variety.

2.2. Homotopy Sheaves. An inconvenience in \mathbb{A}^1 -homotopy theory is varying availability of basepoints. By a “basepoint” we might mean a map $p : S \rightarrow \mathcal{X}$ over S , viz., a morphism from the terminal object of the category $\mathbf{sPre}(\mathbf{Sm}_S)$. Unfortunately, since \mathcal{X} amounts to a family of simplicial sets, parametrized by $\mathbf{Sm}_S^{\text{op}}$, for certain objects $U \rightarrow S$ of $\mathbf{Sm}_S^{\text{op}}$, there may be path components of $\mathcal{X}(U)$ that are not in the image of the map $\pi_0 \mathcal{X}(S) \rightarrow \pi_0 \mathcal{X}(U)$. For instance, if V is the closed subscheme of \mathbb{A}^2 over \mathbb{R} determined by the vanishing of $x^2 + y^2 + 1$, then $V(\mathbb{R})$ is empty, while $V(\mathbb{C})$ is a discrete set of uncountably many points.

In order to handle this technicality, one must allow “basepoints” $x_0 \in \mathcal{X}(U)$ that are not in the image of $\mathcal{X}(S) \rightarrow \mathcal{X}(U)$. That is, for any object U of \mathbf{Sm}_S , and any $x_0 \in (\mathcal{X}(U))_0$ one defines a sequence of presheaves on the slice category \mathbf{Sm}_S/U by

$$(3) \quad (V \xrightarrow{f} U) \mapsto \pi_n(|\mathcal{X}(V)|, f^*(x_0)).$$

Here $|\mathcal{X}(V)|$ denotes the geometric realization of the simplicial set.

The Nisnevich sheaves associated to the functors of (3) are the *homotopy sheaves* of \mathcal{X} for the basepoint x_0 , and will be denoted $\pi_n(\mathcal{X}, x_0)$. If $n \geq 1$, then the sheaves are sheaves of groups, and they are abelian if $n \geq 2$. One may define an unpointed $\pi_0(\mathcal{X})$ similarly.

We define $\pi_n^{\mathbb{A}^1}$, the \mathbb{A}^1 -*homotopy sheaves*, by first replacing \mathcal{X} by an \mathbb{A}^1 local object—for instance, $R_{\mathbb{A}^1} \mathcal{X}$, then calculating π_n of the resulting object. Both the homotopy sheaves and the \mathbb{A}^1 -homotopy sheaves satisfy a Whitehead theorem (in fact, this is taken as the definition of local weak equivalence in [Jar87]).

Proposition 2.3. *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism in $\mathbf{sPre}(\mathbf{Sm}_S)$, then f is an \mathbb{A}^1 -weak equivalence if and only if the morphisms*

$$f_* : \pi_0^{\mathbb{A}^1}(\mathcal{X}) \rightarrow \pi_0^{\mathbb{A}^1}(\mathcal{Y})$$

and

$$f_* : \pi_n^{\mathbb{A}^1}(\mathcal{X}, x_0) \rightarrow \pi_n^{\mathbb{A}^1}(\mathcal{Y}, f(x_0))$$

are isomorphisms for all choices of basepoint $x_0 \in (\mathcal{X}(U))_0$.

The story is much simpler in the case where \mathcal{X} is \mathbb{A}^1 -connected, i.e., when $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is a singleton. In this case, non-globally-defined basepoints need not be considered. This version appears as [MV99, Proposition 2.14, §3].

One can define homotopy groups of \mathcal{X} by taking sections of the homotopy sheaves. One has the following

Proposition 2.4. *Let k be a field and let \mathcal{X} be an object of $\mathbf{sPre}(\mathbf{Sm}_k)$. Choose a basepoint $x_0 \in (\mathcal{X}(k))_0$. Then*

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}, x_0)(k) = [S^n \wedge (\mathrm{Spec} k)_+, \mathcal{X}]_{\mathbb{A}^1}$$

2.3. Spheres. Let \mathbb{G}_m denote the punctured affine line $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, given the basepoint 1, and let S^1 denote the pointed simplicial circle. The role of the spheres S^n in classical algebraic topology is now played by smash products of the spaces S^1 and \mathbb{G}_m . We use the notation

$$S^{p+q\alpha} = S^{p+q, q} := (S^1)^{\wedge p} \wedge \mathbb{G}_m^{\wedge q},$$

and such spaces will be called *spheres*. The presence of two different indexing conventions for spheres in \mathbb{A}^1 -homotopy theory can be confusing, but both appear in the literature, so we give both as well. The notation ΣX (or $\Sigma_{S^1} X$ if there is possible confusion) denotes $X \wedge S^1$. The question of which schemes have the \mathbb{A}^1 -homotopy type of spheres has been studied in [ADF17], and the most common examples are the following.

Example 2.5. The scheme \mathbb{P}^1 is \mathbb{A}^1 -homotopy equivalent to $S^{1+\alpha} = S^{2,1}$. Specifically, the pushout diagram

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

and the weak equivalence $\mathbb{A}^1 \simeq *$, induce a weak equivalence

$$\mathbb{P}^1 \simeq \Sigma \mathbb{G}_m.$$

Example 2.6. The scheme $\mathbb{A}^n - \{0\}$ is \mathbb{A}^1 -homotopy equivalent to $S^{n-1+n\alpha} = S^{2n-1, n}$. The case $n = 1$ is immediate, and then one may proceed by induction on n : the pushout diagram

$$\begin{array}{ccc} \mathbb{G}_m \times (\mathbb{A}^{n-1} - \{0\}) & \longrightarrow & \mathbb{A}^1 \times (\mathbb{A}^{n-1} - \{0\}) \\ \downarrow & & \downarrow \\ \mathbb{G}_m \times \mathbb{A}^n & \longrightarrow & \mathbb{A}^n - \{0\} \end{array}$$

and the weak equivalences $\mathbb{G}_m \times \mathbb{A}^n \simeq \mathbb{G}_m$ and $\mathbb{A}^1 \times (\mathbb{A}^{n-1} - \{0\}) \simeq \mathbb{A}^{n-1} - \{0\}$, induce a weak equivalence between $\mathbb{A}^n - \{0\}$ and the homotopy pushout of the diagram $Y \leftarrow X \times Y \rightarrow X$ of projection maps for $X = \mathbb{G}_m$ and $Y = \mathbb{A}^{n-1} - \{0\}$. Since this latter homotopy pushout is identified with $\Sigma(X \wedge Y)$ by pushing out the rows of the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \vee Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X \wedge Y & \longrightarrow & *, \end{array}$$

the result follows.

An important related example is the identification of the \mathbb{A}^1 -homotopy type of $\mathbb{P}_k^n/\mathbb{P}_k^{n-1}$ with $S^{n+n\alpha} = S^{2n,n}$.

Example 2.7. There is a standard closed immersion¹ $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ sending homogeneous coordinates $[x_0 : \dots : x_n]$ to $[0; x_0 : \dots : x_n]$. The homotopy type of the quotient space $\mathbb{P}^n/\mathbb{P}^{n-1}$ is identified by the chain of weak equivalences

$$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/(\mathbb{P}^n - \{0\}) \simeq \mathbb{A}^n/(\mathbb{A}^n - \{0\}) \simeq \Sigma(\mathbb{A}^n - \{0\})$$

with $\Sigma(\mathbb{A}^n - \{0\})$. Therefore, by example 2.6, we have $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq S^{n+n\alpha}$.

2.4. Thom spaces and purity. Let $V \rightarrow X$ be a vector bundle and let $X \hookrightarrow V$ be the zero section.

Definition 2.8. The *Thom space*, denoted $\mathrm{Th}(V)$ or X^V , of V is defined

$$\mathrm{Th}(V) := V/(V - X).$$

A parametrized version of Example 2.7 shows that the \mathbb{A}^1 -homotopy type of the Thom space over a scheme can be alternatively described as the cofiber of a closed immersion. Namely, let 1 denote the trivial bundle on X . For any vector bundle V on X , let $\mathbb{P}(V)$ denote the projective space bundle given by the fiberwise projectivization. As in Example 2.7, there is a standard closed immersion $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V \oplus 1)$. The complementary open subscheme is isomorphic to V .

Proposition 2.9. *There is an \mathbb{A}^1 -weak equivalence $\mathrm{Th}(V) \simeq \mathbb{P}(V \oplus 1)/\mathbb{P}(V)$.*

Proof. By excision, there is an \mathbb{A}^1 -weak equivalence $\mathrm{Th}(V) \simeq \mathbb{P}(V \oplus 1)/(\mathbb{P}(V \oplus 1) - X)$. The scheme $\mathbb{P}(V \oplus 1) - X$ is the total space of the vector bundle $\mathcal{O}(1)$ over $\mathbb{P}(V)$, giving the claimed weak equivalence. \square

The analogue of the tubular neighborhood theorem from classical topology is Morel–Voevodsky’s Purity theorem.

Purity Theorem 2.10 (Morel, Voevodsky). *Let $Z \hookrightarrow X$ be a closed immersion in \mathbf{Sm}_S . Then there is a natural \mathbb{A}^1 -weak equivalence*

$$\mathrm{Th}(N_Z X) \simeq X/(X - Z)$$

where $N_Z X \rightarrow Z$ denotes the normal bundle.

The original proof is [MV99, Section 3 Theorem 2.23 p.115], and a particularly accessible exposition, done over an algebraically closed field, may be found in [AE17, Section 7], using unpublished notes of A. Asok and [Hoy17].

¹We use the algebra-geometric term “closed immersion” for a map isomorphic to the inclusion of a closed subscheme. The usual term in topology is “closed embedding”, which is also used in [MV99], but is not widespread in the literature on \mathbb{A}^1 -homotopy theory.

Here, we will give an outline of a proof and draw some pictures in the case where $Z \hookrightarrow X$ is the inclusion of the point into the affine line $\{0\} \hookrightarrow \mathbb{A}_k^1$.

First, we explain the concept of “blow up” in algebraic geometry briefly. For a detailed treatment, we refer to [Har77]. Given a closed immersion $Z \hookrightarrow X$, the blow-up $\pi : \text{Bl}_Z X \rightarrow X$ of X along Z is a map satisfying the property that the restriction of π to $\text{Bl}_Z X - \pi^{-1}(Z)$ is an isomorphism and the inverse image of Z , called *the exceptional divisor*, is $\pi^{-1}Z \cong \mathbb{P}N_Z X$ the projectivization of the normal bundle. In other words, we cut out Z and glue in a projectivization of the normal bundle. In the topology of manifolds, this can be accomplished by first removing an open tubular neighborhood of Z from X , so that one has introduced a boundary component S homeomorphic to a sphere bundle in the normal bundle $N_Z X$, and then taking a fibrewise quotient of S to replace it by a projectivization of $N_Z X$. Algebraically, one takes the sheaf of ideals \mathcal{I} determining the closed immersion $Z \rightarrow X$, forms the sheaf of graded algebras $\bigoplus_{i=0}^{\infty} \mathcal{I}^i$ and takes the relative $\underline{\text{Proj}}$ construction, i.e., $\text{Bl}_Z X \rightarrow X$ is the canonical map $\underline{\text{Proj}}_X \bigoplus_{i=0}^{\infty} \mathcal{I}^i \rightarrow X$. The morphism $\pi : \text{Bl}_Z X \rightarrow X$ satisfies the property that the sheaf of ideals $(\pi^{-1}\mathcal{I})\mathcal{O}_{\text{Bl}_Z X}$ associated to $\pi^{-1}Z$ is the sheaf of sections of a line bundle, and any other morphism $Y \rightarrow X$ with this property factors through π , [Har77, II 7.14].

Example 2.11. The blow-up $\text{Bl}_0 \mathbb{A}^n$ of affine n -space \mathbb{A}^n at the origin is the closed subscheme of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ determined by $\langle x_i y_j - x_j y_i \rangle$ where (x_1, \dots, x_n) are the coordinates on \mathbb{A}^n and $[y_1, \dots, y_n]$ are homogeneous coordinates on \mathbb{P}^{n-1} . The restriction to $\text{Bl}_0 \mathbb{A}^n$ of the projection $\mathbb{A}^n \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ is the projection of the tautological bundle, denoted $\mathcal{O}(-1)$, on \mathbb{P}^{n-1} to \mathbb{P}^{n-1} . See Figure 1 for the case $n = 2$. The green line is the exceptional divisor \mathbb{P}^1 , and the red and blue lines are both fibers of the line bundle $\mathcal{O}(-1)$. Blow-ups of \mathbb{A}^2 are sometimes used as jungle gyms. See Figure 2.

