CLASSIFICATION OF PROBLEMATIC SUBGROUPS OF $U(n)$

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Abstract. Let $\mathcal{L}_n$ denote the topological poset of decompositions of $\mathbb{C}^n$ into mutually orthogonal subspaces. We classify p-toral subgroups of $U(n)$ that can have noncontractible fixed points under the action of $U(n)$ on $\mathcal{L}_n$.

1. Introduction

Throughout the paper, let p denote a fixed prime. Let $\mathcal{P}_n$ be the Σ_n-space given by the nerve of the poset category of proper nontrivial partitions of the set $\{1, \ldots, n\}$. In [ADL16], the authors compute the Bredon homology and cohomology groups of $\mathcal{P}_n$ with certain p-local coefficients. The computation is part of a program to give a new proof of the Whitehead Conjecture and the collapse of the homotopy spectral sequence for the Goodwillie tower of the identity, one that does not rely on the detailed knowledge of homology used in [Kuh82], [KP85], and [Beh11]. For appropriate p-local coefficients, the Bredon homology and cohomology of $\mathcal{P}_n$ turn out to be trivial when n is not a power of the prime p, and nontrivial in only one dimension when $n = p^k$ ([ADL16] Theorem 1.1 and Corollary 1.2). A key ingredient of the proof is the identification of the fixed point spaces of p-subgroups of Σ_n acting on $\mathcal{P}_n$. If a p-subgroup $H \subseteq \Sigma_n$ has noncontractible fixed points, then H gives an obstruction to triviality of Bredon homology, so it is "problematic" (see Definition 1.1 below). It turns out that only elementary abelian p-subgroups of Σ_n with free action on $\{1, \ldots, n\}$ can have noncontractible fixed points on $\mathcal{P}_n$ ([ADL16] Proposition 6.2). The proof of the main result of [ADL16] then proceeds by showing that, given appropriate conditions on the coefficients, these problematic subgroups can be "pruned" or "discarded" in most cases, resulting in sparse Bredon homology and cohomology.

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In this paper, we carry out the fixed point calculation analogous to that of [ADL16] in the context of unitary groups. The calculation is part of a program to establish the conjectured *bu*-analogue of the Whitehead Conjecture and the conjectured collapse of the homotopy spectral sequence of the Weiss tower for the functor $V \mapsto BU(V)$ (see [AL07] Section 12). Let $\mathcal{L}_n$ denote the (topological) poset category of decompositions of $\mathbb{C}^n$ into nonzero proper orthogonal subspaces. The category $\mathcal{L}_n$ was first introduced in [Aro02]. It is internal to the category of topological spaces: both its object set and its morphism set have a topology. (See [BJL$^+$15] for a detailed discussion and examples.) The action of the unitary group $U(n)$ on $\mathbb{C}^n$ induces a natural action of $U(n)$ on $\mathcal{L}_n$, and the Bredon homology of the $U(n)$-space $\mathcal{L}_n$ plays an analogous role in the unitary context to the part played by the Bredon homology of the Σ_n-space $\mathcal{P}_n$ in the classical context. The complex $\mathcal{L}_n$ has a similar flavor to the "stable building" or "common basis complex" studied in [Rog92, Rog].

In passing from finite groups to compact Lie groups, one usually replaces the notion of finite p-groups with that of p-toral groups (i.e., extensions of finite p-groups by tori). Our goal in this paper is to identify the p-toral subgroups of $U(n)$ that have noncontractible fixed point spaces on $\mathcal{L}_n$. These groups will be the obstructions to triviality of the Bredon homology and cohomology of $\mathcal{L}_n$ with suitable p-local coefficients. Hence we make the following definition.

Definition 1.1. A closed subgroup $H \subseteq U(n)$ is called *problematic* if the fixed point space of H acting on the nerve of $\mathcal{L}_n$ is not contractible.

In this paper, we give a complete classification of problematic p-toral subgroups of $U(n)$. To state the result, we recall that there is a family of p-toral subgroups $\Gamma_k \subseteq U(p^k)$ with the following key properties: (i) Γ_k acts irreducibly on $\mathbb{C}^{p^k}$, and (ii) Γ_k is an extension of the central $S^1 \subseteq U(p^k)$ by an elementary abelian p-group. The subgroups Γ_k have appeared in numerous works, such as [Gri91], [JMO92], [Oli94], [Aro02], [AL07], and [AGMV08]. The structure of $\Gamma_k \subset U(p^k)$ is described explicitly in Section 6, and we call the corresponding action of Γ_k on $\mathbb{C}^{p^k}$ the *standard action* of Γ_k.

For any $n = mp^k$, we can consider Γ_k acting on $\mathbb{C}^{mp^k}$ by the m-fold multiple of the standard action. That is, we choose a fixed isomorphism $\mathbb{C}^n \cong \mathbb{C}^m \otimes \mathbb{C}^{p^k}$, and we let Γ_k act trivially on $\mathbb{C}^m$, and by the standard action of Γ_k on $\mathbb{C}^{p^k}$. A subgroup $H \subseteq \Gamma_k$ is coisotropic if it contains the subgroup of Γ_k that centralizes it, $C_{\Gamma_k}(H) \subseteq H$. (These are, in fact, the p-centric subgroups of Γ_k.) When we write S^1, we always mean the center of $U(n)$.

Theorem 1.2. *Suppose that $n = mp^k$, where m is coprime to p, and suppose that $H \subseteq U(n)$ is a p-toral subgroup of $U(n)$ that contains S^1. Let Γ_k act on $\mathbb{C}^n$ by m copies of the standard representation of Γ_k on $\mathbb{C}^{p^k}$.*

(1) *If $m = 1$, then H is problematic if and only if H is conjugate to a subgroup of Γ_k.*
(2) *If $m > 1$, then H is problematic if and only if m is a power of a prime different from p and H is conjugate to a coisotropic subgroup of Γ_k.*

Note that there is no loss of generality in assuming that $S^1 \subseteq H$ in Theorem 1.2. This is because S^1 acts on $\mathbb{C}^n$ via multiplication by scalars, and fixes any object of $\mathcal{L}_n$. Further, H is a p-toral subgroup of $U(n)$ if and only if HS^1 is p-toral, and $(\mathcal{L}_n)^{HS^1} = (\mathcal{L}_n)^H$. Hence H is problematic if and only if HS^1 is problematic.

The remainder of the introduction is devoted to outlining the proof of Theorem 1.2. We begin with the special case $H = S^1$. Since $(\mathcal{L}_n)^{S^1} = \mathcal{L}_n$, the question of whether S^1 is problematic is asking if $\mathcal{L}_n$ is contractible. We only need one new ingredient to answer this question.

Theorem 3.1. *If $n \geq 3$, then $\mathcal{L}_n$ is simply connected.*

Homology considerations then give us the following corollary, which says that S^1 is problematic if and only if n is a power of a prime.

Corollary 3.2. *The space $\mathcal{L}_n$ is contractible if and only if n is not a power of a prime.*

For p-toral subgroups $H \subseteq U(n)$ that strictly contain S^1, the proof of Theorem 1.2 has two major components. The first is a general argument to establish that if $n = mp^k$ (with m and p coprime) and H is a problematic p-toral subgroup of $U(n)$, then H is conjugate to a subgroup of $\Gamma_k \subset U(n)$. The second consists of checking which of these remaining possibilities are actually problematic, by analyzing the resulting fixed point spaces.

To outline the reduction to subgroups of $\Gamma_k \subset U(n)$, we need a little more terminology. The center of $U(n)$ is S^1, acting on $\mathbb{C}^n$ via scalar multiples of the identity matrix, and the *projective unitary group* $PU(n)$ is the quotient $U(n)/S^1$. We say that a closed subgroup $H \subseteq U(n)$ is a *projective elementary abelian p-group* if the image of H in $PU(n)$ is an elementary abelian p-group. Lastly, if H is a closed subgroup of $U(n)$, we write χ_H for the character of H acting on $\mathbb{C}^n$ through the standard action of $U(n)$ on $\mathbb{C}^n$.

Theorem 5.4. *Let H be a problematic p-toral subgroup of $U(n)$ that contains S^1. Then*

(1) *H is a projective elementary abelian p-group, and*
(2) *the character of H is* $\begin{cases} \chi_H(h) = 0 & h \notin S^1 \\ \chi_H(h) = nh & h \in S^1. \end{cases}$

The first part of Theorem 5.4 was the principal result of [BJL+15]. For completeness, we give a streamlined proof in the current work, in order to obtain the second part of the theorem, the subgroup's character.

The character data of Theorem 5.4 allows us to narrow down the problematic p-toral subgroups of $U(n)$ to a very small, explicitly described collection. The subgroups $\Gamma_k \subseteq U\big(p^k\big)$ (see Section 6) satisfy the conclusions of Theorem 5.4. The next step is to show that they generate all the possibilities.

Theorem 8.3. *Let H be a problematic p-toral subgroup of $U(n)$ that contains S^1, and suppose that $n = mp^k$, where m and p are coprime. Then H is conjugate to a subgroup of $\Gamma_k \subset U(n)$.*

The strategy of the proof of Theorem 8.3 is to use the first part of Theorem 5.4 and bilinear forms to classify H up to abstract group isomorphism. Then character theory, together with the second part of Theorem 5.4, allows us to pinpoint the actual conjugacy class of H in $U(n)$.

After this, the remaining question for the classification theorem is whether all of the groups named in Theorem 8.3 are, in fact, problematic. Our strategy is to compute the fixed point spaces fairly explicitly. Let $X * Y$ denote the join of the spaces X and Y. The following formula was suggested to us by Gregory Arone.