An essential component of the proof of the Purity Theorem 2.10 is the “deformation to the normal bundle”, a pre-existing idea in intersection theory [Ful84, Chapter 5]. The input to this construction is a closed immersion $Z \hookrightarrow X$ in \mathbf{Sm}_k and the output is a family of closed immersions over \mathbb{A}^1 :

$$(4) \quad \begin{array}{ccc} Z \times \mathbb{A}^1 & \xrightarrow{\quad} & D_Z X \\ & \searrow & \swarrow \\ & \mathbb{A}^1 & \end{array}$$

such that over $t = 1$, the fiber of the family is the original closed immersion $Z \hookrightarrow X$, and over $t = 0$, the fiber of the family is the inclusion of the zero section into $N_Z X$.

To form this family, first consider $\text{Bl}_{Z \times 0}(X \times \mathbb{A}^1) \xrightarrow{\pi} X \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. The inverse image of $t = 0$ under the composition is $\pi^{-1}(X \times \{0\})$. It is the union of the exceptional divisor $\mathbb{P}(N_{Z \times 0}(X \times \mathbb{A}^1))$ and a copy of $\text{Bl}_{Z \times 0}(X \times 0)$. These are glued by identifying the copy of $\mathbb{P}N_Z X$ embedded in $\text{Bl}_{Z \times 0}(X \times 0)$ as the exceptional divisor, and the following copy of $\mathbb{P}N_Z X$ in $\mathbb{P}(N_{Z \times 0}(X \times \mathbb{A}^1))$: the bundle $N_{Z \times 0}(X \times \mathbb{A}^1)$ is canonically isomorphic to $N_Z X \oplus 1$. Thus $\mathbb{P}(N_{Z \times 0}(X \times \mathbb{A}^1))$ is the union of $N_Z X$ and $\mathbb{P}N_Z X$, and the latter $\mathbb{P}N_Z X$ is the copy we seek. Then let

$$D_Z X := \text{Bl}_{Z \times 0}(X \times \mathbb{A}^1) - \text{Bl}_{Z \times 0}(X \times 0).$$

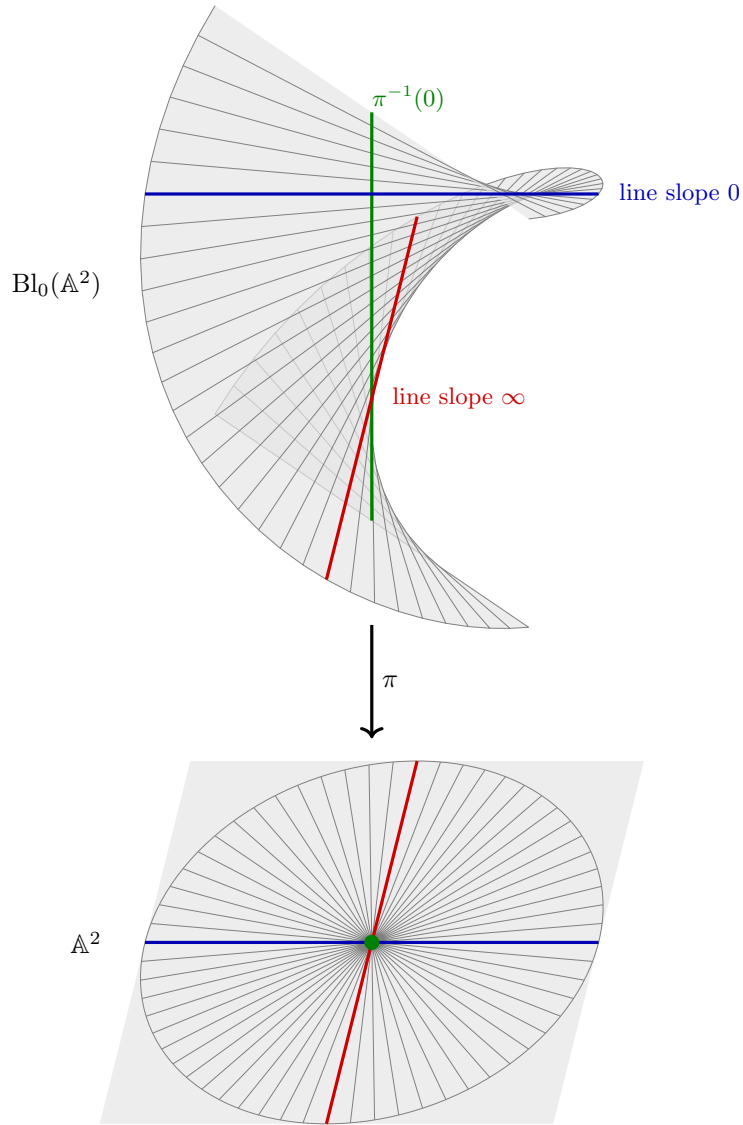


FIGURE 1.
Blow-up

Since the copy of $Z \times \mathbb{A}^1$ -in $\text{Bl}_{Z \times 0}(X \times \mathbb{A}^1)$ intersects the exceptional divisor as the zero section of $N_Z X$ in $\mathbb{P}(N_{Z \times 0}(X \times \mathbb{A}^1))$, the scheme $D_Z X$ provides the claimed family. $D_0 \mathbb{A}^1$ is shown in Figure 3.

To prove purity, one shows that the fibers of (4) above $t = 0$ and $t = 1$ induce \mathbb{A}^1 -weak equivalences

$$i_0 : N_Z X / (N_Z X - Z) \rightarrow D_Z X / (D_Z X - Z \times \mathbb{A}^1_k)$$



FIGURE 2. $\text{Bl}_0 \mathbb{A}^2$ Jungle Gym

and

$$i_1 : X/(X - Z) \rightarrow D_Z X/(D_Z X - Z \times \mathbb{A}^1),$$

respectively. Like the étale topology, the Nisnevich topology is fine enough to reduce many arguments about closed immersions in \mathbf{Sm}_S to arguments for the standard inclusion $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+c}$. A precise theorem enabling such reductions may be found in [GR71, II 4.10].

In the case where $Z \hookrightarrow X$ is $0 \hookrightarrow \mathbb{A}^n$, the deformation to the normal bundle

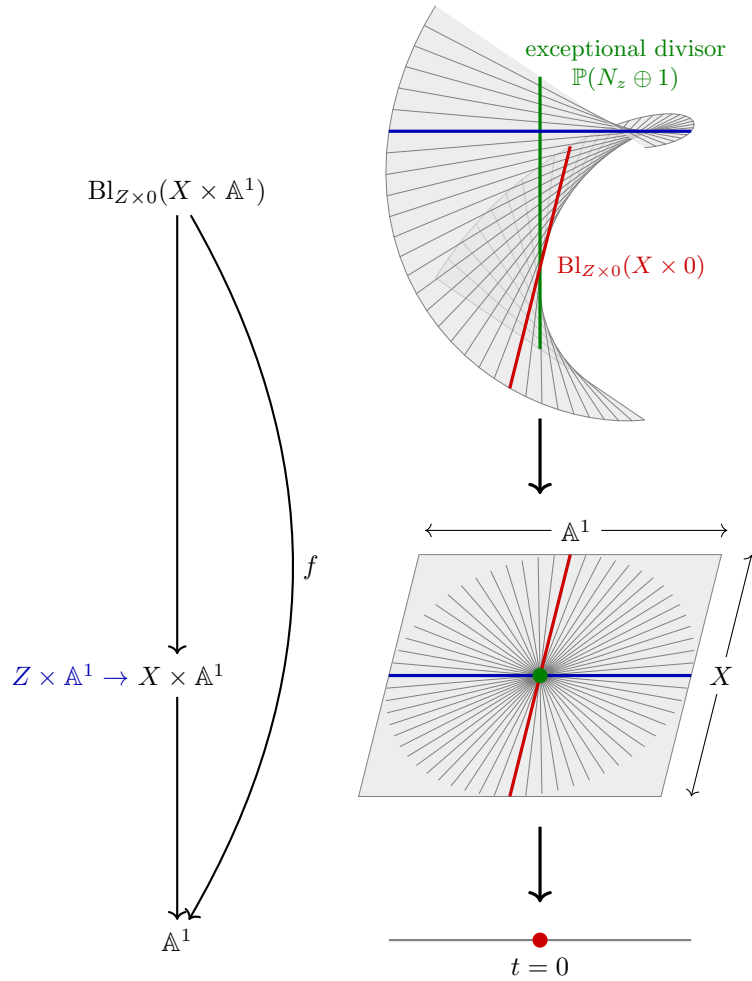
$$D_Z X = \text{Bl}_0(\mathbb{A}^n \times \mathbb{A}^1) - \text{Bl}_0(\mathbb{A}^n \times 0) \cong \mathcal{O}_{\mathbb{P}_k^n}(-1) - \mathcal{O}_{\mathbb{P}_k^{n-1}}(-1) \cong \mathcal{O}_{\mathbb{P}_k^n}(-1)|_{\mathbb{A}_k^n}$$

is the restriction of the total space of the tautological bundle on \mathbb{P}_k^n restricted to the standard copy of \mathbb{A}^n . The fiber above 0 is the zero-section of this bundle. The fiber above 1 also defines a section. We can see directly that i_0 and i_1 are \mathbb{A}^1 -weak equivalences.

3. REALIZATIONS

3.1. Complex Realization. Suppose $k \subset \mathbb{C}$ is a subfield of the complex numbers, then one may view the category \mathbf{Sm}_k of smooth k -schemes as a category of complex varieties. Let us abuse notation and write $X(\mathbb{C})$ for the complex manifold determined by a smooth k -scheme \mathbf{Sm}_k . The object $X(\mathbb{C})$, being a manifold, is the sort of thing to which classical homotopy theory is well adapted. One therefore might hope for a functor $|\cdot| : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{Top}$ with good homotopical properties. Such a functor can be constructed in the most obvious way: for a smooth k -scheme X , one defines $|X| = X(\mathbb{C})$, as discussed above. The functor

$$|\cdot| : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{Top}$$



$$D_z(X) = \text{Bl}_{Z \times 0}(X \times \mathbb{A}^1) - \text{Bl}_{Z \times 0}(X \times 0)$$

FIGURE 3.
Deformation to the normal bundle

is then extended to all simplicial presheaves in such a way as to preserve (homotopy) colimits. For formal reasons, if K is a simplicial set, viewed as a constant simplicial presheaf in $\mathbf{sPre}(\mathbf{Sm}_k)$, then this realization $|K|$ agrees with the usual topological realization of the simplicial set K .

The technical details in showing that this realization procedure works out is to be found in [Dug01], [DHI04] and [DI04]. The first paper shows that the motivic projective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ is an \mathbb{A}^1 -localization of a “universal” model structure, and then the last two show that the relations imposed in the motivic projective model structure, namely making hypercovers and

trivial line bundles into weak equivalences, are relations that yield ordinary homotopy equivalences after realization. The upshot is a Quillen adjunction

$$|\cdot| : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows \mathbf{Top} : S,$$

when the left hand side is endowed with the motivic projective model structure. The approach outlined above is an improvement on a result of [MV99] where a functor $t^{\mathbb{C}} : \mathcal{H}(\mathbb{C}) \rightarrow \mathcal{H}$ is produced, but only on the level of homotopy categories.

Being a left adjoint, the realization functor constructed in this way is not compatible with fiber sequences, but one sees that it is compatible with products, in that $|X \times Y| \approx |X| \times |Y|$, [PPR09, Section A.4] and as a consequence, one may deduce

$$|B(X, G, Y)| \approx B(|X|, |G|, |Y|)$$

for two-sided bar constructions. This may suffice in certain cases.

Since realization is homotopically well behaved, one obtains maps $[X, Y]_{\mathbb{A}^1_{\mathbb{C}}} \rightarrow [|X|, |Y|]$. This map is understood when $X = S^{n+m\alpha}$ and $Y = S^{n+j\alpha}$, assuming $n \geq 2$ and $j \geq 1$, in which case one has maps

$$(5) \quad \mathbf{K}_{j-m}^{\text{MW}}(\mathbb{C}) \rightarrow \pi_{n+m}(S^{n+j}).$$

For dimensional reasons, this map vanishes when $j > m$. When $j = m$, the identity map generates both source and target, and one has an isomorphism $\mathbb{Z} = \text{GW}(\mathbb{C}) \rightarrow \mathbb{Z} = \pi_{n+m}(S^{n+m})$. In the cases where $j < m$, one knows from [Mor12] that $\mathbf{K}_{j-m}^{\text{MW}}(\mathbb{C}) = W(\mathbb{C})$ contains a single nontrivial class, that of η^{j-m} , the iterated Hopf map, and so the image of the realization map (5) is the group generated by η_{top}^{j-m} , which vanishes if $j - m$ is sufficiently large.

3.2. Real Realization. Suppose now that $k = \mathbb{R}$. Take $X(\mathbb{C}) = \text{Mor}_{\mathbb{R}}(\text{Spec } \mathbb{C}, X)$ and recall that the Galois action $C_2 = \text{Gal}(\mathbb{C}/\mathbb{R})$ endows this manifold with a C_2 action. Extending the functor $X \mapsto X(\mathbb{C})$ in such a way as to preserve homotopy colimits, one produces an adjunction:

$$|\cdot|_{\text{equi}} : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows C_2 - \mathbf{Top} : S_{\mathbb{R}}.$$

A map $f : X \rightarrow Y$ of C_2 -spaces is a weak equivalence if and only if it induces weak equivalences after taking H fixed points $f^H : X^H \rightarrow Y^H$ for both $H = C_2$ and $H = \{e\}$. With this definition, the functor $|\cdot|_{\text{equi}}$ is compatible with the homotopy—specifically [DI04, Section 5.3] establishes that $|\cdot|_{\text{equi}}$ is a left Quillen functor when the source is endowed with the motivic projective model structure, the arguments being largely the same as in the complex case.

Composing $|\cdot|_{\text{equi}}$ with the functor taking C_2 fixed points, which is a right adjoint, one arrives at a functor $X \mapsto |X|_{\text{equi}}^{C_2} := |X|_{\mathbb{R}}$, and if X is a scheme, then $|X|_{\mathbb{R}}$ is the analytic space $X(\mathbb{R})$ endowed with the obvious topology. The functor $|\cdot|_{\mathbb{R}}$ cannot be a left adjoint, since it involves the taking of fixed points, it does not preserve all colimits, but it does preserve smash products. Notably, $|S^{n+m\alpha}|_{\mathbb{R}} \simeq S^n \wedge |\mathbb{G}_m^{\wedge m}|_{\mathbb{R}} \simeq S^n$, and so the induced maps

$$\pi_{n+m\alpha}(S^{n+j\alpha})(\mathbb{R}) = \mathbf{K}_{j-n}^{\text{MW}}(\mathbb{R}) \rightarrow \pi_n(S^n) = \mathbb{Z}$$

are quite different in character from those in the complex case of (5) above. The ring $\mathbf{K}_*^{\text{MW}}(\mathbb{R})$ is generated by classes $[a] \in \mathbf{K}_1^{\text{MW}}$ where $a \in \mathbb{R}^\times$ and $\eta \in \mathbf{K}_{-1}^{\text{MW}}$. Under \mathbb{R} -realization, $[a] \mapsto 0$ if $a > 0$, and $[a] \mapsto 1$ if $a < 0$, and $\eta \mapsto -2$, for a proof, see [AFW18][Proposition 3.1.3].

3.3. Étale realization. In the case of an arbitrary Noetherian base scheme, S , there is an *étale realization* functor, constructed in [Isa04], operating along similar lines to the above two functors. In this case, one begins with a functor $\mathbf{Sm}_S \rightarrow \text{pro-}\mathbf{sPre}(\mathbf{Sm}_k)$, taking a smooth S -scheme X to the étale topological type of X , as defined in [Fri82]. One then extends this functor to all of $\mathbf{sPre}(\mathbf{Sm}_S)$ by requiring it to commute with homotopy colimits. The result is again a left Quillen functor

$$|\cdot|_{\text{ét}} : \mathbf{sPre}(\mathbf{Sm}_S) \rightleftarrows \text{pro-}\mathbf{Top} : S$$

with a certain extra complication since the target category is a category of pro-objects.

It might be hoped that if $S = \text{Spec } k$ is a field, then $\text{Gal}(\bar{k}/k)$ should act on $|X|_{\text{ét}}$ and the target category might be enriched to $\text{Gal}(\bar{k}/k)$ -equivariant-pro-spaces. The extent to which this can be done is not known, owing in part to the technical difficulty of working with equivariant pro-spaces, and in part because it is known that not all desirable properties for such a functor are attainable: [KW18].

4. DEGREE

4.1. The Grothendieck–Witt group. Equipped with definitions of spheres and \mathbb{A}^1 -homotopy classes of maps, we consider again Morel’s \mathbb{A}^1 -degree homomorphism

$$\text{deg} : [S^n \wedge \mathbb{G}_m^{\wedge n}, S^n \wedge \mathbb{G}_m^{\wedge n}] \rightarrow \text{GW}(k)$$

where the target is the Grothendieck–Witt group.

The Grothendieck–Witt group $\text{GW}(k)$ is both complicated enough to support interesting invariants and simple enough to allow explicit computations. It is defined to be the group completion of the semi-ring of isomorphism classes of k -valued, non-degenerate, symmetric, bilinear forms on finite dimensional k -vector spaces. In more detail: The isomorphism classes of non-degenerate, symmetric, bilinear forms $\beta : V \times V \rightarrow k$ over k , where V is a finite dimensional k -vector space, admit the operations of perpendicular direct sum \oplus and tensor product \otimes . These operations give the set of such isomorphism classes the structure of a semi-ring. Taking the group completion, i.e., introducing formal differences of isomorphism classes, defines the ring $\text{GW}(k)$.

$\text{GW}(k)$ has a presentation [Lam05, II Theorem 4.1 pg 39] given by generators $\langle a \rangle$ for a in k^\times and the relations

- (1) $\langle a \rangle = \langle ab^2 \rangle$ for all $a, b \in k^\times$.
- (2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for all $a, b \in k^\times / (k^\times)^2$
- (3) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for all $a, b \in k^\times$ such that $a + b \neq 0$.

The relations (2) and (3) imply that

$$\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: h$$

for all $a \in k^\times / (k^\times)^2$, where h is the hyperbolic form, defined $h = \langle 1 \rangle + \langle -1 \rangle$.

As mentioned above, the generator $\langle a \rangle$ corresponds to the bilinear form $k \times k \rightarrow k$ taking (x, y) to axy . The fact that $\{\langle a \rangle : a \in k^\times / (k^\times)^2\}$ is a set of generators is thus equivalent to the statement from bilinear algebra that a symmetric bilinear form can be stably diagonalized, meaning that after potentially adding another form (which is only necessary in characteristic 2), the original form is a perpendicular direct sum of bilinear forms on 1-dimensional vector spaces.

The *rank* of the bilinear form $\beta : V \times V \rightarrow k$ is the dimension of V as a k -vector space. So for example, the generators $\langle a \rangle$ for a in $k^\times / (k^\times)^2$ are the isomorphism classes of all non-degenerate rank one forms. The rank defines a ring homomorphism

$$\text{rank} : \text{GW}(k) \rightarrow \mathbb{Z}.$$

The kernel of the rank homomorphism

$$I := \text{Ker}(\text{rank})$$

is called the fundamental ideal and gives rise to a filtration of GW

$$\text{GW} \supseteq I \supseteq I^2 \supseteq \dots$$

related to étale cohomology and Milnor K-theory by the Milnor Conjecture [Mil70], which is now a theorem of Voevodsky proven using \mathbb{A}^1 -homotopy theory [Voe03a] [Voe03b]. Specifically, assume that the characteristic of k is not 2 and let $H^n(k, \mathbb{Z}/2)$ denote the étale cohomology of $\text{Spec } k$ with $\mathbb{Z}/2$ coefficients, or equivalently, the (continuous) cohomology of the Galois group of the separable closure of k with $\mathbb{Z}/2$ coefficients. Let $K_n^M(k)$ denote the degree- n summand of the Milnor K-theory of k , [Mil70]. For present purposes, we only state that $K_n^M(k)$ is the quotient of the n -fold tensor product $k^\times \otimes k^\times \otimes \dots \otimes k^\times$ by the group generated by tensors $a_1 \otimes a_2 \otimes \dots \otimes a_n$ where there is some i such that $a_i + a_{i+1} = 1$. In degrees 0, 1 one has $K_0^M(k) = \mathbb{Z}$ and $K_1^M(k) = k^\times$. The Kummer map $k^\times \rightarrow H^1(k, \mathbb{Z}/2)$ is the first boundary map obtained by taking the Galois cohomology of the exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow k_s^\times \xrightarrow{z \mapsto z^2} k_s^\times \rightarrow 1,$$

where k_s denotes the separable closure of k . It extends to a ring homomorphism $K_*^M(k) \otimes \mathbb{Z}/2 \rightarrow H^*(k, \mathbb{Z}/2)$. The Milnor conjecture says there are isomorphisms

$$I^n / I^{n+1} \cong K_n^M(k) \otimes \mathbb{Z}/2 \cong H^n(k, \mathbb{Z}/2),$$

where the second isomorphism is induced by the map just discussed coming from the Kummer map and the first sends

$$a_1 \otimes a_2 \otimes \dots \otimes a_n$$

in $K_n^M(k)$ to

$$(\langle a_1 \rangle - \langle 1 \rangle)(\langle a_2 \rangle - \langle 1 \rangle) \cdots (\langle a_n \rangle - \langle 1 \rangle)$$

in I^n / I^{n+1} .

The maps $I^n \rightarrow I^n/I^{n+1} \rightarrow H^n(k, \mathbb{Z}/2)$ define invariants on bilinear forms in I^n . The 0th of these is by definition the rank. The first is the *discriminant*

$$\text{Disc} : \text{GW}(k) \rightarrow H^1(k, \mathbb{Z}/2) \cong k^\times / (k^\times)^2,$$

which sends the bilinear form $\beta : V \times V \rightarrow k$ to the determinant of a Gram matrix for β

$$\text{Disc}(\beta) = (\beta(v_i, v_j))_{i,j}$$

where $\{v_1, \dots, v_n\}$ is a basis of V . The second is the Hasse–Witt invariant

$$\gamma : \text{GW}(k) \rightarrow H^2(k, \mathbb{Z}/2) \cong K_2^M(k) \otimes \mathbb{Z}/2,$$

which takes $\langle a_1 \rangle + \langle a_2 \rangle + \dots + \langle a_m \rangle$ to the image of

$$\sum_{i < j} a_i \otimes a_j$$

in $K_2^M(k) \otimes \mathbb{Z}/2$. If we identify $H^2(k, \mathbb{Z}/2)$ with the 2-torsion subgroup of the Brauer group of equivalence classes of central simple k -algebras, the class $\gamma(\langle a_1 \rangle + \langle a_2 \rangle + \dots + \langle a_m \rangle)$ is that of the tensor product of the quaternion algebras (a_i, a_j) as (i, j) runs over pairs of integers from 1 to m with $i < j$. The third is the Arason invariant. The higher invariants are not explicitly named, but do not vanish in general. For example, in the Grothendieck–Witt group of the field $\mathbb{C}((x_1, x_2, \dots, x_n))$, the n th quotient I^n/I^{n+1} is not 0.

For many fields k , there are explicit computations of $\text{GW}(k)$ and algorithms for deciding when two sums of the given generators represent the same element. See [Lam05].

Example 4.1. When $k = \mathbb{C}$ or any algebraically closed field, $k^\times / (k^\times)^2$ is the one element group, so by the previously given presentation of $\text{GW}(k)$, we see that $\text{rank} : \text{GW}(k) \rightarrow \mathbb{Z}$ is an isomorphism, e.g., $\text{GW}(\mathbb{C}) \cong \mathbb{Z}$.

Example 4.2. For a bilinear form $\beta : V \times V \rightarrow \mathbb{R}$ over the real numbers, Sylvester’s law of inertia states that there is a basis of V so that β is diagonal with only 1’s, -1 ’s and 0’s on the diagonal. The *signature* of β is the number of 1’s minus the number of -1 ’s. The signature determines a homomorphism $\text{sign} : \text{GW}(k) \rightarrow \mathbb{Z}$. Non-degenerate bilinear forms over \mathbb{R} are classified by their rank and signature, and thus the homomorphism

$$\text{rank} \times \text{sign} : \text{GW}(\mathbb{R}) \rightarrow \mathbb{Z} \times \mathbb{Z}$$

determines an isomorphism from $\text{GW}(\mathbb{R})$ onto its image $\text{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$.