Theorem 9.2. *Suppose that $n = mp^k$ with $k \geq 1$. Then there is an equivalence of $U(m)$-spaces*

$$\mathcal{L}_m * \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} \to \left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}.$$

The remainder of the proof of Theorem 1.2 is then a homology calculation, based on the formula of Theorem 9.2.

The organization of the paper is as follows.

In the first part of the paper, Section 2 and Section 3, we describe some elementary properties of $\mathcal{L}_n$ and prove the simple connectivity result for $n > 2$.

The second part of the paper is devoted to establishing the list of candidates for problematic subgroups. In Section 4 we give a normal subgroup condition from which one can deduce contractibility of a fixed point set $(\mathcal{L}_n)^H$. We follow up in Section 5 with the proof of Theorem 5.4, by finding an appropriate normal subgroup unless H is a projective elementary abelian p-subgroup of $U(n)$. Section 6 is expository and discusses the salient properties of the subgroup $\Gamma_k \subseteq U\big(p^k\big)$. The projective elementary abelian p-subgroups of $U(n)$ are classified up to abstract group isomorphism using bilinear forms in Section 7. They are shown to be isomorphic to $\Gamma_s \times \Delta_t$ where Δ_t denotes $(\mathbb{Z}/p)^t$. In Section 8, we prove Theorem 8.3, allowing us to view this abstract group isomorphism as an isomorphism of representations.

The third part of the paper is devoted to checking the candidates identified in Theorem 8.3. Section 9 calculates the $U(m)$-equivariant homotopy type of the fixed points of the action of Γ_s on $\mathcal{L}_{mp^s}$ (Theorem 9.2). This allows us to compute the fixed points of the action of $\Gamma_s \times \Delta_t$ on $\mathcal{L}_{mp^s}$ by first computing the fixed points under Γ_s, reducing the problem to the computation of Δ_t-fixed points. The bulk of Section 10 consists of analyzing the problem of Δ_t-fixed points, after which we assemble the pieces to prove Theorem 1.2.

Definitions, Notation, and Terminology

When we speak of a subgroup of a Lie group, we always mean a closed subgroup. If we speak of $S^1 \subseteq U(n)$ without any further description, we mean the center of $U(n)$.

We generally do not distinguish in notation between a category and its nerve, since the context will make clear which we mean.

We are concerned with actions of subgroups $H \subseteq U(n)$ on $\mathbb{C}^n$; we write ρ_H for the restriction of the standard representation of $U(n)$ to H, and χ_H for the corresponding character. We apply the standard terms from representation theory to H if they apply to ρ_H. For example, we say that H is *irreducible* if ρ_H is irreducible, and we say that H is *isotypic* if ρ_H is the sum of all isomorphic irreducible representations of H. We also introduce a new term: if H is not isotypic, we say that H is *polytypic*, as a succinct way to say that "the action of H on $\mathbb{C}^n$ is not isotypic."

A *decomposition* λ of $\mathbb{C}^n$ is an (unordered) decomposition of $\mathbb{C}^n$ into mutually orthogonal, nonzero subspaces. We say that λ is *proper* if it consists of subspaces properly contained in $\mathbb{C}^n$. If $v_1, \ldots, v_m$ are the subspaces in λ, then we call $v_1, \ldots, v_m$ the *components* or *classes* of λ; we write $\mathrm{cl}(\lambda)$ for $\{v_1, \ldots, v_m\}$ when we want to emphasize the set of subspaces in the decomposition λ.

The action of $U(n)$ on $\mathbb{C}^n$ induces a natural action of $U(n)$ on $\mathcal{L}_n$, and we are interested in fixed points of this action. If λ consists of the subspaces $v_1,\ldots,v_m$, we say that λ is *weakly fixed* by a subgroup $H \subseteq U(n)$ if for every $h \in H$ and every v_i, there exists a j such that $hv_i = v_j$. (We use the word "weakly fixed" here to contrast with the notion of "strongly fixed," below.) We write $(\mathcal{L}_n)^H$ for the full subcategory of weakly H-fixed objects of $\mathcal{L}_n$. Note that $\text{Nerve}\left((\mathcal{L}_n)^H\right) \cong \left(\text{Nerve}(\mathcal{L}_n)\right)^H$, and we abuse notation by writing $(\mathcal{L}_n)^H$ for either the subcategory or its nerve, depending on context.

By contrast, we say that λ is *strongly fixed* by $H \subseteq U(n)$ if for all i, we have $Hv_i = v_i$, that is, each v_i is a representation of H. We write $(\mathcal{L}_n)^H_{\text{st}}$ for the full subcategory of strongly H-fixed objects of $\mathcal{L}_n$ (and for its nerve).

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2. The topological category $\mathcal{L}_n$

The goal of this section is to offer descriptions and examples of the object and morphism spaces of $\mathcal{L}_n$, to build intuition for the category. We devote some attention to the fundamental groups of the connected components of the object and morphism spaces, since the goal of Section 3 is to establish simple connectivity of the nerve of $\mathcal{L}_n$. We also refer the reader to [BJL$^+$15], where some low-dimensional cases were studied explicitly. For example, it was established that $\mathcal{L}_2$ is homeomorphic to $\mathbb{R}P^2$.

From time to time, we refer to the category of unordered partitions of an integer n, by which we mean the quotient of the action of Σ_n

on the set of partitions of the set $\mathbf{n} = \{1, \ldots, n\}$. We sometimes use a notation for unordered partitions where sizes of components in the partition are underlined, and the multiplicity of components of a certain size is indicated as scalar multiplication. For example, the unordered partition $7 = 1 + 3 + 3$ is denoted $\underline{1} + 2 \cdot \underline{3}$, because the partition has two components of size 3 and one component of size 1.

We begin with the object space of $\mathcal{L}_n$, denoted by $\mathrm{Obj}\,(\mathcal{L}_n)$. The elements are (unordered) decompositions of $\mathbb{C}^n$ into proper, mutually orthogonal subspaces. There is a natural action of $U(n)$ on $\mathrm{Obj}\,(\mathcal{L}_n)$, and we topologize $\mathrm{Obj}\,(\mathcal{L}_n)$ as the disjoint union of orbits of the $U(n)$ action. (This description and another equivalent one are given in [BJL$^+$15].)

Definition 2.1.

(1) If $\lambda \in \mathrm{Obj}\,(\mathcal{L}_n)$, then the *type* of λ, denoted by $t(\lambda)$, is the unordered partition of the integer n given by the dimensions of the components of λ.
(2) For $\lambda \in \mathrm{Obj}\,(\mathcal{L}_n)$, we write $K_\lambda \subseteq U(n)$ for the isotropy subgroup of λ under the action of $U(n)$, i.e. K_λ contains those elements of $U(n)$ that weakly fix λ.
(3) If τ is an unordered partition of the integer n, we gather the decompositions with this type by defining

$$C_\tau = \{\lambda \mid t(\lambda) = \tau\} \subseteq \mathrm{Obj}\,(\mathcal{L}_n)\,.$$

Since $\mathrm{Obj}\,(\mathcal{L}_n)$ is topologized as the disjoint union of $U(n)$-orbits, the transitive action of $U(n)$ on n-frames for $\mathbb{C}^n$ gives us the following result.

Lemma 2.2. *The subspace $C_{t(\lambda)}$ is a homogeneous space; specifically,*

$$C_{t(\lambda)} \cong U(n)\,/K_\lambda.$$

The connected components of $\mathrm{Obj}\,(\mathcal{L}_n)$ *are given by the spaces C_τ as τ ranges over unordered partitions of the integer n.*

Example 2.3. Consider decompositions of $\mathbb{C}^n$ into n mutually orthogonal lines. An element of the isotropy group of such a decomposition can act by $U(1)$ on each of the lines, and can also permute the lines, so the isotropy group is $U(1) \wr \Sigma_n$. The connected component of $\mathrm{Obj}\,(\mathcal{L}_n)$ consisting of all decompositions into lines is

$$C_{n\cdot\underline{1}} \cong U(n)\,/\,(U(1) \wr \Sigma_n)\,.$$

Before moving on, we record a generalization of the computation of Example 2.3.

Lemma 2.4. *If* $\lambda \in \operatorname{Obj}(\mathcal{L}_n)$ *has type* $t(\lambda) = k_1 \cdot \underline{n}_1 + \cdots + k_j \cdot \underline{n}_j$, *then*

$$K_\lambda \cong (U(n_1) \wr \Sigma_{k_1}) \times \cdots \times \left(U(n_j) \wr \Sigma_{k_j}\right).$$

Next we consider $\operatorname{Morph}(\mathcal{L}_n)$, the morphism space of $\mathcal{L}_n$. Between any two objects of $\operatorname{Obj}(\mathcal{L}_n)$, there is at most one morphism. Hence the source and target maps give a $U(n)$-equivariant monomorphism

$$\operatorname{Morph}(\mathcal{L}_n) \longrightarrow \operatorname{Obj}(\mathcal{L}_n) \times \operatorname{Obj}(\mathcal{L}_n),$$

and we give $\operatorname{Morph}(\mathcal{L}_n)$ the subspace topology.

Definition 2.5.

(1) If $m\colon \lambda \to \mu$ is a morphism in $\mathcal{L}_n$, the *type* of m, denoted by $t(m)$ or $t(\lambda \to \mu)$, is the morphism that m induces in the category of unordered partitions of the integer n.