Example 4.3. For a finite field \mathbb{F}_q , the rank and the discriminant define a group isomorphism $\text{GW}(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$. When q is odd, note that this means that the Grothendieck–Witt group is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$.

Springer’s theorem [Lam05, VI Theorem 1.4] computes the Grothendieck–Witt group of \mathbb{Q}_p and $\mathbb{C}((x_1, x_2, \dots, x_n))$. A theorem of Milnor gives the analogous computation for function fields [Mil70, Theorem 5.3]. For $\text{GW}(\mathbb{Q})$, see [Lam05, VI 4]. In summary, the Grothendieck–Witt group is interesting and computable.

Morel's degree has the agreeable feature that for subfields k of \mathbb{R} , it records the ordinary topological degrees of both the complex and real realizations. Namely, the following diagram commutes.

$$\begin{array}{ccccc}
\mathbf{Top} & \xleftarrow{|\cdot|_{\mathbb{R}}} & \mathbf{sPre}(\mathbf{Sm}_k) & \xrightarrow{|\cdot|} & \mathbf{Top} \\
\downarrow \text{deg} & & \downarrow \text{deg} & & \downarrow \text{deg} \\
\mathbb{Z} & \xleftarrow{\text{rank}} & \mathbf{GW}(k) & \xrightarrow{\text{sign}} & \mathbb{Z}
\end{array}$$

4.2. Local Degree. As in the case of the degree morphism from classical algebraic topology, there is an associated notion of “local degree”, defined as follows. Let U and W be Zariski open subsets of \mathbb{A}_k^n , and let $f : U \rightarrow W$ be a map. Suppose x is a closed point of U such that $y = f(x)$ is a k -point (meaning that the coordinates of y are elements of k), and such that x is isolated in $f^{-1}(y)$. By possibly shrinking U , we may assume that f maps $U - \{x\}$ to $W - \{y\}$, and therefore f induces a map

$$f : U/(U - \{x\}) \rightarrow W/(W - \{y\}).$$

The Purity Theorem identifies the quotients $W/(W - \{y\})$ and $U/(U - \{x\})$ with the sphere $(\mathbb{P}^1)^{\wedge n}$ and the smash product $(\mathbb{P}^1)^{\wedge n} \wedge \text{Spec } k(x)_+$, respectively. The *local degree* $\text{deg}_x f$ of f at x is the degree of the composite

$$\mathbb{P}^n/\mathbb{P}^{n-1} \rightarrow \mathbb{P}^n/(\mathbb{P}^n - \{x\}) \cong U/(U - \{x\}) \xrightarrow{f} W/(W - \{y\}) \cong (\mathbb{P}^1)^{\wedge n} \cong \mathbb{P}^n/\mathbb{P}^{n-1}$$

in $\mathbf{GW}(k)$. If x is also a k -point, it is more natural to take the degree of the composite

$$(\mathbb{P}^1)^{\wedge n} \rightarrow U/(U - \{x\}) \xrightarrow{f} W/(W - \{y\}) \rightarrow (\mathbb{P}^1)^{\wedge n}.$$

These definitions are equivalent [KW16, Proposition 11]. In the presence of Nisnevich local coordinates (see [KW17, Definition 17]) and appropriate orientations, the definition of local degree can be generalized.

There is an explicit algorithm to compute the local degree [EL77] [Eis78] [KW16], which can be implemented with a computer algebra package, or even by hand in good circumstances. Namely, consider a map $f : U \rightarrow W$, and a point x in U as above. Since W is a subset of \mathbb{A}_k^n , we can express f as an n -tuple of functions $f = (f_1, \dots, f_n)$ defined on U . Since U is an open subset of \mathbb{A}_k^n , the point x corresponds to a prime ideal p in $k[x_1, \dots, x_n]$. The f_i can be viewed as elements of the localization $k[x_1, \dots, x_n]_p$. The assumption that x is isolated in $f^{-1}(y)$ implies that the quotient

$$Q = \frac{k[x_1, \dots, x_n]_p}{\langle f_1, \dots, f_n \rangle}$$

is finite dimensional as a k -vector space. The \mathbb{A}^1 -local degree of f at p is represented by the following bilinear form on Q . The Jacobian

$$J = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j}$$

determines an element of Q . For simplicity, we assume that p is a k -point, and that $\text{rank}_k Q$ is not divisible by the characteristic of k . Choose any k -linear function

$$\eta : Q \rightarrow k$$

such that

$$\eta(J) = \text{rank}_k Q.$$

Then define the bilinear form $\omega^{\text{EKL}} : Q \times Q \rightarrow k$ by

$$\omega^{\text{EKL}}(g, h) = \eta(gh).$$

It can be shown that the isomorphism class of ω^{EKL} does not depend on the choice of η [EL77, Proposition 3.5] and that the class of ω^{EKL} in $\text{GW}(k)$ is the \mathbb{A}^1 -local degree [KW16]. For the analogous algorithm without the two assumptions, see the work of Scheja and Storch [SS75, Section 3].

As in classical algebraic topology, the degree of a map can be expressed as a sum of local degrees. For example, let $f : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ be a finite map. Let \mathbb{A}_k^n denote the Zariski open subset of \mathbb{P}_k^n which is the complement of the standard closed immersion $\mathbb{P}_k^{n-1} \hookrightarrow \mathbb{P}_k^n$ described in Example 2.7. Suppose that $f^{-1}(\mathbb{A}_k^n) = \mathbb{A}_k^n$, so there is an induced map

$$F : \mathbb{P}_k^n / \mathbb{P}_k^{n-1} \rightarrow \mathbb{P}_k^n / \mathbb{P}_k^{n-1}.$$

Then, for any k -point y of \mathbb{A}_k^n ,

$$\deg F = \sum_{x \in f^{-1}(y)} \deg_x f.$$

This is well-known and the proof is very similar to a classical argument. A reference is [KW16, Proposition 13].

4.3. Milnor–Witt groups. The ring $\text{GW}(k)$ is the degree 0 subring of a graded ring $\mathbf{K}_*^{\text{MW}}(k)$ developed by Morel and M. J. Hopkins [Mor12] to describe part of the motivic homotopy groups of spheres. For a field k , the ring $\mathbf{K}_*^{\text{MW}}(k)$ is the associative, graded ring with the following presentation. The generators consist of one element η of degree -1 and elements $[a]$ of degree 1 for each a in k^\times . The hyperbolic element h is defined $h := \eta[-1] + 2$. These generators are subject to the relations

- (Steinberg relation) $[a][1 - a] = 0$ for each a in $k - \{0, 1\}$.
- $[ab] = [a] + [b] + \eta[a][b]$ for every a, b in k^\times .
- $[a]\eta = \eta[a]$ for every a in k^\times .
- $\eta h = 0$.

One can identify $\text{GW}(k)$ with $\mathbf{K}_0^{\text{MW}}(k)$ by identifying $\langle a \rangle$ with $1 + \eta[a]$,

$$\langle a \rangle = 1 + \eta[a],$$

and it follows that the two definitions $h = \eta[-1] + 2$ and $h = \langle 1 \rangle + \langle -1 \rangle$ agree. Morel proves an isomorphism between \mathbb{P}^1 -stable \mathbb{A}^1 -homotopy classes of maps and $\mathbf{K}_*^{\text{MW}}(k)$

$$[S^0, \mathbb{G}_m^{\wedge n}]_{\mathbf{Spt}(\mathbf{Sm}_k)} \cong \mathbf{K}_n^{\text{MW}}(k).$$

Moreover, there is an analogous unstable result:

$$[S^n \wedge \mathbb{G}_m^{\wedge r}, S^n \wedge \mathbb{G}_m^{\wedge j}] \cong \mathbf{K}_{j-r}^{\text{MW}}(k)$$

for $n \geq 2$, which extends to an isomorphism of sheaves discussed in Section 5. For $n = 1$, we are lead to consider the unstable \mathbb{A}^1 -homotopy classes of maps $\mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$, also discussed in Section 5; see Equation (14).

The fact that $\mathrm{GW}(k)$ and $\mathbf{K}_*^{\mathrm{MW}}(k)$ are global sections of stable \mathbb{A}^1 -homotopy sheaves $\pi_*^{\mathbb{A}^1, s}$ implies the existence of extra structure, for example residue maps, which are closely related to the unramified sheaves discussed in more detail in the next section, and transfer maps, which exist for 2-fold \mathbb{G}_m -loop sheaves of \mathbb{A}^1 -homotopy sheaves $\pi_*^{\mathbb{A}^1}$ for $* \geq 2$, and in particular exist for stable \mathbb{A}^1 -homotopy sheaves [Mor12, Chapter 4]). We discuss these residue and transfer maps in more detail now.

We have a residue map in the following situation. Let $v : K \rightarrow \mathbb{Z} \cup \infty$ be a valuation on a field K containing k , and let \mathcal{O}_v denote the corresponding valuation ring of elements of non-negative valuation. The residue field $k(v)$ is the quotient of \mathcal{O}_v by its maximal ideal consisting of the elements of K of positive valuation. Let π be a uniformizer (i.e., an element such that $v(\pi) = 1$). Then there is a unique homomorphism ∂_v^π called a *residue map*

$$\partial_v^\pi : \mathbf{K}_*^{\mathrm{MW}}(K) \rightarrow \mathbf{K}_{*-1}^{\mathrm{MW}}(k(v))$$

commuting with η and such that

$$\begin{aligned} \partial_v^\pi([\pi][a_1][a_2] \dots [a_n]) &= [\overline{a_1}][\overline{a_2}] \dots [\overline{a_n}], \\ \partial_v^\pi([a_0][a_1][a_2] \dots [a_n]) &= 0, \end{aligned}$$

where $a_i \in \mathcal{O}_v^*$ and $\overline{a_i}$ denotes the image of a_i in $k(v)^*$. The kernel of the residue maps define the sections of the sheaf $\mathbf{K}_*^{\mathrm{MW}}$ over $\mathrm{Spec} \mathcal{O}_v$, see the discussion following [Mor12][Lemma 3.19].

We have a transfer map in the following situation. Let $K \subseteq L$ be a field extension of finite rank, where K is finite type over k .

We first note that the inclusion $K \subseteq L$ corresponds to a map of schemes (or spaces) $\mathrm{Spec} L \rightarrow \mathrm{Spec} K$ in the opposite direction. The sheaf property (which comes from pullback of maps) gives restriction maps $\mathbf{K}_*^{\mathrm{MW}}(K) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(L)$. These restriction maps are, not surprisingly, $[a] \mapsto [a]$, $\eta \mapsto \eta$, and correspond to pullback of bilinear forms when restricted to GW .

Transfer maps for such field extensions can be constructed by producing a stable map

$$\mathrm{Spec} K \rightarrow \mathrm{Spec} L$$

in the direction opposite to the map of spaces, closely analogous to the Becker–Gottlieb transfer [Mor12, 4.2]. Namely, when L is generated by a single element over K , we can choose a closed point z of \mathbb{P}_K^1 with residue field L , or equivalently, the data of a closed immersion

$$z : \mathrm{Spec} L \hookrightarrow \mathbb{P}_K^1.$$

Using Purity, we then have the cofiber sequence

$$(\mathbb{P}_K^1 - \{z\}) \rightarrow \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1 / (\mathbb{P}_K^1 - \{z\}) \cong \mathrm{Th}(N_z \mathbb{P}_K^1) \cong \mathbb{P}_k^1 \wedge (\mathrm{Spec} L)_+.$$

The quotient map, or Thom collapse map, in this sequence

$$\mathbb{P}_K^1 \cong \mathbb{P}_k^1 \wedge (\mathrm{Spec} K)_+ \rightarrow \mathbb{P}_k^1 \wedge (\mathrm{Spec} L)_+$$

is the desired stable map $\mathrm{Spec} K \rightarrow \mathrm{Spec} L$, and induces a transfer map

$$\tau_{L/K}^z : \mathbf{K}_*^{\mathrm{MW}}(L) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(K),$$

called the *geometric transfer*. This transfer depends on the chosen generator of L over K . Although L may not be generated over K by a single element, we can always choose a finite list of generators. Given an ordered such list, we define a transfer $\mathbf{K}_*^{\mathrm{MW}}(L) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(K)$ by composing the transfers just constructed.