(2) We define $M_{t(\lambda\to\mu)} \subseteq \operatorname{Morph}(\mathcal{L}_n)$ by

$$M_{t(\lambda\to\mu)} = \{\lambda' \to \mu' \mid t(\lambda' \to \mu') = t(\lambda \to \mu)\}.$$

Example 2.6. Let $\lambda = \{a, b, c, v\}$ be an orthogonal decomposition of $\mathbb{C}^5$ into lines a, b, and c and a two-dimensional subspace v. There are two different types of morphisms from λ to objects in $C_{\underline{2}+\underline{3}}$. The first is:

$$\begin{gathered}\{a, b, c, v\} \longrightarrow \{a + b + c, v\}\\ (\underline{1} + \underline{1} + \underline{1}) + \underline{2} \longrightarrow \underline{3} + \underline{2}.\end{gathered}$$

The second type takes the sum of two of the lines, and adds the third line to the two-dimensional subspace. There are three morphisms of this type that have λ as their source:

$$\begin{gathered}\{a, b, c, v\} \longrightarrow \{a + b, c + v\}\\ \{a, b, c, v\} \longrightarrow \{a + c, b + v\}\\ \{a, b, c, v\} \longrightarrow \{b + c, a + v\}\\ (\underline{1} + \underline{1}) + (\underline{1} + \underline{2}) \longrightarrow \underline{2} + \underline{3}.\end{gathered}$$

However, the three morphisms of the second type are in the same orbit of the action of $U(n)$ on $\operatorname{Morph}(\mathcal{L}_n)$, whereas the first type of morphism is in a different $U(n)$-orbit.

We have a parallel result to that of Lemma 2.2.

Lemma 2.7. *The subspace* $M_{t(\lambda\to\mu)}$ *is a homogeneous space; specifically,*

$$M_{t(\lambda\to\mu)} \cong U(n) \,/\, (K_\lambda \cap K_\mu).$$

The connected components of Morph $(\mathcal{L}_n)$ *are given by the spaces* $M_{t(\lambda\to\mu)}$ *as the type of* $\lambda \to \mu$ *ranges over the morphisms of unordered partitions of the integer* n.

Proof. We must show that $U(n)$ acts transitively on $M_{t(\lambda\to\mu)}$. Suppose that $t(\lambda' \to \mu') = t(\lambda \to \mu)$. Then μ and μ' have the same type, so there exists $u \in U(n)$ such that $u\,\mu = \mu'$. It is not necessarily the case that $u\lambda = \lambda'$. However, $u\lambda$ does have the same type as λ', and both are refinements of μ'. Hence there exists $u' \in K_{\mu'}$ such that $u'u\lambda = \lambda'$.

To compute the isotropy group of $\lambda \to \mu$, we recall that there is at most one morphism between any two objects of $\mathcal{L}_n$. Hence $\lambda \to \mu$ is fixed by $u \in U(n)$ if and only if u fixes both λ and μ. □

3. $\mathcal{L}_n$ IS SIMPLY CONNECTED

In [BJL+15], we proved that the nerve of $\mathcal{L}_3$ is simply connected by exhibiting the object space (which has two connected components) and the morphism space (which has one connected component), and using the Van Kampen Theorem. In this section, we use similar methods to show more generally that $\mathcal{L}_n$ is simply connected when $n \geq 3$. Since $\mathcal{L}_1$ is empty, and we know $\mathcal{L}_2 \cong \mathbb{R}P^2$ [BJL+15, Prop. 2.1], the following theorem completes the understanding of the fundamental group of $\mathcal{L}_n$ for all n.

Theorem 3.1. *If* $n \geq 3$*, then* $\mathcal{L}_n$ *is simply connected.*

Given Theorem 3.1, the contractibility of $\mathcal{L}_n$ is determined by homology.

Corollary 3.2. *The space* $\mathcal{L}_n$ *is contractible if and only if* n *is not a power of a prime.*

Proof. The mod p homology of $\mathcal{L}_{p^k}$ is nonzero [Aro02, Theorem 4(b)], so certainly $\mathcal{L}_{p^k}$ is not contractible. If n is not a prime power, then $n > 2$, so $\mathcal{L}_n$ is simply connected by Theorem 3.1. But if n is not a power of a prime, then $\mathcal{L}_n$ is acyclic both rationally and at all primes by another result of Arone (see, for example, [AL07, Prop. 9.6]). The result follows. □

We build on the results of Section 2 to give an elementary approach to proving Theorem 3.1. We begin by considering the fundamental group of each connected component of Obj $(\mathcal{L}_n)$.

Lemma 3.3. *Suppose that* $\lambda \in$ Obj $(\mathcal{L}_n)$ *has type* $k_1\underline{n}_1 + \cdots + k_j\underline{n}_j$. *Then* $\pi_1 C_\lambda \cong \prod_{i=1}^{j} \Sigma_{k_i}$.

Proof. Recall from Lemmas 2.2 and 2.4 that $C_\lambda \cong U(n)/K_\lambda$ and $K_\lambda \cong \prod_{i=1}^{j}(U(n_i) \wr \Sigma_{k_i})$ However, if G is a compact Lie group with maximal torus T, then $\pi_1 T \to \pi_1 G$ is surjective. (See, for example, Corollary 5.17 in [MT91].) The subgroup K_λ contains a maximal torus of $U(n)$, so $\pi_1 K_\lambda \to \pi_1 U(n)$ is an epimorphism. The lemma then follows from the long exact sequence of homotopy groups associated to $K_\lambda \to U(n) \to U(n)/K_\lambda$, because $\pi_0 K_\lambda \cong \prod_{i=1}^{j}\Sigma_{k_i}$ and $U(n)$ is connected. □

We need an easily-applied criterion for the Van Kampen Theorem to yield simple connectivity, which is the purpose of the following algebraic lemma.

Lemma 3.4. *Consider a diagram of groups*

$$A \xleftarrow{f} C \xrightarrow{g} B.$$

Let $K = \ker(C \xrightarrow{g} B)$ *and assume the following conditions.*

1. *The normal closure of* $\mathrm{im}(K \xrightarrow{f} A) \subseteq A$ *is* A*.*
2. *The normal closure of* $\mathrm{im}(C \xrightarrow{g} B) \subseteq B$ *is* B*.*

Then $A *_C B$ *is the trivial group.*

Proof. We first show that any element of A, regarded as an element of $A *_C B$, is actually trivial. By definition, elements of the free product of A and B that have the form $f(c)g(c)^{-1}$ are null in the amalgamated product $A *_C B$. Since the normal closure of K in A is A itself, any element of A can be written as a product of elements of the form $y = af(k)a^{-1}$ where $a \in A$ and $k \in K$. However, $g(k) = e$, so $y = af(k)a^{-1} = af(k)g(k)^{-1}a^{-1}$, and y becomes null in $A *_C B$.

Likewise, any element of B can be written as the product of elements $z = bg(c)b^{-1}$ where $b \in B$ and $c \in C$. We can write $z = bg(c)b^{-1} = bf(c)\left[f(c)^{-1}g(c)\right]b^{-1}$, that is, $z = bf(c)b^{-1}$ in the amalgamated product. But there exist $k \in K$ and $a \in A$ such that $f(c) = af(k)a^{-1}$. Hence, in the amalgamated product $z = b\left(af(k)a^{-1}\right)b^{-1}$. But since $k \in K$, we know that $f(k) = e$ in the amalgamated product, from which it follows that z also becomes trivial in $A *_C B$. □

Let $\mathrm{sk}^i(\mathcal{L}_n)$ denote the i-skeleton of the nerve of $\mathcal{L}_n$. It is sufficient to show that the map

$$\pi_1 \mathrm{sk}^1(\mathcal{L}_n) \to \pi_1 \mathrm{sk}^2(\mathcal{L}_n)$$

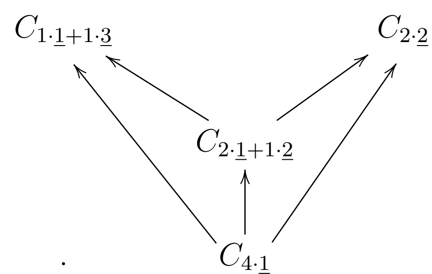
is trivial, so we begin by analyzing paths in $\mathrm{sk}^1(\mathcal{L}_n)$ in detail.

Given a decomposition $\lambda \in \mathrm{Obj}(\mathcal{L}_n)$, recall that $C_{t(\lambda)}$ is the connected component of $\mathrm{Obj}(\mathcal{L}_n)$ consisting of all objects with the same type as λ (Definition 2.1 (3) and Lemma 2.2). Likewise, $M_{t(\lambda\to\mu)}$ is the connected component of $\mathrm{Morph}\,(\mathcal{L}_n)$ containing morphisms of the same type as $\lambda\to\mu$ (Definition 2.5 (3) and Lemma 2.7). Finally, we write $\mathrm{Cyl}\,(\lambda\to\mu)$ for the homotopy pushout of the diagram

$$\begin{array}{ccc} M_{t(\lambda\to\mu)} & \xrightarrow{\text{target}} & C_\mu \\ {\scriptstyle\text{source}}\downarrow & & \\ C_\lambda & & \end{array} \qquad . \tag{3.5}$$

The space $\mathrm{sk}^1(\mathcal{L}_n)$ can be written as the (finite) union of spaces $\mathrm{Cyl}\,(\lambda\to\mu)$ where $\lambda\to\mu$ ranges over a set of representatives of the connected components of $\mathrm{Morph}\,(\mathcal{L}_n)$. Note that these cylinders are not disjoint. For example, the following diagram shows connected components of $\mathrm{Obj}\,(\mathcal{L}_4)$, with arrows between those that are connected by morphisms:

(3.6)



Thus $\mathrm{sk}^1(\mathcal{L}_4)$ is the union of five non-degenerate cylinders, three of which contain the object component $C_{4\cdot\underline{1}}$.

Using the Van Kampen Theorem, the fundamental group of any double mapping cylinder $\mathrm{Cyl}\,(\lambda\to\mu)$ can be expressed as a quotient of the free product of $\pi_1 C_\lambda$ and $\pi_1 C_\mu$. We illustrate with a special case that turns out to do much of the work to prove Theorem 3.1.

Proposition 3.7. *Let μ be any decomposition containing at least one subspace of dimension greater than one, and let ϵ be a refinement of μ into lines. Then $\mathrm{Cyl}\,(\epsilon\to\mu)$ is simply connected.*

Proof. Suppose that μ has type $k_1\underline{n}_1+\cdots+k_j\underline{n}_j$. Then

$$K_\mu \cong (U(n_1)\wr\Sigma_{k_1})\times\cdots\times\left(U(n_j)\wr\Sigma_{k_j}\right),$$

and by Lemma 3.3 we have $\pi_1 C_\mu\cong\prod_{i=1}^{j}\Sigma_{k_i}$. Likewise

$$K_\epsilon\cong U(1)\wr\Sigma_n$$

and by Lemma 3.3 we have $\pi_1 C_\epsilon\cong\Sigma_n$. Finally, $U(n)$ acts transitively on $M_{t(\epsilon\to\mu)}$, with isotropy group $K_\epsilon\cap K_\mu$ (Lemma 2.7). We can compute

$$K_\epsilon\cap K_\mu\cong(U(1)\wr\Sigma_{n_1})\wr\Sigma_{k_1})\times\cdots\times\left(U(1)\wr\Sigma_{n_j})\wr\Sigma_{k_j}\right),$$

and as a result, by a similar argument to Lemma 3.3, we have

$$\pi_1 M_{t(\epsilon\to\mu)} \cong \left(\Sigma_{n_1} \wr \Sigma_{k_1}\right) \times \cdots \times \left(\Sigma_{n_j} \wr \Sigma_{k_j}\right).$$

Hence applying π_1 to diagram (3.5) gives us

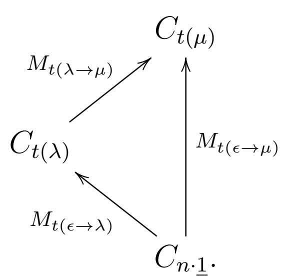
$$\begin{array}{ccc} \left(\Sigma_{n_1} \wr \Sigma_{k_1}\right) \times \cdots \times \left(\Sigma_{n_j} \wr \Sigma_{k_j}\right) & \longrightarrow & \Sigma_{k_1} \times \cdots \times \Sigma_{k_j} \\ \downarrow & & \\ \Sigma_n & & , \end{array}$$

where the top map uses the projection for each of the wreath products, and the vertical map is inclusion (recall that $n_1k_1 + \cdots + k_jn_j = n$). It now follows from Lemma 3.4 that the fundamental group of $\mathrm{Cyl}\,(\epsilon \to \mu)$ is trivial. □

If we apply Proposition 3.7 to diagram (3.6), we see that we have simple connectivity of each of the three cylinders containing $C_{4\cdot\underline{1}}$. However, it is possible to make homotopically essential loops in the 1-skeleton by making circuits of the triangles in the diagram, so we focus attention on such triangles.

Given a morphism $\lambda \to \mu$ in $\mathcal{L}_n$, choose a refinement ϵ of λ into one-dimensional components, and consider the "triangle"

(3.8)



We define

$$T_{t(\lambda\to\mu)} = \mathrm{Cyl}\,(\epsilon \to \mu) \cup \mathrm{Cyl}\,(\epsilon \to \lambda) \cup \mathrm{Cyl}\,(\lambda \to \mu)\,,$$

a path-connected space that depends only on the morphism type of $\lambda \to \mu$. Consider the following paths:

$$(3.9)\qquad \begin{cases} \alpha_1(t) = (\epsilon \to \lambda, t) & \text{in } \mathrm{Cyl}\,(\epsilon \to \lambda) \\ \alpha_2(t) = (\lambda \to \mu, t) & \text{in } \mathrm{Cyl}\,(\lambda \to \mu) \\ \alpha_3(t) = (\epsilon \to \mu, t) & \text{in } \mathrm{Cyl}\,(\epsilon \to \mu) \end{cases}$$

Let $\alpha = \alpha_1 * \alpha_2 * \alpha_3^{-1}$ be the concatenation of the three paths to form a loop in $T_{t(\lambda\to\mu)}$ based at ϵ.

Lemma 3.10. *If λ contains at least one subspace of dimension greater than one, then $\pi_1\left(T_{t(\lambda\to\mu)}, \epsilon\right)$ is generated by the loop α given above.*

Proof. Any loop based at ϵ can be written as a finite concatenation of paths in $\mathrm{Cyl}\,(\epsilon \to \lambda)$, $\mathrm{Cyl}\,(\lambda \to \mu)$, and $\mathrm{Cyl}\,(\epsilon \to \mu)$. Furthermore, path-connectedness of $C_{t(\lambda)}$, $C_{t(\mu)}$, and $C_{n\cdot\underline{1}}$ means that we can assume that the paths in the concatenation have endpoints at ϵ, λ, or μ. Hence it is sufficient to analyze paths in the cylinders with those endpoints.

By Proposition 3.7, $\mathrm{Cyl}\,(\epsilon \to \lambda)$ is simply connected. Hence any path in $\mathrm{Cyl}\,(\epsilon \to \lambda)$ beginning at ϵ and ending at λ is path homotopic to α_1. Similarly, any path in $\mathrm{Cyl}\,(\epsilon \to \mu)$ beginning at ϵ and ending at μ is path homotopic to α_3.

To finish the proof, we need to show that any path in $\mathrm{Cyl}\,(\lambda \to \mu)$ from λ to μ is path homotopic in $T_{t(\lambda\to\mu)}$ to α_2. Note that we need the entire triangle. Unlike the previous cases, it is not generally true that a path from λ to μ in $\mathrm{Cyl}\,(\lambda \to \mu)$ is path homotopic to α_2 by a path homotopy that stays inside $\mathrm{Cyl}\,(\lambda \to \mu)$.

It is sufficient to prove that the inclusion $\mathrm{Cyl}\,(\lambda \to \mu) \subseteq \mathrm{T}_{\mathrm{t}(\lambda\to\mu)}$ induces the trivial map on fundamental groups. Observe that by the Van Kampen Theorem, $\pi_1\mathrm{Cyl}\,(\lambda \to \mu)$ is a quotient of the free product $\pi_1 C_{t(\lambda)} * \pi_1 C_{t(\mu)}$. However, simple connectivity of $\mathrm{Cyl}\,(\epsilon \to \lambda)$ means that $\pi_1 C_{t(\lambda)}$ maps trivially to $\pi_1 T_{t(\lambda\to\mu)}$. Likewise, simple connectivity of $\mathrm{Cyl}\,(\epsilon \to \mu)$ means that $\pi_1 C_{t(\mu)}$ maps trivially to $\pi_1 T_{t(\lambda\to\mu)}$.

The proposition follows, because any loop based at ϵ is path homotopic to a concatenation of the paths α_1, α_2, α_3, and their inverses. □

Having identified a generator for the potentially nontrivial fundamental group of a triangle, we establish that the generator becomes null in the two-skeleton. (In fact, the fundamental group of $T_{t(\lambda\to\mu)}$ is nontrivial for $n > 2$ if and only if ϵ, λ, and μ all have different types, but we will not need this fact.)

Proposition 3.11. *Suppose that $\epsilon \to \lambda \to \mu$ are composable morphisms in $\mathcal{L}_n$, where ϵ has type $n \cdot \underline{1}$ and λ has at least one subspace of dimension greater than one. Then $T_{t(\lambda\to\mu)} \hookrightarrow \mathrm{sk}^2(\mathcal{L}_n)$ induces the zero map on fundamental groups.*

Proof. By Lemma 3.10, the fundamental group $\pi_1 T_{t(\lambda\to\mu)}$ is generated by the loop α in (3.9). But in $\mathrm{sk}^2(\mathcal{L}_n)$, a null-homotopy for α is provided by the 2-simplex corresponding to the composable morphisms $\epsilon \to \lambda \to \mu$, thus proving the proposition. □

By breaking a loop up as a concatenation of loops in triangles, we obtain the desired simple connectivity result.

Proof of Theorem 3.1. We can write $\mathrm{sk}^1(\mathcal{L}_n)$ as the finite union of the path connected spaces $T_{t(\lambda\to\mu)}$. Therefore any path α based at $\epsilon \in C_{n\cdot\underline{1}}$

can be written as a finite concatenation $\alpha_1 * \cdots * \alpha_k$, where each α_i is a path in a triangle $T_{t(\lambda_i\to\mu_i)}$. However, the intersection of neighboring triangles $T_{t(\lambda_i\to\mu_i)} \cap T_{t(\lambda_{i+1}\to\mu_{i+1})}$ is path connected and contains $C_{n\cdot\underline{1}}$, so $\alpha_i(1) = \alpha_{i+1}(0)$ can be connected to the basepoint ϵ by a path β_i within $T_{t(\lambda_i\to\mu_i)} \cap T_{t(\lambda_{i+1}\to\mu_{i+1})}$. Then α is path homotopic to

$$(\alpha_1 * \beta_1) * \left(\beta_1^{-1} * \alpha_2 * \beta_2\right) * \cdots * \left(\beta_{k-1}^{-1} * \alpha_k\right),$$

which is a concatenation of loops at ϵ completely contained within a triangles.