For simplicity, assume that the characteristic of k is not 2. The dependency of the transfer map on the chosen generators can be eliminated by modifying the definition in the following manner: Suppose z is a generator of L over K . The monic minimal polynomial of z can be expressed in a canonical manner as $P(x^{p^m})$ where P is a separable polynomial. The derivative $P'(z^{p^m})$ of P evaluated at z^{p^m} is an element of L^* . The *cohomological transfer*

$$(6) \quad \mathrm{Tr}_{L/K} : \mathbf{K}_*^{\mathrm{MW}}(L) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(K)$$

is then defined

$$\mathrm{Tr}_{L/K}(\beta) = \tau_{L/K}^z(\langle P'(z^{p^m}) \rangle \beta),$$

and this map is independent of the chosen generator [Mor12, Theorem 4.27]. More generally, for any finite extension $K \subseteq L$, one has a cohomological transfer (6) by composing the the transfers just defined for a sequence of generators, and the resulting map is again independent of the chosen sequence. This independence of the choice of generators is more naturally understood terms of twisted Milnor–Witt K-theory, where the twist is by the canonical sheaf.

From the algebraic perspective, there are many possible transfers $\mathrm{GW}(L) \rightarrow \mathrm{GW}(K)$ for $[L : K] \leq \infty$ as follows: for any nonzero K -linear map $f : L \rightarrow K$, and non-degenerate bilinear form $\beta : V \times V \rightarrow L$, the composite

$$f \circ \beta : V \times V \xrightarrow{\beta} L \xrightarrow{f} K$$

is a non-degenerate bilinear form. Thus we may define a map $\mathrm{Tr}_f : \mathrm{GW}(L) \rightarrow \mathrm{GW}(K)$ which takes the isomorphism class of an appropriate β to the isomorphism class of $f \circ \beta$, where in the former case, V is viewed as an L -vector space, and in the latter case, V is viewed as a K -vector space.

This abundance of transfers for GW also implies the same for $\mathbf{K}_*^{\mathrm{MW}}$ as the latter can be expressed as the fiber product

$$\begin{array}{ccc} \mathbf{K}_*^{\mathrm{MW}}(K) & \longrightarrow & I(K)^n \\ \downarrow & & \downarrow \\ \mathbf{K}_n^{\mathrm{M}}(K) & \longrightarrow & I(K)^n / I(K)^{n+1}, \end{array}$$

and there is a canonical transfer on Milnor K-theory.

When $K \subset L$ is a finite separable extension, there is a canonical choice of such an f , namely, the trace map $L \rightarrow K$ from Galois theory, given by summing the Galois conjugates of an element of L . The resulting transfer map is the cohomological transfer (6) defined above. This transfer arises naturally when studying the local degree at non- k -rational points. For example, let $f : U \rightarrow W$ be a map between open subsets of \mathbb{A}_k^n . Suppose that x in U maps to a k -point y of W and that x is an isolated in $f^{-1}(y)$, so the local degree $\deg_x^{\mathbb{A}^1} f$ exists. If the extension $k \subseteq k(x)$ is separable and f is étale at x , then

$$\deg_x^{\mathbb{A}^1} f = \mathrm{Tr}_{k(x)/k} \langle J(x) \rangle,$$

where J denotes the Jacobian of f . A proof of this is in [KW16, Proposition 14], and this proof relies on Hoyois's work in [Hoy14].

Example 4.4. $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \langle 1 \rangle = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \langle 1 \rangle + \langle -1 \rangle = h$, where the central expression is the Gram matrix with respect to the \mathbb{R} -basis $\{1, i\}$ of \mathbb{C} .

4.4. Euler class. Given an oriented vector bundle of rank r on an oriented \mathbb{R} -manifold of dimension r , one can define an Euler number. It is an element of \mathbb{Z} that counts the number of zeros of a section in the following sense: Given a section σ with an isolated zero x , we can define a local index (or degree) of σ at x by the following procedure. First, choose local coordinates around x and a local trivialization of the vector bundle such that both are compatible with the appropriate orientations. With these choices, the section can be viewed as a function $\mathbb{R}^r \rightarrow \mathbb{R}^r$. The index $\mathrm{ind}_x \sigma$ is then the local degree of this function at x , where x is viewed as a point of \mathbb{R}^r , by a slight abuse of notation. The Euler number e is then

$$(7) \quad e = \sum_{x: \sigma(x)=0} \mathrm{ind}_x \sigma,$$

when σ has only isolated zeros.

With Morel's $\mathrm{GW}(k)$ -valued degree, this Euler number can be enriched to an element of $\mathrm{GW}(k)$. Namely, one can again define a local index using the local degree and then use (7) to define the Euler number, now an element of $\mathrm{GW}(k)$ [KW17]. There are subtleties involved in choosing coordinates and identifying σ with a function, due to difficulties in defining algebraic functions. One must then show that the Euler number is well-defined. There are other approaches avoiding these difficulties, using oriented Chow or Chow–Witt groups of Barge and Morel [BM00]. Barge and Morel construct an Euler class (loc. cit.) in oriented Chow and it can be pushed forward under certain conditions to land in $\mathrm{GW}(k)$. See the work of Jean Fasel [Fas08] and Marc Levine [Lev17]. Morel gives an alternate construction of an Euler class using obstruction theory [Mor12, Section 8.2]. This Euler class lies in the same oriented Chow group, and will be discussed further in Section 6.3. Some comparison results are available between these two Euler classes [AF16a] [Lev17, Proposition 11.6]. Further approaches to defining an Euler class using \mathbb{A}^1 -homotopy theory are found in [DJK18] [LR18].

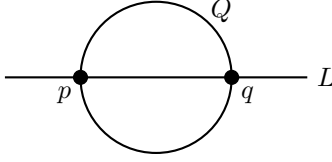


FIGURE 4.
 $T_p X \cap X \subset T_p X$

4.5. Applications to enumerative geometry. Questions in enumerative geometry ask to count a set of algebro-geometric objects satisfying certain conditions, or more generally, to describe this set. A classical example is the question “How many lines intersect four general lines in three dimensional space?” The questions are typically posed so there is a fixed answer, such as “2,” as opposed to “sometimes 2 and sometimes 0,” and to get such “invariance of number,” one needs to work over an algebraically closed field; the number of solutions to a polynomial equation of degree n is always n if one works over \mathbb{C} , but not over \mathbb{R} , and the same phenomenon appears in the classical question quoted above about the number of lines. However, a feature of \mathbb{A}^1 -homotopy theory is its applicability to general fields k . Moreover, there are classical theorems in enumerative geometry where a count of geometric objects is identified with an Euler number, which is then computed using characteristic class techniques. Equipped with an enriched Euler class in $\text{GW}(k)$, as in Section 4.4, we may thus take these theorems and hope to perform the analogous counts over other fields.

What sorts of results appear? Here is an example taken from [KW17]. A cubic surface X is the zero locus in \mathbb{P}_k^3 of a degree 3 homogenous polynomial f in $k[x_0, x_1, x_2, x_3]$. It is a lovely 19th century result of Salmon and Cayley that when $k = \mathbb{C}$ and X is smooth, the number of lines in X is exactly 27. Over the real numbers, the number of real lines is either 3, 7, 15, or 27, and in particular, the number depends on the surface. A classification was obtained by Segre [Seg42], but it is a recent observation of Benedetti–Silhol [BS95], Finashin–Kharlamov [FK13], Horev–Solomon [HS12] and Okonek–Teleman [OT14] that a certain signed count of lines is always 3. Specifically, Segre distinguished between two types of real lines on $X = \{[x_0, x_1, x_2, x_3] \in \mathbb{R}\mathbb{P}^3 : f(x_0, x_1, x_2, x_3) = 0\}$, called *hyperbolic* and *elliptic*. The distinction is as follows. Let L be a real line on X . For every point p of L , consider the intersection $T_p X \cap X$ of the tangent plane to the cubic surface at p with the cubic surface itself. Since $T_p X$ is a plane, the intersection is a curve in the plane, which by Bézout’s theorem has degree $3 \cdot 1 = 3$. The line L must be contained in this plane curve. Algebraic curves can be decomposed into irreducible components, and it follows that we can express $T_p X \cap X$ as a union $T_p X \cap X = L \cup Q$, where Q is a plane curve of degree $3 - 1 = 2$. Applying Bézout’s theorem again shows that the intersection $L \cap Q$ of the plane curves L and Q is degree 2, or in other words, generically consists of two points, which we call $\{p, q\}$. See Figure 4. We may thus define an involution $I : L \rightarrow L$ by sending p to the unique point q .

The points of the intersection $L \cap Q = \{p, q\}$ are precisely the points x on L such that $T_x X = T_p X$. To see this, note that $T_q X$ contains the span of a vector along L and a vector along Q . At

least generically, it follows that $T_q X$ contains a 2-dimensional subspace of $T_p X$. Since $T_q X$ is 2-dimensional (X is smooth), it follows that $T_q X = T_p X$. Similar reasoning applies in the reverse direction as well. So we can characterize the involution I as the unique map exchanging points on L with the same tangent space to X .

An automorphism of $L \cong \mathbb{R}P^1$ is a conjugacy class of element of $\mathbb{P}GL_2 \mathbb{R}$, and the elements of $\mathbb{P}GL_2 \mathbb{R}$ are classified as elliptic, hyperbolic, or parabolic by the behavior of the fixed points. If I has a complex conjugate pair of fixed points, I is elliptic, if I has two real fixed points, it is hyperbolic, and if I has one fixed point, it is parabolic. Involutions are never parabolic. Segre classified the line L as elliptic (respectively hyperbolic) if I is.

Another description of this distinction involves (S)pin structures. The tangent plane $T_p X$ rotates around L as p travels along L , describing a loop in the frame bundle of \mathbb{P}^3 . L is hyperbolic if this loop lifts to the double cover, and elliptic if it does not.

The signed count referred to above is that

$$(8) \quad \#\{\text{hyperbolic lines}\} - \#\{\text{elliptic lines}\} = 3.$$

Results which are true over \mathbb{C} and \mathbb{R} may be realizations of a more general result in \mathbb{A}^1 -homotopy theory. In the case of the count of lines on cubic surface, this is indeed the case: Consider a cubic surface X over a field k . Suppose L is a line in \mathbb{P}_k^3 which lies in X . The coefficients of L determine a field extension $k(L)$. Moreover, the previously given definition of the involution I carries over in this generalized situation, determining an involution I of the line, and thus a conjugacy class in $\mathbb{P}GL_2 k(L)$. Such a conjugacy class has a well-defined determinant $\det I$ in $k(L)^*/(k(L)^*)^2$. The *Type* of L is $\text{Type}(L) = \langle \det I \rangle$ in $\text{GW}(k(L))$. Alternatively, the type of L may be described as $\text{Type}(L) = \langle D \rangle$, where D is the unique element of $k(L)^*/(k(L)^*)^2$ so that the fixed points of the involution I are a conjugate pair of points defined over the field $k(L)[\sqrt{D}]$. There is a third description of the type as $\text{Type}(L) = \langle -1 \rangle \deg I$ the multiplication of the \mathbb{A}^1 -degree of I and $\langle -1 \rangle$ in $\text{GW}(k(L))$. The theorem of Salmon and Cayley and (8) then are realizations of the following theorem [KW17, Theorem 1].

Theorem 4.5. *Let X be a smooth cubic surface over a field k of characteristic not 2. Then*

$$\sum_{\text{lines } L \text{ in } X} \text{Tr}_{k(L)/k} \text{Type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle.$$

This is proven by identifying the left hand side with the Euler class of the third symmetric power of the dual tautological bundle on the Grassmannian of lines in \mathbb{P}^3 .

Other results along these lines include [Hoy14], [Lev17], [Lev18a], [Lev18b], [Wen18], [SW18], [BKW18], and this is an active area of research.

5. HOMOTOPY SHEAVES AND THE CONNECTIVITY THEOREM

In the monograph [Mor12], Morel establishes a number of extremely strong results describing the \mathbb{A}^1 -homotopy sheaves of objects of $\mathbf{sPre}(\mathbf{Sm}_k)$ when k is an infinite perfect field. These are sheaves on the big Nisnevich site \mathbf{Sm}_k . The assumptions on the field are probably both unnecessary, but the literature does not currently contain proofs of certain necessary statements for finite or imperfect fields.