But by Proposition 3.11, any loop in $T_{t(\lambda\to\mu)}$ maps to zero in $\pi_1\mathrm{sk}^2(\mathcal{L}_n)$, which proves the theorem. □

4. A contractibility criterion for fixed points

Recall that we call a subgroup of $U(n)$ *polytypic* if its action on $\mathbb{C}^n$ is not isotypic. This section establishes that subgroups $H \subseteq U(n)$ with certain normal, polytypic subgroups have contractible fixed point sets on $\mathcal{L}_n$. The main result is Proposition 4.1 below, which first appeared as Theorem 4.5 in [BJL+15]; its proof is given at the end of the section. Our goal in Section 5 will be to find normal subgroups J satisfying the hypotheses of Proposition 4.1.

Proposition 4.1. *Let H be a subgroup of $U(n)$, and suppose H has a normal subgroup J with the following properties:*

(1) *J is polytypic.*
(2) *For every $\lambda \in (\mathcal{L}_n)^H$, the action of J on $\mathrm{cl}(\lambda)$ is not transitive.*

Then $(\mathcal{L}_n)^H$ is contractible.

To prove Proposition 4.1, we need two constructions that will give natural retractions between various subcategories of $\mathcal{L}_n$.

Definition 4.2. Let $J \subseteq U(n)$ be a subgroup, and let λ be a weakly J-fixed decomposition of $\mathbb{C}^n$.

(1) The decomposition $\lambda_{\mathrm{st}(J)}$ is defined as follows: $w \subseteq \mathbb{C}^n$ is a component of $\lambda_{\mathrm{st}(J)}$ if and only if $w = \Sigma_{j\in J}\, jv$ for some $v \in \mathrm{cl}(\lambda)$. That is, a component of $\lambda_{\mathrm{st}(J)}$ is formed by taking the sum of all components that are in the same orbit of the action of J on the set of components of λ.
(2) If λ is strongly J-fixed, let $\lambda_{\mathrm{iso}(J)}$ be the refinement of λ obtained by breaking each component of λ into its canonical J-isotypical summands.

A routine check establishes the following properties of the operations above.

Lemma 4.3.

(1) *The decomposition* $\lambda_{\mathrm{st}(J)}$ *is strongly* J*-fixed, natural in* λ*, and minimal among coarsenings of* λ *that are strongly* J*-fixed.*

(2) *The decomposition* $\lambda_{\mathrm{iso}(J)}$ *is natural in strongly* J*-fixed decompositions* λ *and is maximal among refinements of* λ *whose classes are isotypical representations of* J*.*

The following two lemmas give criteria for $\lambda_{\mathrm{st}(J)}$ and $\lambda_{\mathrm{iso}(J)}$ to be weakly fixed by a supergroup $H \subseteq U(n)$ of J. Lemma 4.4 is a straightforward check, and Lemma 4.5 involves some basic representation theory.

Lemma 4.4. *Let* $J \triangleleft H$*, and suppose that* λ *is weakly fixed by* J*. If* $h \in H$*, then* $(h\lambda)_{\mathrm{st}(J)} = h\left(\lambda_{\mathrm{st}(J)}\right)$*. In particular, if* λ *is weakly fixed by* H*, then* $\lambda_{\mathrm{st}(J)}$ *is also weakly fixed by* H*.*

Proof. We show that a component of $h\left(\lambda_{\mathrm{st}(J)}\right)$ is in fact a component of $(h\lambda)_{\mathrm{st}(J)}$. Let $v \in \mathrm{cl}(\lambda)$, and consider the component $w \in \mathrm{cl}\left(\lambda_{\mathrm{st}(J)}\right)$ given by $w = \Sigma_{j\in J}\, jv$. We can compute hw as

$$\begin{aligned} hw &= h\left(\Sigma_{j\in J}\, jv\right) \\ &= \Sigma_{j\in J}\left(hjh^{-1}\right)(hv) \\ &= \Sigma_{j'\in J}\, j'(hv). \end{aligned}$$

Here j' runs over all elements of J because conjugation by H is an automorphism. Therefore hw is a component in $(h\lambda)_{\mathrm{st}(J)}$. □

Lemma 4.5. *Let* $J \triangleleft H$*, and suppose that* λ *is strongly fixed by* J*. If* $h \in H$*, then* $(h\lambda)_{\mathrm{iso}(J)} = h\left(\lambda_{\mathrm{iso}(J)}\right)$*. In particular, if* λ *is weakly fixed by* H*, then* $\lambda_{\mathrm{iso}(J)}$ *is also weakly fixed by* H*.*

Proof. Let $v \in \mathrm{cl}(\lambda)$, and let $h \in H$. Because λ is strongly J-fixed, we know that v is stabilized by J. The subspace hv is a component of $h\lambda$, and because $J \triangleleft H$, we can check that hv is also stabilized by J. Hence $v \in \mathrm{cl}(\lambda)$ and $hv \in \mathrm{cl}(h\lambda)$ are both J-representations.

But as a representation of J, the subspace hv is conjugate to the representation v of J. Thus if $w \subseteq v$ is the isotypical summand of v for an irreducible representation ρ of J, then hw is the isotypical summand of hv for the conjugate of ρ by h. We conclude that h maps the canonical J-isotypical summands of v to the canonical J-isotypical summands of hv, which are components of $(h\lambda)_{\mathrm{iso}(J)}$. Since this result is true for every $v \in \mathrm{cl}(\lambda)$, we find that $(h\lambda)_{\mathrm{iso}(J)} = h\left(\lambda_{\mathrm{iso}(J)}\right)$. □

For $J \triangleleft H$, the constructions $\lambda \mapsto \lambda_{\mathrm{st}(J)}$ and $\lambda \mapsto \lambda_{\mathrm{iso}(J)}$ will allow us to retract $(\mathcal{L}_n)^H$ onto subcategories that in many cases have a terminal object and hence contractible nerve. As a first step, we need to verify continuity of the constructions. The proofs use an explicit identification of the path components of $(\mathcal{L}_n)^J$. We need the following lemma and its corollary, which we learned from S. Costenoble.

Lemma 4.6. [May99, Lemma 1.1] *Let G be a compact Lie group with closed subgroups J and K. Let $p : \alpha \to \beta$ be a G-homotopy between G-maps $G/J \longrightarrow G/K$. Then p factors as the composite of α and a homotopy $c\colon G/J\times I \to G/J$ such that $c\,(eJ,t) = c_t J$, where $c_0 = e$ and the values c_t specify a path in the identity component of the centralizer $C_G(J)$ of J in G.*

Corollary 4.7. *Let $J \subseteq U(n)$ be a closed subgroup, and let $C_0(J)$ denote the identity component of the centralizer of J in $U(n)$.*

(1) *The path components of* $\operatorname{Obj}(\mathcal{L}_n)^J$ *are $C_0(J)$-orbits.*
(2) *Any $C_0(J)$-equivariant map from* $\operatorname{Obj}(\mathcal{L}_n)^J$ *to itself is continuous.*

Proof. The space $\operatorname{Obj}(\mathcal{L}_n)$ is topologized as the disjoint union of $U(n)$-orbits $U(n)/K_\lambda$, so we can apply Lemma 4.6 with $G = U(n)$ to each path component of $\operatorname{Obj}(\mathcal{L}_n)$ to obtain the first result. The second result then follows from the fact that $\operatorname{Obj}(\mathcal{L}_n)^J$ is topologized as the disjoint union of $C_0(J)$-orbits. □

As a consequence, we obtain continuity of $\lambda \mapsto \lambda_{\mathrm{st}(J)}$.

Lemma 4.8. *If $J \triangleleft H$, then the function $\lambda \mapsto \lambda_{\mathrm{st}(J)}$ is continuous on* $\operatorname{Obj}(\mathcal{L}_n)^H$.

Proof. Since $\operatorname{Obj}(\mathcal{L}_n)^H$ is a subspace of $\operatorname{Obj}(\mathcal{L}_n)^J$, it suffices to show the lemma for $H = J$. We need only verify that the hypothesis of Corollary 4.7 (2) holds, that is, that the assignment $\lambda \mapsto \lambda_{\mathrm{st}(J)}$ is $C_0(J)$-equivariant. However, Lemma 4.4 tells us the stronger condition that $\lambda \mapsto \lambda_{\mathrm{st}(J)}$ is actually equivariant with respect to the normalizer of J, hence necessarily with respect to $C_0(J)$ as well. □

We handle isotypical refinement in a similar way.

Lemma 4.9. *If $J \triangleleft H$, then the function $\lambda \mapsto \lambda_{\mathrm{iso}(J)}$ is continuous on* $\operatorname{Obj}(\mathcal{L}_n)^H_{\mathrm{st}}$.

Proof. Since $\operatorname{Obj}(\mathcal{L}_n)^H$ is a subspace of $\operatorname{Obj}(\mathcal{L}_n)^J$, it suffices to show the lemma for $H = J$. First we observe that $\operatorname{Obj}(\mathcal{L}_n)^J_{\mathrm{st}}$ and $\operatorname{Obj}(\mathcal{L}_n)^J_{\mathrm{iso}}$ are both stabilized by the action of $C_0(J)$, and hence both are unions of path components of $\operatorname{Obj}(\mathcal{L}_n)^J$. As in the previous lemma, the result follows from Corollary 4.7 (2). □

With continuity established, we can present the key retraction result.

Proposition 4.10. *Let H be a subgroup of $U(n)$, and suppose $J \lhd H$. Then the inclusion functor*

$$\iota_1 \colon \left(\mathcal{L}_n\right)^J_{\text{iso}} \,\cap\, \left(\mathcal{L}_n\right)^H \longrightarrow \left(\mathcal{L}_n\right)^J_{\text{st}} \,\cap\, \left(\mathcal{L}_n\right)^H$$

induces a homotopy equivalence on nerves. If J further has the property that $\lambda_{\text{st}(J)}$ is proper for every $\lambda \in \left(\mathcal{L}_n\right)^H$, then the inclusion functor

$$\iota_2 \colon \left(\mathcal{L}_n\right)^J_{\text{st}} \,\cap\, \left(\mathcal{L}_n\right)^H \longrightarrow \left(\mathcal{L}_n\right)^H$$

induces a homotopy equivalence on nerves.

Proof. By Lemmas 4.5 and 4.9, a continuous, functorial retraction of ι_1 is given by $r_1 \colon \lambda \mapsto \lambda_{\text{iso}(J)}$. The coarsening morphism $\lambda_{\text{iso}(J)} \to \lambda$ provides a natural transformation from $\iota_1 r_1$ to the identity, establishing the first desired equivalence.

Similarly, by Lemmas 4.4 and 4.8, a continuous, functorial retraction of ι_2 is given by $r_2 \colon \lambda \mapsto \lambda_{\text{st}(J)}$, because by assumption $\lambda_{\text{st}(J)}$ is always a proper decomposition of $\mathbb{C}^n$. The coarsening morphism $\lambda \to \lambda_{\text{st}(J)}$ provides a natural transformation from the identity to $\iota_2 r_2$, which establishes the second desired equivalence. □

Proof of Proposition 4.1. Let μ denote the decomposition of $\mathbb{C}^n$ into the canonical isotypical components of J. If J is polytypic, then μ has more than one component and thus is proper, and μ is terminal in $\left(\mathcal{L}_n\right)^J_{\text{iso}}$.

We assert that μ is a terminal object in $\left(\mathcal{L}_n\right)^J_{\text{iso}} \,\cap\, \left(\mathcal{L}_n\right)^H$, and we need only establish that μ is weakly H-fixed. But μ is the J-isotypical refinement of the indiscrete decomposition of $\mathbb{C}^n$, i.e., the decomposition consisting of just $\mathbb{C}^n$ itself. Since the indiscrete decomposition is certainly H-fixed, Lemma 4.5 tells us that μ is weakly H-fixed.

The result now follows from Proposition 4.10, because the assumption that the action of J on $\text{cl}(\lambda)$ is always intransitive means that $\lambda_{\text{st}(J)}$ is always proper. □

5. Finding a normal subgroup

Throughout this section, suppose that $H \subseteq U(n)$ is a p-toral subgroup of $U(n)$, and let $\overline{H}$ denote the image of H in $PU(n)$. Note that although $PU(n)$ does not act on $\mathbb{C}^n$, it does act on $\mathcal{L}_n$, because the central $S^1 \subseteq U(n)$ stabilizes any subspace of $\mathbb{C}^n$. Hence, for example, if a decomposition λ is weakly H-fixed, we can speak of the action of $\overline{H}$ on $\text{cl}(\lambda)$.

Our goal is to prove Theorem 5.4, which says that if H is problematic, then $\overline{H}$ is elementary abelian and χ_H is zero except on elements that are in the central S^1 of $U(n)$. The plan for the proof is to apply Proposition 4.1 after locating a relevant normal polytypic subgroup of H. The following lemma gives us a starting point, and the proof of Theorem 5.4 appears at the end of the section.

Lemma 5.1. *If $J \subseteq U(n)$ is abelian, then either J is polytypic or $J \subseteq S^1$.*

Proof. Decompose $\mathbb{C}^n$ into a sum of J-irreducible representations, all of which are necessarily one-dimensional because J is abelian. If J is isotypic, then an element $j \in J$ acts on every one-dimensional summand by multiplication by the same scalar, so $j \in S^1$. □

Lemma 5.1 places an immediate restriction on problematic subgroups.

Lemma 5.2. *If H is a problematic p-toral subgroup of $U(n)$, then $\overline{H}$ is discrete.*

Proof. We assert that, without loss of generality, we can assume that H actually contains S^1. Because $(\mathcal{L}_n)^H = (\mathcal{L}_n)^{HS^1}$, if H is problematic, then so is HS^1. Likewise, H is p-toral if and only if HS^1 is p-toral. Further, H and HS^1 have the same image in $PU(n)$. Hence we can assume that $S^1 \subseteq H$, by replacing H by HS^1 if necessary.

The group $\overline{H}$ is p-toral (e.g., [BJL+15], Lemma 3.3), so its identity component, denoted by $\overline{H}_0$, is a torus. Let J denote the inverse image of $\overline{H}_0$ in H; thus $J \lhd H$. Since we have a fibration $S^1 \to J \to \overline{H}_0$, we know that J is connected. Further, J is a torus, because it is a connected closed subgroup of the identity component of H, which is a torus.

If $\overline{H}$ is not discrete, then J is not contained in S^1, and thus J is polytypic by Lemma 5.1. Since J is connected, its action on the set of equivalence classes of any proper decomposition is trivial. The lemma follows from Proposition 4.1. □

In terms of progress towards Theorem 5.4, we now know that if H is a problematic p-toral subgroup of $U(n)$, then $\overline{H}$ must be a finite p-group. The next part of our strategy is to show that if $\overline{H}$ is not an elementary abelian p-group, then $\overline{H}$ has a normal subgroup V satisfying the conditions of the following lemma.

Lemma 5.3. *Let $H \subseteq U(n)$, and assume there exists $V \lhd \overline{H}$ such that $V \cong \mathbb{Z}/p$ and V does not act transitively on $\mathrm{cl}(\lambda)$ for any $\lambda \in (\mathcal{L}_n)^H$. Then $(\mathcal{L}_n)^H$ is contractible.*

Proof. Let J be the inverse image of V in H. Then $J \lhd H$, and because the action of J on $\mathcal{L}_n$ factors through V, the action of J on $\mathrm{cl}(\lambda)$ is not transitive for any $\lambda \in (\mathcal{L}_n)^H$. Further, J is abelian, because a routine splitting argument shows $J \cong V \times S^1$ (see [BJL+15], Lemma 3.1). Therefore J is polytypic by Lemma 5.1, and the lemma follows from Proposition 4.1. □

Before we prove our first theorem, we recall that if G is a finite p-group, then its Frattini subgroup, $\Phi(G)$, is generated by the commutators $[G, G]$ and the p-fold powers G^p. It has the property that $G \to G/\Phi(G)$ is initial among maps from G to elementary abelian p-groups.

We now have all the ingredients we require to prove Theorem 5.4. We note that both the statement and the proof are closely related to those of Proposition 6.1 of [ADL16].

Theorem 5.4. *Let H be a problematic p-toral subgroup of $U(n)$ that contains S^1. Then*

(1) *H is a projective elementary abelian p-group, and*

(2) *the character of H is* $\begin{cases} \chi_H(h) = 0 & h \notin S^1 \\ \chi_H(h) = nh & h \in S^1. \end{cases}$

Proof. For (1), we must show that $\overline{H}$ is an elementary abelian p-group. By Lemma 5.2, we know that $\overline{H}$ is a finite p-group. If $\overline{H}$ is not elementary abelian, then the kernel of $\overline{H} \to \overline{H}/\Phi(\overline{H})$ is a nontrivial normal p-subgroup of $\overline{H}$, and thus has nontrivial intersection with the center of $\overline{H}$. Choose $V \cong \mathbb{Z}/p$ with

$$V \subseteq \ker \left[\overline{H} \to \overline{H}/\Phi(\overline{H}) \right] \cap Z\left(\overline{H}\right).$$

We assert that V satisfies the conditions of Lemma 5.3. Certainly $V \lhd \overline{H}$, because V is contained in the center of $\overline{H}$. Suppose that there exists $\lambda \in (\mathcal{L}_n)^H$ such that V acts transitively on $\mathrm{cl}(\lambda)$. The set $\mathrm{cl}(\lambda)$ must then have exactly p elements. The action of $\overline{H}$ on $\mathrm{cl}(\lambda)$ induces a map $\overline{H} \to \Sigma_p$, and since $\overline{H}$ is a p-group, the image of this map must lie in a Sylow p-subgroup of Σ_p. But then the map $\overline{H} \to \Sigma_p$ factors as

$$\overline{H} \to \overline{H}/\Phi(\overline{H}) \to \mathbb{Z}/p \hookrightarrow \Sigma_p,$$

with $V \subseteq \overline{H}$ mapping nontrivially to $\mathbb{Z}/p \subseteq \Sigma_p$. We have thus contradicted the assumption that $V \subseteq \ker\left[\overline{H} \to \overline{H}/\Phi(\overline{H})\right]$. We conclude that $\overline{H}$ is, in fact, an elementary abelian p-group, and so (1) is proved.

For (2), first note that if $h \in S^1$, then its matrix representation in $U(n)$ is hI, hence $\chi_H(h) = \mathrm{tr}(hI) = nh$. Suppose that $\overline{H}$ is an elementary abelian p-group, and consider an arbitrary element $h \in H$ such that $h \notin S^1$. The image of h in $\overline{H}$ generates a subgroup $V \cong \mathbb{Z}/p \subseteq \overline{H}$, and V is a candidate for applying Lemma 5.3. Since $(\mathcal{L}_n)^H$ is not contractible, there must be a decomposition $\lambda \in (\mathcal{L}_n)^H$ such that V acts transitively on $\mathrm{cl}(\lambda)$. The action of V on $\mathrm{cl}(\lambda)$ is necessarily free because $V \cong \mathbb{Z}/p$, so a basis for $\mathbb{C}^n$ can be constructed that is invariant under V and consists of bases for the subspaces in $\mathrm{cl}(\lambda)$. This action represents h as a fixed-point free permutation of a basis of $\mathbb{C}^n$. Hence $\chi_H(h) = 0$. □

6. THE SUBGROUPS Γ_k OF $U(p^k)$

Theorem 5.4 tells us that if H is a problematic p-toral subgroup of $U(n)$, then H is a projective elementary abelian p-group and the character of H is zero away from the center of $U(n)$. In fact, there are well-known subgroups of $U(n)$ that satisfy these conditions, namely the subgroups $\Gamma_k \subseteq U(p^k)$ that arise in, for example, [Gri91], [JMO92], [Oli94], [Aro02], [AL07], [AGMV08], and others. In this section, we review background on the groups Γ_k.