The \mathbb{A}^1 -homotopy sheaves $\pi_i^{\mathbb{A}^1}(X)$ —when $i \geq 1$ —must satisfy a strong \mathbb{A}^1 -invariance property. For all smooth k -schemes U , the map on cohomology

$$(9) \quad \mathrm{H}^n(U, \pi_i^{\mathbb{A}^1}(X)) \rightarrow \mathrm{H}^n(U \times \mathbb{A}^1_k, \pi_i^{\mathbb{A}^1}(X))$$

must be an isomorphism for all applicable n , to wit, $n \in \{0, 1\}$ for all sheaves of groups, and $n \in \{0, 1, \dots\}$ if the group is abelian. This condition is known as *strong \mathbb{A}^1 -invariance* in the general case, and *strict \mathbb{A}^1 -invariance* in the case of an abelian sheaf of groups. Over an infinite perfect field, a strongly \mathbb{A}^1 -invariant sheaf of abelian groups is strictly \mathbb{A}^1 -invariant, [Mor12, Theorem 5.46], so for us the distinction is chiefly useful for defining a category of “strictly \mathbb{A}^1 -invariant sheaves”, which necessarily consist of abelian groups.

We remark that the case of $n = 0$ in (9) is merely \mathbb{A}^1 -invariance. It is possible to give an example of an (abelian) group that is \mathbb{A}^1 -invariant but not strongly \mathbb{A}^1 -invariant—for instance, if one defines $\tilde{\mathbb{Z}}[\mathbb{G}_m]$ to be the abelian group generated by the sheaf of sets \mathbb{G}_m , modulo the relation that $1 \in \mathbb{G}_m$ is identified with the 0 element, then $\tilde{\mathbb{Z}}[\mathbb{G}_m]$ is \mathbb{A}^1 -invariant—since \mathbb{G}_m itself is \mathbb{A}^1 invariant—but the free strongly \mathbb{A}^1 -invariant sheaf of abelian groups generated by $\tilde{\mathbb{Z}}[\mathbb{G}_m]$ is $\mathbf{K}_1^{\mathrm{MW}}$ by [Mor12, Theorem 3.37].

A sheaf of groups, \mathcal{G} , is strongly \mathbb{A}^1 -invariant if and only if it appears as an \mathbb{A}^1 -homotopy sheaf $\pi_1^{\mathbb{A}^1}$, since it appears as the \mathbb{A}^1 -fundamental sheaf of $B\mathcal{G}$. The strongly \mathbb{A}^1 -invariant sheaves \mathcal{G} are *unramified* sheaves, which is to say, briefly, that $\mathcal{G}(X)$ is the product of $\mathcal{G}(X_\alpha)$ as X_α range over the irreducible components, and that for any dense open $U \subset X$, $\mathcal{G}(X) \rightarrow \mathcal{G}(U)$ is an injection which is even an isomorphism if $X - U$ is everywhere of codimension at least 2 in X , see [Mor12, Section 2.1]. A feature of unramified sheaves \mathcal{G} in this sense is that \mathcal{G} can be recovered from the values $\mathcal{G}(F)$ where F ranges over fields of finite transcendence degree over k , along with subsets $\mathcal{G}(\mathcal{O}_v) \subset \mathcal{G}(F)$ associated to any discrete valuation v on F , and specialization maps $\mathcal{G}(\mathcal{O}_v) \rightarrow \mathcal{G}(\kappa(v))$ mapping to the residue fields, all satisfying certain compatibility axioms, [Mor12, Section 2.1]. Most strikingly, if $\mathcal{G}(F)$ vanishes for all field extensions of k , then $\mathcal{G} = 0$.

The category of strictly \mathbb{A}^1 -invariant sheaves forms a full abelian subcategory of the category of sheaves of abelian groups on \mathbf{Sm}_k , being the heart of a t -structure—this appears as [Mor12, Corollary 6.24].

One can rephrase the statement that the sheaves $\pi_i^{\mathbb{A}^1}(X)$ are strongly \mathbb{A}^1 -invariant for $i \geq 1$ as follows: if the \mathbb{A}^1 -localization map $X \rightarrow L_{\mathbb{A}^1}X$ is a simplicial equivalence, i.e., if X is \mathbb{A}^1 -local, then the sheaves $\pi_i^{\mathbb{A}^1}(X)$ for $i \geq 1$ are strongly \mathbb{A}^1 -invariant. There is a partial converse:

Theorem 5.1. *Suppose X is a pointed connected object of $\mathbf{sPre}(\mathbf{Sm}_k)$. Then X is \mathbb{A}^1 -local if and only if $\pi_i(X)$ is strongly \mathbb{A}^1 -invariant for all $i \geq 1$.*

This holds, loosely speaking, because X is \mathbb{A}^1 -local if the functor $\mathrm{Map}(\cdot, X)$ does not distinguish between U and $U \times \mathbb{A}^1$ up to homotopy. For a connected X , the calculation of maps $\mathrm{Map}(\cdot, X)$ can be reduced to the calculation of $\mathrm{Map}(\cdot, K(\pi_i(X), j))$ for varying i, j by means of the Postnikov tower, and strong \mathbb{A}^1 -invariance of $\pi_i(X)$ amounts to the same thing as \mathbb{A}^1 -locality of $K(\pi_i(X), j)$. The result is stated as [Mor12, Corollary 6.3].

The assumption of connectivity, while it may seem mild at first, is a great inconvenience. In contrast to classical homotopy theory, where one can generally work component-by-component, it is a feature of the homotopy of sheaves that π_0 , a sheaf of sets, can be extremely complicated.

In fact, the difficulties at π_0 are substantial. If X is a discrete sheaf of groups, viewed as a space in dimension 0, then X is \mathbb{A}^1 -local if and only if X is \mathbb{A}^1 -invariant. If X is \mathbb{A}^1 -invariant without being strongly \mathbb{A}^1 -invariant, then $L_{\mathbb{A}^1}BX$ cannot be (simplicially) equivalent to BX , and so $\pi_0\Omega L_{\mathbb{A}^1}BX \not\cong \pi_0X$. In particular, BX does not have the \mathbb{A}^1 -homotopy type of a delooping of X . It follows from this that \mathbb{A}^1 -homotopy theory does correspond to an ∞ -topos in the sense of [Lur09, Section 6.1]. This observation, due to J. Lurie, appears as [SO12, Remark 3.5].

As a heuristic, aside from problems at π_0 , results that hold in ∞ -topoi (or in the “model topoi” of C. Rezk [TV05]) can generally be expected to hold in \mathbb{A}^1 -homotopy theory. For instance:

Theorem 5.2. *Suppose*

$$(10) \quad F \rightarrow E \rightarrow B$$

is a simplicial homotopy fiber sequence in $\mathbf{sPre}(\mathbf{Sm}_k)$ where $\pi_0(B)$ is \mathbb{A}^1 -invariant. Then (10) is an \mathbb{A}^1 -homotopy fiber sequence.

This is a corollary of [AHW18, Theorem 2.1.5] (see also Remark 2.1.6), a development of [Mor12, Theorem 6.53].

5.1. Contractions. The following construction appears originally in [Voe00], but applied there only to “presheaves with transfers”. In the current context, it is due to [Mor12][Section 2.2, pp. 33–36]. Given a presheaf of groups \mathcal{G} , define

$$\mathcal{G}_{-1}(U) = \ker(\mathcal{G}(\mathbb{G}_m \times U) \xrightarrow{ev_1} \mathcal{G}(U)).$$

Here the map is induced inclusion of $U \times \{1\}$ in $U \times \mathbb{G}_m$; when the presheaf is applied, this appears as a kind of “evaluation at 1”, hence the notation. This construction is functorial in \mathcal{G} .

The functor $\mathcal{G} \mapsto \mathcal{G}_{-1}$ is sometimes called *contraction*. It may be iterated, in which case one writes \mathcal{G}_{-n} . The functor $(\cdot)_{-1}$ has a number of excellent properties. It restricts to give a functor of (Nisnevich) sheaves, for instance, and it preserves the property of being abelian. It is also left exact in general. When applied to \mathbb{A}^1 -homotopy sheaves, it has the following striking description, which appears as [Mor12, Theorem 6.13]

Theorem 5.3 (Morel). *If X is a pointed, connected \mathbb{A}^1 -local space, then the (derived) mapping space $\Omega_{\mathbb{G}_m} := \text{Map}(\mathbb{G}_m, X)$ is also pointed, connected and \mathbb{A}^1 -local, and there is a canonical isomorphism*

$$\pi_n^{\mathbb{A}^1}(\Omega_{\mathbb{G}_m} X) \rightarrow \pi_n^{\mathbb{A}^1}(X)_{-1}.$$

This theorem, of course, is known only for homotopy sheaves over \mathbf{Sm}_k where k is a field satisfying the running assumptions of [Mor12].

Any short exact sequence of strictly \mathbb{A}^1 -invariant sheaves may be realized as the homotopy sheaves of an \mathbb{A}^1 homotopy fiber sequence, for instance

$$0 \rightarrow \pi_2^{\mathbb{A}^1}(K(\mathcal{A}, 2)) \rightarrow \pi_2^{\mathbb{A}^1}(K(\mathcal{B}, 2)) \rightarrow \pi_2^{\mathbb{A}^1}(K(\mathcal{C}, 2)) \rightarrow 0$$

and since the (derived) mapping space $\text{Map}(\mathbb{G}_m, \cdot)$ preserves homotopy fibre sequences, one deduces that the functor $\mathcal{A} \mapsto \mathcal{A}_{-1}$ is exact on the category of strictly \mathbb{A}^1 -invariant sheaves.

It is immediate that if \mathcal{C} is a constant presheaf of abelian groups, then $\mathcal{C}_{-1} = 0$. One might fantasize that if \mathcal{G} is strictly \mathbb{A}^1 -invariant, then $\mathcal{G}_{-1} = 0$ should imply that \mathcal{G} is constant, but this is far from the case. In fact, from the analysis furnished by [Mor12, Chapter 2] of $(\cdot)_{-1}$, one can deduce that $\mathcal{G}_{-1} = 0$ if and only if \mathcal{G} is *birational* in the sense of converting dense open inclusions of schemes $U \rightarrow X$ into isomorphisms $\mathcal{G}(X) \rightarrow \mathcal{G}(U)$. In [AM11, Section 6], a study is made of such sheaves, and the category of all such sheaves over \mathbf{Sm}_k is seen to be equivalent to a very large category of functors on the category of field extensions of k .

5.2. Unramified K -theories. Certain groups that were previously known to be functors of fields and field extensions are known to extend to give strictly \mathbb{A}^1 -invariant sheaves on the Nisnevich site of \mathbf{Sm}_k . For instance, there is a strictly \mathbb{A}^1 -invariant sheaf \mathbf{K}_n^Q such that $\mathbf{K}_n^Q(F) = K_n^Q(F)$, Quillen's K -theory—here the field extension F/k is supposed to be a separable extension of an extension of finite transcendence degree. The sheaf \mathbf{K}_n^Q arises as $\pi_n^{\mathbb{A}^1}(BGL_N)$ for $N \geq n + 2$, this being a consequence of the representability of K -theory for smooth schemes as proved in [MV99, Theorem 3.13, p140]. Analogous constructions for hermitian K -groups are made in [ST15].

Another notable sheaf is \mathbf{K}_n^M , which can be recovered as a quotient of \mathbf{K}_n^{MW} , the *unramified Milnor–Witt K -theory sheaf* as constructed by Morel in [Mor12, Section 3.2]. In this case, the group of sections for a field is $K_n^M(F)$, as defined in Section 4.

In the cases of \mathbf{K}_n^Q , \mathbf{K}_n^{MW} and \mathbf{K}_n^M , the phenomenon of \mathbb{P}^1 -stability for the associated theories implies $(\mathbf{K}_n^Q)_{-1} = \mathbf{K}_{n-1}^Q$ and similarly for the other two theories; a proof is outlined in [AF14a, Lemma 2.7, Proposition 2.9].

One may define $\mathbf{K}_3^{\text{ind}}$, the cokernel of a natural map $\phi : \mathbf{K}_3^M \rightarrow \mathbf{K}_3^Q$. By virtue of Matsumoto's theorem, one knows that

$$(\phi)_{-1} : \mathbf{K}_2^M \rightarrow \mathbf{K}_2^Q$$

is an isomorphism, so that $(\mathbf{K}_3^{\text{ind}})_{-1} \cong 0$. It is known, for instance by [MS90], that $\mathbf{K}_3^{\text{ind}}$ is not constant, so this furnishes a specific example of a nonconstant strictly \mathbb{A}^1 -invariant sheaf, the contraction of which is 0.