We begin with the discrete analogue of Γ_k. Let $n = p^k$ and choose an identification of the elements of $(\mathbb{Z}/p)^k$ with the set $\{1, ..., n\}$. The action of $(\mathbb{Z}/p)^k$ on itself by translation identifies $(\mathbb{Z}/p)^k$ as a transitive elementary abelian p-subgroup of Σ_{p^k}, denoted by Δ_k. Up to conjugacy, Δ_k is the unique transitive elementary abelian p-subgroup of Σ_{p^k}. Note that every nonidentity element of Δ_k acts without fixed points. The embedding

$$\Delta_k \hookrightarrow \Sigma_{p^k} \hookrightarrow U(p^k)$$

given by permuting the standard basis elements is the regular representation of Δ_k, and has character $\chi_{\Delta_k} = 0$ except at the identity.

In the unitary context, the projective elementary abelian p-subgroup $\Gamma_k \subseteq U(p^k)$ is generated by the central $S^1 \subseteq U(p^k)$ and two different embeddings of Δ_k in $U(p^k)$, which we denote by $\mathcal{A}_k$ and $\mathcal{B}_k$ and describe momentarily. Just as Δ_k is the unique (up to conjugacy) elementary abelian p-subgroup of Σ_{p^k} with transitive action, it turns out that Γ_k is the unique (up to conjugacy) projective elementary abelian p-subgroup of $U(p^k)$ containing the central S^1 and having irreducible action (see, for example, [Zol02]). For the explicit description of Γ_k, we follow [Oli94].

The subgroup $\mathcal{B}_k \cong \Delta_k$ of $U(p^k)$ is given as follows. Consider $\mathbb{Z}/p \subseteq U(p)$ acting by the regular representation, and let $\mathcal{B}_k$ be the group $(\mathbb{Z}/p)^k$ acting on the k-fold tensor power $(\mathbb{C}^p)^{\otimes k}$. (This action is, in fact, the regular representation of $\Delta_k \cong (\mathbb{Z}/p)^k$.) Explicitly, for any $r = 0, 1, \ldots, k-1$, let $\sigma_r \in \Sigma_{p^k}$ denote the permutation defined by

$$\sigma_r(i) = \begin{cases} i + p^r & \text{if } i \equiv 1, \ldots, (p-1)p^r \mod p^{r+1}, \\ i - (p-1)p^r & \text{if } i \equiv (p-1)p^r + 1, \ldots, p^{r+1} \mod p^{r+1}. \end{cases}$$

For each r, let $B_r \in U\big(p^k\big)$ be the corresponding permutation matrix,

$$(B_r)_{ij} = \begin{cases} 1 & \text{if } \sigma_r(i) = j \\ 0 & \text{if } \sigma_r(i) \neq j. \end{cases}$$

For later purposes, we record the following lemma.

Lemma 6.1. *The character $\chi_{\mathcal{B}_k}$ is zero except at the identity.*

Proof. Every nonidentity element of $\mathcal{B}_k$ acts by a fixed-point free permutation of the standard basis of $\mathbb{C}^{p^k}$, and so has only zeroes on the diagonal. □

Our goal is to define Γ_k with irreducible action on $\mathbb{C}^{p^k}$, but $\mathcal{B}_k$ alone does not act irreducibly: being abelian, the subgroup $\mathcal{B}_k$ has only one-dimensional irreducibles. In fact, since $\mathcal{B}_k$ is acting on $\mathbb{C}^{p^k}$ by the regular representation, each of its p^k irreducible representations is present exactly once. The role of the other subgroup $\mathcal{A}_k \cong \Delta_k \subseteq \Gamma_k$ is to permute the irreducible representations of $\mathcal{B}_k$. To be specific, let $\zeta = e^{2\pi i/p}$, and consider $\mathbb{Z}/p \subseteq U(p)$ generated by the diagonal matrix with entries $1, \zeta, \zeta^2, ..., \zeta^{p-1}$. Then $\mathcal{A}_k \subseteq U\big(p^k\big)$ is the group $(\mathbb{Z}/p)^k$ acting on the k-fold tensor power $(\mathbb{C}^p)^{\otimes k}$. Explicitly, for $r = 0, \ldots, k-1$ define $A_r \in U\big(p^k\big)$ by

$$(A_r)_{ij} = \begin{cases} \zeta^{[(i-1)/p^r]} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where $[-]$ denotes the greatest integer function. The matrices A_0,...,A_{k-1} commute, are of order p, and generate a rank k elementary abelian p-group $\mathcal{A}_k$.

Lemma 6.2. *The character $\chi_{\mathcal{A}_k}$ is zero except at the identity.*

Proof. The character of $\mathbb{Z}/p \subseteq U(p)$ generated by the diagonal matrix with entries $1, \zeta, \zeta^2, ..., \zeta^{p-1}$ is zero away from the identity by direct computation, because

$$1 + \zeta + \zeta^2 + ... + \zeta^{p-1} = \frac{\zeta^p - 1}{\zeta - 1} = 0.$$

The same is true for $\mathcal{A}_k$, because the character is obtained by multiplying together the characters of the individual factors. □

Since the characters $\chi_{\mathcal{A}_k}$ and $\chi_{\mathcal{B}_k}$ are the same, we obtain the following corollary.

Corollary 6.3. *The subgroups $\mathcal{A}_k$ and $\mathcal{B}_k$ are conjugate in $U\big(p^k\big)$.*

Although $\mathcal{A}_k$ and $\mathcal{B}_k$ do not quite commute with each other in $U\big(p^k\big)$, the commutator relations are simple and follow from examining the actions of $\mathcal{B}_k \cong (\mathbb{Z}/p)^k$ and $\mathcal{A}_k \cong (\mathbb{Z}/p)^k$ on $(\mathbb{C}^p)^{\otimes k}$. If $r \neq s$, then A_r and B_s are acting on different tensor factors in $(\mathbb{C}^p)^{\otimes k}$, and hence they commute. The commutator of A_r and B_r, which both act on the rth tensor factor of $\mathbb{C}^p$, can be computed by an explicit computation in $U(p)$. As a result, we obtain the following relations:

$$\begin{aligned} &[A_r, A_s] = I = [B_r, B_s], && \text{for all } r, s \\ &[A_r, B_s] = I, && \text{for all } r \neq s \\ &[B_r, A_r] = \zeta I, && \text{for all } r. \end{aligned} \tag{6.4}$$

Definition 6.5. The subgroup $\Gamma_k \subseteq U\big(p^k\big)$ is generated by the subgroups $\mathcal{A}_k$, $\mathcal{B}_k$, and the central $S^1 \subseteq U\big(p^k\big)$.

Lemma 6.6. *There is a short exact sequence*

$$1 \to S^1 \to \Gamma_k \to (\Delta_k \times \Delta_k) \to 1. \tag{6.7}$$

where S^1 is the center of $U\big(p^k\big)$.

Proof. The subgroups $\mathcal{A}_k$ and $\mathcal{B}_k$ can be taken as the preimages of the two copies of Δ_k in Γ_k. The commutator relations (6.4) show that $\mathcal{A}_k$ and $\mathcal{B}_k$ do not generate any noncentral elements. □

Remark. When $k = 0$ we have $\Gamma_0 = S^1 \subseteq U(1)$, and Δ_0 is trivial, so Lemma 6.6 is true even for $k = 0$.

For later purposes, we record the following lemma.

Lemma 6.8. *The subgroup $\Gamma_k \subseteq U\big(p^k\big)$ contains subgroups isomorphic to $\Gamma_s \times \Delta_t$ for all nonnegative integers s and t such that $s + t \leq k$.*

Proof. The required subgroup is generated by S^1, the matrices $A_0, \ldots,$ A_{s+t-1}, and the matrices $B_0, \ldots, B_{s-1}$. □

A consequence of Lemma 6.6 is that $\Gamma_k \subseteq U\big(p^k\big)$ is an example of a p-toral subgroup that satisfies the first conclusion of Theorem 5.4. The next lemma says that Γ_k satisfies the second conclusion as well.

Lemma 6.9. *The character of Γ_k is nonzero only on the elements of S^1.*

Proof. The character of Γ_k on elements of $\mathcal{A}_k$ and $\mathcal{B}_k$ is zero by Lemmas 6.1 and 6.2. Multiplying any of these matrices by an element of S^1, i.e., a scalar, gives a matrix that also has zero trace. Finally, products of nonidentity elements of $\mathcal{B}_k$ with elements of $\mathcal{A}_k S^1$ are obtained by multiplying matrices in $\mathcal{B}_k$, which have only zero entries on the diagonal, by diagonal matrices. The resulting products likewise have no nonzero diagonal entries and thus have zero trace. □

Remark. We note that, by inspection of the commutator relations, $S^1 \times \mathcal{A}_k$ and $S^1 \times \mathcal{B}_k$ normalize each other in Γ_k. Suppose we decompose $\mathbb{C}^{p^k}$ by the p^k one-dimensional irreducible representations of $\mathcal{A}_k$. That decomposition is weakly fixed by $\mathcal{B}_k$, and further, $\mathcal{B}_k$ acts transitively on the classes in the decomposition because Γ_k is irreducible. Likewise, the decomposition of $\mathbb{C}^{p^k}$ by the p^k one-dimensional irreducible representations of $\mathcal{B}_k$ is weakly fixed by $\mathcal{A}_k$, which has transitive action on the classes.