5.3. **\mathbb{A}^1 -homology and the connectivity theorem.** In [Mor12], Morel defines an “ \mathbb{A}^1 -homology theory” $H_n^{\mathbb{A}^1}(X)$, by means of the following: for any simplicial (pre)sheaf X one may define $\mathbb{Z}(X)$, the free abelian group on X , which one converts to a chain complex $C_*(X)$ via the Dold–Kan correspondence. One can take the category of (pre)sheaves of chain complexes on \mathbf{Sm}_k , viewed as a setting for homotopy theory in its own right, and then localize with respect to \mathbb{A}^1 , that is, with respect to the maps $C_* \otimes C_*(\mathbb{A}^1) \rightarrow C_*$. One may now replace chain complexes $C_*(X)$ by “abelian \mathbb{A}^1 -local replacements”—for Morel, these are the fibrant objects in the localized model category—denoted $C_*^{\mathbb{A}^1}(X)$. The homology of such an object is $H_*^{\mathbb{A}^1}(X)$, and these homology sheaves are all strictly \mathbb{A}^1 -invariant.

This homology theory is quite distinct from the “motivic homology theory” defined by Voevodsky [Voe00]; it is much closer to the unstable \mathbb{A}^1 -homotopy theory, and little is known about it. It does enjoy the following three properties:

- (1) It is S^1 stable: $H_{n+1}^{\mathbb{A}^1}(X \wedge S^1) \cong H_n^{\mathbb{A}^1}(X)$.
- (2) If \mathcal{F} is a discrete sheaf, then $H_0^{\mathbb{A}^1}(\mathcal{F})$ is the strictly \mathbb{A}^1 -invariant sheaf freely generated by the sheaf \mathcal{F} , in the sense that this construction is left adjoint to an obvious forgetful functor.
- (3) For any \mathbb{A}^1 -simply-connected pointed object X , a Hurewicz isomorphism holds. If $\pi_i^{\mathbb{A}^1}(X) = 0$ for $i \leq n - 1$, and $n \geq 2$, then a natural map $\pi_n^{\mathbb{A}^1}(X) \rightarrow H_n^{\mathbb{A}^1}(X)$ is an isomorphism. A modification of this holds for π_1 , involving abelianization and “ \mathbb{A}^1 -strictification”. See [Mor12, Theorems 6.35, 6.37].

It is this theory, and the properties above, that allows Morel to compute the unstable \mathbb{A}^1 -homotopy sheaves of the spheres. Specifically, provided $n \geq 2$, so that $\pi_n^{\mathbb{A}^1}$ is known to be abelian & strictly \mathbb{A}^1 invariant, then $\pi_n^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge m})$ is the free strictly \mathbb{A}^1 -invariant sheaf generated by the set $\mathbb{G}_m^{\wedge m}$. That is

$$(11) \quad \pi_n(S^n \wedge \mathbb{G}_m^{\wedge m}) = \mathbf{K}_m^{\text{MW}}$$

provided $n \geq 2$.

Combined with further results of Morel’s on the contractions of $\mathbf{K}_\bullet^{\text{MW}}$, one deduces

$$(12) \quad \pi_{n+i\alpha}^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge m}) = \mathbf{K}_{m-i}^{\text{MW}}$$

provided $n \geq 2$ and $m \geq 1$. These calculations appear as [Mor12, Remark 6.42].

The following result of Morel is known as the “unstable connectivity theorem”.

Theorem 5.4. *Let $n > 0$ be an integer and let X be a pointed $(n - 1)$ -connected object in $\mathbf{sPre}(\mathbf{Sm}_k)$. Then its \mathbb{A}^1 -localization is simplicially $(n - 1)$ -connected.*

6. APPLICATION TO VECTOR BUNDLES

6.1. \mathbb{A}^1 -classifying spaces. A recurring distinction in \mathbb{A}^1 -homotopy theory is the existence of two different notions of classifying space of an algebraic group scheme G . In both cases, one wishes to begin with a contractible object EG on which G acts, and then to form the quotient $(EG)/G$. The difference arises because there are two distinct notions of quotient.

The *simplicial* or *Nisnevich* classifying space BG is constructed by taking a contractible object EG on which G acts freely, and then forming the quotient $BG = (EG)/G$ in the category of simplicial Nisnevich sheaves. Standard homotopical devices for the construction of BG , such as found in [May75] for instance, will invariably produce this classifying space, up to homotopy.

One may produce a new space, $B_{\text{ét}}G$, by taking a (derived) pushforward of the pullback of BG along the morphism of sites

$$(\mathbf{Sm}_k)_{\text{ét}} \rightarrow (\mathbf{Sm}_k)_{\text{Nis}},$$

see [MV99, Section 4.1].

The *geometric* classifying space, $B_{gm}G$, is constructed in [MV99, Section 4.2], using what they term an *admissible gadget*, which is a generalization of the construction of [Tot99]. The construction of [MV99] applies to étale sheaves of groups over a base S , but for the sake of the exposition here, we restrict to a reductive algebraic group G over a field k . While EG cannot be constructed as a variety, one may construct a sequence $U_i \hookrightarrow U_{i+1}$ of increasingly highly- \mathbb{A}^1 -connected varieties on which G acts freely and compatibly. The construction is accomplished by considering a suitable family of larger and larger representations of G on affine spaces over k , and discarding the locus where the action of G is not free. The quotients U_i/G may then be taken in the category of algebraic spaces, or if one is particularly careful, in schemes, and $B_{gm}G$ is then defined to be the colimit of the ind-algebraic-space $\{U_i/G\}$. The quotient that is constructed here is ‘geometric’, in that it agrees with classically-existing notions of quotient of one variety by another, as in e.g., [MFK94]; it is also the quotient in the big site of étale sheaves. It is then the case, [MV99, Proposition 4.2.6], that $B_{\text{ét}}G$ is \mathbb{A}^1 -equivalent to $B_{gm}G$.

If G is a sheaf of étale group schemes, e.g., if G is representable, and if $H_{\text{Nis}}^1(\cdot, G) \cong H_{\text{ét}}^1(\cdot, G)$, then the constructions of $B_{\text{ét}}G$ and BG above are naturally simplicially equivalent, [MV99, Lemma 4.1.18]. In particular, in the case of the *special* algebraic groups of [Ser95, Section 4.1], including GL_n , SL_n and Sp_n , all notions of classifying space considered above are \mathbb{A}^1 -weakly equivalent.

In contrast, in the case of a nontrivial finite group, both BG and $B_{\text{ét}}G$ are \mathbb{A}^1 -local objects already, [MV99, Section 4.3], and one may easily construct a G -torsor $\pi : U \rightarrow U/G$ where both U and U/G are smooth k schemes but where π is not Nisnevich-locally trivial, it follows from [MV99, Lemma 4.1.8] again that for a nontrivial finite group, $BG \rightarrow B_{\text{ét}}G$ is never an \mathbb{A}^1 -equivalence.

In the study of vector bundles, or equivalently of GL_n -bundles, the spaces $B_{gm}GL_n$ appearing above are approximated by the ordinary Grassmanians, $\text{Gr}_n(\mathbb{A}^r)$. The notation Gr_n is adopted for

the colimit as $r \rightarrow \infty$, and the \mathbb{A}^1 -weak equivalences

$$\mathrm{Gr}_n \simeq_{\mathbb{A}^1} B \mathrm{GL}_n$$

can be interpreted as a relation between a geometric construction on the left and a homotopical construction on the right.

6.2. Classification of Vector Bundles. The following metatheorem lies at the heart of the applications of \mathbb{A}^1 -homotopy to the classification of vector bundles:

Theorem 6.1. *Let k be a sufficiently pleasant base ring, and let $r \in \mathbb{N}$. The functor assigning to a smooth affine k -scheme X its set of rank- r vector bundles, denoted $X \mapsto V_r(X)$, is represented in the \mathbb{A}^1 -homotopy category by the infinite Grassmannian of r -planes, Gr_n . That is, there is a natural bijection of sets*

$$(13) \quad V_r(X) \leftrightarrow \mathrm{Mor}_{\mathcal{H}(k)}(X, \mathrm{Gr}_r).$$

This metatheorem was proved by Morel as [Mor12, Theorem 8.1] in the case where $n \neq 2$ and subject to the running assumptions of [Mor12], namely, that k be an infinite perfect field.² Subsequently this was vastly generalized, to the case where k is a regular noetherian ring over a Dedekind domain with perfect residue fields—this includes, in particular, all fields—and allowing the case $n = 2$ in [AHW17a, Theorem 5.2.3].

We remark in passing that the restriction to smooth affine k -schemes cannot be weakened. Once the “affine” hypothesis is dropped, the set of isomorphism classes of vector bundles is no longer \mathbb{A}^1 -invariant, and so cannot be represented in the \mathbb{A}^1 -homotopy category. One can consult [AD08] for a proof that $V_r(\mathbb{P}^1) \rightarrow V_r(\mathbb{P}^1 \times \mathbb{A}^1)$ is not an isomorphism or that there exists a smooth quasiprojective $X \simeq_{\mathbb{A}^1} *$ such that $V_r(X) \neq V_r(*)$.

One can use the metatheorem to deduce facts about $V_r(X)$. The main computational tool is the existence of a Postnikov tower in \mathbb{A}^1 -homotopy theory; which reduces the calculation of maps $X \rightarrow \mathrm{Gr}_n$, where X is a smooth affine variety, to a succession of lifting problems. It should be noted that $\pi_1^{\mathbb{A}^1}(\mathrm{Gr}_n) \cong \mathbb{G}_m$, and this π_1 acts nontrivially on the higher homotopy sheaves of Gr_r , so that the lifting problems one encounters for GL_n are \mathbb{G}_m -twisted lifting problems. The essentials of the twisted sheaf-theoretic obstruction are set out in [AF14b, Section 6] and [Mor12, Appendix B].

Once the obstruction theory has been set up, two related problems remain: the calculation of $\pi_i^{\mathbb{A}^1}(\mathrm{Gr}_n)$ for various values of i —including the \mathbb{G}_m -action—and the interpretation of the resulting lifting problems.

For the calculational problem, the following is known

$$\mathbb{A}^n - \{0\} \longrightarrow \mathrm{Gr}_{n-1} \longrightarrow \mathrm{Gr}_n$$

²Although the case $n = 2$ can presumably be proved using similar methods, the work on the subject has not been published.

is an \mathbb{A}^1 -homotopy fiber sequence, [Mor12, Remark 8.15]. Let $X = \text{Spec } A$ denote a smooth affine k -scheme and let V denote a rank n vector bundle on X , i.e., a projective module. The “splitting problem” is to determine necessary and sufficient conditions for there to be an isomorphism $V \cong V' \oplus A$.

6.3. The splitting problem. Since $\mathbb{A}^n - \{0\}$ is $n - 2$ -connected, by results of [Mor12, Chapter 6], it follows by induction that if the dimension of $d < n$, then one can always split $V \cong V' \oplus A$. This gives a “geometric” argument for a result of Serre, [Ser58].

More interesting is what happens when the obstruction-theory problem is not trivial for dimensional reasons. The problem is slightly easier if all vector bundles appearing are assumed to be equipped with a trivialization of their determinant bundles, so that $B\text{SL}_n$ may be substituted for $\text{Gr}_n = B\text{GL}_n$. The first obstruction to the splitting problem is an *Euler class* class $\tilde{c}_n(V) \in H_{\text{Nis}}^n(X; \mathbf{K}_n^{\text{MW}})$, as calculated in [Mor12, Theorem 8.14], arising directly from the obstruction theory in \mathbb{A}^1 -homotopy theory, and if the dimension of the base, d , is equal to the rank n of the bundle, then this is the only obstruction.

Related to the above, Nori gave a definition of an “Euler class group” $E(A)$ of a noetherian ring A , appearing in [MS96, Section 1] and of the Euler class $e(P)$ of a projective module, and Bhatwadekar & Sridharan [BS00] established that the vanishing of the Euler class is precisely equivalent to the splitting off of a trivial summand of P . There exists a surjective map $\eta : E(A) \rightarrow \widehat{CH}^n(\text{Spec } A)$, where A is a smooth k -algebra of dimension n , [Fas08, Chapitre 17], taking one Euler class to the other. According to recent, not yet published, work of Asok and Fasel [AF16b], the map η is an isomorphism for smooth affine varieties over an infinite perfect field of characteristic different from 2.

6.4. Other results on vector bundles. Obstruction theory in \mathbb{A}^1 -homotopy theory, applied to the representing space for V_r , as established in (13), is a powerful technique for deducing results about vector bundles.

The main result of [AF14b] is the following:

Theorem 6.2. *Suppose X is a smooth affine 3-fold over an algebraically closed field k having characteristic unequal to 2. The map assigning to a rank-2 vector bundle E on X the pair $(c_1(E), c_2(E))$ of Chern classes gives a pointed bijection*

$$V_2(X) \xrightarrow{\cong} \text{Pic}(X) \times CH^2(X).$$

The surjectivity of the map above had previously been established in [KM82]. The method of proof is a calculation—at least in part—of the first three nontrivial \mathbb{A}^1 -homotopy groups of GL_n . Of these, two are calculated by Morel in [Mor12]. Specifically $\pi_0^{\mathbb{A}^1}(\text{GL}_n) \cong \mathbb{G}_m$, for all $n \geq 0$: the reason being that SL_n is \mathbb{A}^1 -connected [Mor12, 6.52], so that

$$1 \rightarrow \text{SL}_n \rightarrow \text{GL}_n \rightarrow \mathbb{G}_m \rightarrow 1$$

is an \mathbb{A}^1 -homotopy fibre sequence. It is moreover split, and \mathbb{G}_m is strongly \mathbb{A}^1 -invariant, so it follows that for $n \geq 1$

$$\pi_i^{\mathbb{A}^1}(\mathrm{GL}_n) = \begin{cases} \mathbb{G}_m & \text{if } i = 0, \\ \pi_i^{\mathbb{A}^1}(\mathrm{SL}_n) & \text{otherwise.} \end{cases}$$

It is then also possible to calculate $\pi_1^{\mathbb{A}^1}(\mathrm{GL}_2) = \pi_1^{\mathbb{A}^1}(\mathrm{SL}_2) = \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\}) \cong \mathbf{K}_2^{\mathrm{MW}}$ —this belonging in the family of calculations of $\pi_n^{\mathbb{A}^1}(S^{n+m\alpha})$ of [Mor12]—the reason the groups $\pi_i^{\mathbb{A}^1}$ are sometimes exceptional is that they are not known in general to be abelian, but here one is calculating a higher homotopy sheaf of a group, so the Eckmann–Hilton argument applies and $\pi_1^{\mathbb{A}^1}(\mathrm{SL}_2)$ is abelian.

The sheaf $\mathbf{K}_2^{\mathrm{M}}$ relates to classical invariants of vector bundles via the “formula of Rost”:

$$\mathrm{H}_{\mathrm{Nis}}^n(X, \mathbf{K}_n^{\mathrm{M}}) = \mathrm{H}_{\mathrm{Zar}}^n(X, \mathbf{K}_n^{\mathrm{M}}) = \mathrm{CH}^n(X),$$

the first identity being [Mor12] and the second being [Ros96, Corollary 6.5]. For a quadratically closed field, the result also holds for $\mathbf{K}_n^{\mathrm{MW}}$, see [AF14b].

Knowing the first two nonvanishing homotopy sheaves of $B\mathrm{GL}_2$, along with the action of $\pi_1^{\mathbb{A}^1}(B\mathrm{GL}_2) = \mathbb{G}_m$ on $\pi_2^{\mathbb{A}^1}$, allows one to construct the second stage of the Postnikov tower, $B\mathrm{GL}_2^{(2)}$. The action of \mathbb{G}_m does have to be taken into account, so the calculation is more involved than simply the observations

$$[X, B\mathbb{G}_m]_{\mathbb{A}^1} = \mathrm{Pic}(X), \quad [X, K(\mathbf{K}_2^{\mathrm{MW}}(\mathcal{L}), 2)]_{\mathbb{A}^1} = \mathrm{CH}^2(X),$$

for a smooth X —here $\mathbf{K}_2^{\mathrm{MW}}(\mathcal{L})$ is a twisted form of $\mathbf{K}_2^{\mathrm{MW}}$ —, but over a quadratically closed field the ultimate calculation is the same. For a 3-fold, there is no further obstruction to lifting to a map $[X, B\mathrm{GL}_2]_{\mathbb{A}^1}$, but what there might be *a priori* is a choice of different liftings. The set of such choices is $\mathrm{H}^3(X, \pi_3^{\mathbb{A}^1}(B\mathrm{GL}_2)(\mathcal{L}))$, and [AF14b] shows that this is trivial over an algebraically closed field. It is worth remarking that the sheaf $\pi_3^{\mathbb{A}^1}(B\mathrm{GL}_2)$ in question—at least in the untwisted form—is isomorphic to $\pi_2^{\mathbb{A}^1}(S^{1+2\alpha})$, that is, it corresponds to the stable 1-stem.

A number of results in a similar vein to the above can be established by the methods outlined. It is a corollary of Theorem 6.2 that a rank-2 vector bundle on a smooth affine 3-fold over an algebraically closed field has a trivial line-bundle summand if and only if c_2 vanishes. In [AF15], the generalization to dimension n is named “Murthy’s Splitting Conjecture”:

“On an affine variety of dimension n over an algebraically closed field, a rank $n - 1$ vector bundle admits a trivial summand if and only if c_{n-1} vanishes.”

The conjecture is proved for $n = 4$, again by analysis of a group in the 1-stem: $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 - \{0\})$. In spite of considerable attention, [AF13], [AF17], [AWW17], the groups $\pi_n^{\mathbb{A}^1}(\mathbb{A}^n - \{0\})$ have not been calculated for $n \geq 4$, and the problem remains open.

The groups GL_n and SL_n are the most studied groups, corresponding to the study of vector bundles, either *per se* or with an orientation. Other groups have also been studied. For instance, Wendt shows in [Wen11] that if G is a smooth split reductive group over an infinite field k , and if $E \rightarrow B$ is a G -torsor satisfying certain local triviality conditions—local triviality in the Zariski topology

and triviality over the basepoint of B —then the fibre sequence

$$G \rightarrow E \rightarrow B$$

is an \mathbb{A}^1 -homotopy fibre sequence. In [AHW18], the authors establish, again over an infinite field, that for an isotropic reductive k group G , the functor $H_{\text{Nis}}^1(\cdot, G)$ of isomorphism classes of Nisnevich-trivial G -torsors is represented in the \mathbb{A}^1 -homotopy category by BG . These results generalize to a wide class of groups the results of Morel’s that are foundational to the study of vector bundles. Unfortunately for the applicability of the theory, however, for many algebraic groups G of interest, the set of torsors of interest is $H_{\text{ét}}^1(\cdot, G)$ rather than $H_{\text{Zar}}^1(\cdot, G)$ or $H_{\text{Nis}}^1(\cdot, G)$; therefore the theory is most interesting in the case of special algebraic groups in the sense of [Ser95]—these groups include SL_n , GL_n and Sp_n . In [AHW17b], the same authors apply their techniques to G_2 , where $H_{\text{Nis}}^1(\cdot, G_2)$ classifies “split octonion algebras”.

The means by which Asok–Hoyois–Wendt establish their representability result is a strong general representability result: working over a general quasicompact, quasiseparated base scheme, and letting \mathcal{F} denote a presheaf such that $\pi_0(\mathcal{F})$ is an \mathbb{A}^1 -invariant functor and such that \mathcal{F} satisfies a certain “affine Nisnevich excision” property—a Mayer–Vietoris style property—then $\pi_0(\mathcal{F})(U) = [U, \mathcal{F}]_{\mathbb{A}^1}$ for affine S -schemes, [AHW17a, Theorem 5.1.3]. If one restricts to an infinite base field k , then [AHW18] shows that if G is an isotropic reductive group scheme, then the simplicial classifying space BG —or, more correctly, a Nisnevich fibrant replacement—satisfies these hypotheses.

6.5. Naïve \mathbb{A}^1 -homotopy. If $\mathcal{F} : \mathbf{Ring} \rightarrow \mathbf{Set}$ is a functor defined on (affine) rings, or some subcategory of rings, then it is possible to define *naïve \mathbb{A}^1 -homotopy classes* of elements in $F(R)$. One considers the two evaluation maps $R[t] \rightarrow R$ given by evaluation at $t = 0$ and $t = 1$. A naïve homotopy between a and b in $F(R)$ is an element $H \in F(R[t])$ such that under the two evaluation maps $H \mapsto a$ and $H \mapsto b$. Of course, there is no reason in general to this should be an equivalence relation, so one takes the equivalence relation generated by it.

The definition above also applies to contravariant functors on schemes, but it is on affine schemes that it works best. When \mathcal{F} is representable, say by a scheme X , then naïve homotopy on $\mathcal{F}(R)$ in the sense above recovers the theory of equivalence classes of maps $\text{Spec } R \rightarrow X$ under the relation generated by declaring the two restrictions of a map $H : \text{Spec } R[t] \rightarrow X$ at $t = 0$ and $t = 1$ to be equivalent.

It is worth noting that the naïve \mathbb{A}^1 -equivalence classes of maps $\text{Spec } R \rightarrow X$ are exactly the path components of the \mathbb{A}^1 singular construction $\text{Sing}^{\mathbb{A}^1} X$ evaluated at R , i.e., $\pi_0(\text{Sing}^{\mathbb{A}^1} X(R))$. In general, $\text{Sing}^{\mathbb{A}^1} X$ is not an \mathbb{A}^1 -local object, so that naïve \mathbb{A}^1 -homotopy is not the same as genuine \mathbb{A}^1 -homotopy. This is perhaps the great challenge in unstable \mathbb{A}^1 -homotopy—the naïve, geometrically appealing construction does not coincide with the genuine construction possessed of good theoretical properties. One consequence of the work of [AHW17a], [AHW18], however, is that in a number of cases, genuine and naïve \mathbb{A}^1 -homotopy agree.

For an example where the naïve and genuine \mathbb{A}^1 -homotopy classes of maps differ, we refer to [Caz08]. In this paper, the naïve \mathbb{A}^1 -homotopy classes of maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ are calculated. In order to put this

in context, we should mention that in [Mor12, Theorem 7.36 and Remark 7.37], Morel calculated

$$(14) \quad [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} = \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times.$$

It is to be remarked that although \mathbb{P}^1 is a motivic sphere, this calculation lies outside the main body of calculations of self-maps of spheres, since $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ is not \mathbb{A}^1 -simply-connected. The main theorem of [Caz08] is that the naïve homotopy classes of maps of spheres make up the submonoid of $\mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$ corresponding to elements in $\mathrm{GW}(k)$ of positive degree, so that the map from the set of naïve \mathbb{A}^1 -homotopy classes of maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ to the set of all \mathbb{A}^1 -homotopy classes is, in fact, a group completion.

The following is a part of [AHW18, Theorem 2.3.2]. As before, the base S is quasicompact and quasiseparated.

Theorem 6.3. *Suppose G is a finitely presented smooth S -group scheme and that H is a finitely presented closed subgroup scheme such that G/H is an S -scheme, and such that $G \rightarrow G/H$ is Nisnevich locally split, and suppose that $H_{\mathrm{Nis}}^1(\cdot, G)$ and $H_{\mathrm{Nis}}^1(\cdot, H)$ are \mathbb{A}^1 -invariant on affine schemes, then for an affine scheme $\mathrm{Spec} R$, the natural map*

$$\pi_0(\mathrm{Spec}^{\mathbb{A}^1}(G/H)(R)) \rightarrow [\mathrm{Spec} R, G/H]_{\mathbb{A}^1}$$

is a bijection.

That is, for G/H , provided the source is affine, the naïve and genuine \mathbb{A}^1 -homotopy classes of maps agree. The restrictions on G , H are not perhaps as inconvenient as might be feared, since, in contrast to the case of the étale topology, the Nisnevich cohomology $H_{\mathrm{Nis}}^1(\cdot, G)$ is liable to be \mathbb{A}^1 -invariant. In fact, in [AHW18, Theorem 3.3.6], the authors establish \mathbb{A}^1 invariance for all isotropic reductive group schemes over infinite fields. The more severe restriction in the theorem is that $G \rightarrow G/H$ should be Nisnevich locally split, but nonetheless, the theorem provides a wide range of notable spaces for which naïve and genuine \mathbb{A}^1 -homotopy are the same.

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