7. Alternating forms

From Theorem 5.4, we know that if H is a problematic p-toral subgroup of $U(n)$, then its image $\overline{H}$ in $PU(n)$ is an elementary abelian p-group. We would like to know the possible group isomorphism types of such subgroups of $U(n)$. For simplicity, we restrict ourselves to subgroups H that contain the central S^1 of $U(n)$. (See the proof of Lemma 5.2.) Our main results are Propositions 7.12 and 7.13, below. Once the group-theoretic classification is complete, we use representation theory in Section 8 to pin down the conjugacy classes of elementary abelian p-subgroups of $U(n)$ that can be problematic.

Before proceeding, we note that the remarkable paper [AGMV08] of Andersen-Grodal-Møller-Viruel classifies non-toral elementary abelian p-subgroups (for odd primes p) of the simple and center-free Lie groups. In particular, Theorem 8.5 of that work contains a classification of all the elementary abelian p-subgroups of $PU(n)$, building on earlier work of Griess [Gri91]. Our approach is independent of this classification, and works for all primes, using elementary methods.

We make the following definition.

Definition 7.1. A p-toral group H is an *abstract projective elementary abelian p-group* if H can be written as a central extension

$$1 \to S^1 \to H \to V \to 1,$$

where V is an elementary abelian p-group.

We begin by recalling some background on forms. Let A be a finite-dimensional $\mathbb{F}_p$-vector space, and let $\alpha\colon A \times A \to \mathbb{F}_p$ be a bilinear form. We say that α is *totally isotropic* if $\alpha(a,a) = 0$ for all $a \in A$. (A totally isotropic form is necessarily skew-symmetric, as seen by expanding $\alpha(a+b, a+b) = 0$, but the reverse is not true for $p = 2$.) If α is not only totally isotropic, but also non-degenerate, then it is called a *symplectic form*. Any vector space with a symplectic form is even-dimensional and has a (nonunique) basis $e_1, \ldots, e_s, f_1, \ldots, f_s$, called a *symplectic basis*, with the property that $\alpha(e_i, e_j) = 0 = \alpha(f_i, f_j)$ for all i, j, and $\alpha(e_i, f_j)$ is 1 if $i = j$ and zero otherwise. All symplectic vector spaces of the same dimension are isomorphic (i.e., there exists a linear isomorphism that preserves the form), and if the vector space has dimension $2s$ we use $\mathbb{H}_s$ to denote the associated isomorphism class of symplectic vector spaces. Let $\mathbb{T}_t$ denote the vector space of dimension t over $\mathbb{F}_p$ with trivial form. We have the following standard classification result.

Lemma 7.2. *Let A be a vector space over $\mathbb{F}_p$ with a totally isotropic bilinear form α. Then there exist s and t such that $A \cong \mathbb{H}_s \oplus \mathbb{T}_t$ by a form-preserving isomorphism.*

Our next task is to relate the preceding discussion to abstract projective elementary abelian p-groups. For the remainder of the section, assume that H is an abstract projective elementary abelian p-group

$$1 \to S^1 \to H \to V \to 1.$$

Choose an identification of $\mathbb{Z}/p$ with the elements of order p in S^1. Given $x, y \in V$, let $\tilde{x}, \tilde{y}$ be lifts of x, y to H. Define the *commutator form associated to H* as the form on V defined by

$$\alpha(x,y) = [\tilde{x}, \tilde{y}] = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}. \tag{7.3}$$

Lemma 7.4. *Let H and α be as above, and suppose that $x, y \in V$. Then*

(1) *$\alpha(x,y)$ is a well-defined element of $\mathbb{Z}/p \subseteq S^1$,*
(2) *α is a totally isotropic bilinear form on V, and*
(3) *isomorphic groups H and H' give isomorphic forms α and α'.*

Proof. Certainly $[\tilde{x}, \tilde{y}] \in S^1$, since x and y commute in V. If $\zeta \in S^1$ then $[\zeta\tilde{x}, \tilde{y}] = [\tilde{x}, \tilde{y}] = [\tilde{x}, \zeta\tilde{y}]$ because S^1 is central in H, which shows that α is independent of the choice of lifts $\tilde{x}$ and $\tilde{y}$, and that α is linear with respect to scalar multiplication.

To show that $[\tilde{x}, \tilde{y}]$ has order p, we note that commutators in a group satisfy the following versions of the Hall-Witt identities, as can be verified by expanding and simplifying:

$$[a, bc] = [a, b] \cdot \big[b, [a, c]\big] \cdot [a, c]$$
$$[ab, c] = \big[a, [b, c]\big] \cdot [b, c] \cdot [a, c].$$

We know that $[H, H]$ is contained in the center of H, so $[H, [H, H]]$ is the trivial group. Hence for H, the identities reduce to

$$\begin{aligned}[a, bc] &= [a, b][a, c] \\ [ab, c] &= [a, c][b, c].\end{aligned} \tag{7.5}$$

In particular, $[\tilde{x}, \tilde{y}]^p = [\tilde{x}, \tilde{y}^p] = e$, since $\tilde{y}^p \in S^1$ commutes with $\tilde{x}$. Bilinearity of α with respect to addition follows directly from (7.5). The form is totally isotropic because an element commutes with itself.

Finally, an isomorphism $H \to H'$ necessarily restricts to an isomorphism on the identity component and induces a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^1 & \longrightarrow & H & \longrightarrow & V & \longrightarrow & 1 \\ & & \cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & S^1 & \longrightarrow & H' & \longrightarrow & V' & \longrightarrow & 1, \end{array}$$

which, in turn, induces an isomorphism of the associated forms α and α' on V and V', respectively. □

Lemma 7.4 shows that (7.3) gives a function from isomorphism classes of projective elementary abelian p-groups to isomorphism classes of totally isotropic forms over $\mathbb{Z}/p$. Conversely, we can start with a form α and directly construct an abstract projective elementary abelian p-group H_α. Let $\alpha\colon V \times V \to \mathbb{Z}/p$ be a totally isotropic bilinear form, which can be regarded as a function $\alpha\colon V \times V \to S^1$. Let

$$\begin{aligned} V_K &= \{v \in V \mid \alpha(v, x) = 0 \ \text{ for all } x \in V\} \\ &= \ker(V \to V^*), \end{aligned}$$

where V^* denotes the dual of V. Then α restricted to V_K is trivial.

Now we make some choices, and we address the issue of the choices a little later in the section. Choose a complement V_c to V_K in V; note V_c is necessarily orthogonal to V_K. Since α must be symplectic on V_c, we can choose a symplectic basis $e_1, \ldots, e_r, f_1, \ldots, f_r$ for V_c; let V_E and V_F denote the spans of $E = \{e_1, \ldots, e_r\}$ and $F = \{f_1, \ldots, f_r\}$, respectively. By construction, we can write any $v \in V$ uniquely as a sum $v = v_K + v_E + v_F$ where $v_K \in V_K$, $v_E \in V_E$, and $v_F \in V_F$.

Let H_α be the set $S^1 \times V$. Given the previous choices, we can endow H_α with the following operation $\alpha(E, F)$:

$$(z, v) *_{\alpha(E,F)} (z', v') = \big(zz'\,\alpha(v_F, v'_E),\ v + v'\big)\,. \tag{7.6}$$

Proposition 7.7. *For an elementary abelian p-group V and a totally isotropic bilinear form $\alpha\colon V\times V \to \mathbb{Z}/p$ as above, we have the following.*

1. *The operation (7.6) gives H_α the structure of an abstract projective elementary abelian p-group with associated commutator pairing α.*
2. *The group isomorphism class of H_α depends only on the isomorphism class of α as a bilinear form.*

Further, non-isomorphic forms α and α' on V give nonisomorphic groups H_α and $H_{\alpha'}$.

Proof. The element $(1,0)$ serves as the identity in H_α. By bilinearity of α, we know $\alpha(v_F,-v_E)+\alpha(v_F,v_E)=0$, which allows us to check that the inverse of (z,v) is $(z^{-1}\alpha(v_F,v_E),-v)$. A straightforward computation verifies associativity, showing that $*_{\alpha(E,F)}$ defines a group law, and another shows that H_α has α for its commutator pairing.

We need to check the effect of the choices we made when we defined the operation $\alpha(E,F)$ on H_α. The subspace $V_K\subseteq V$ is well-defined, but V_E and V_F are not, and they are used in the definition $\alpha(E,F)$. Suppose that E,F and E',F' are two choices for a symplectic basis spanning (not necessarily identical) complements of V_K in V. There is an automorphism of α that takes E to E' and F to F', and then this automorphism defines an isomorphism $(H_\alpha,\alpha(E,F))\cong(H_\alpha,\alpha(E',F'))$. Hence the group isomorphism class of H_α is well-defined, independent of the choices made to define the group operation.

Similarly, if $V\xrightarrow{\cong}V'$ induces an isomorphism of forms α,α', then compatible choices can be made for the symplectic bases $E,F\subseteq V$ and $E',F'\subseteq V'$, and these choices will induce an isomorphism of $(H_\alpha,\alpha(E,F))$ with $(H_{\alpha'},\alpha'(E',F'))$.

Lastly, if α and α' are not isomorphic, then by Lemma 7.2 their trivial components must be of different dimensions. It follows that the centers of H_α and $H_{\alpha'}$ are not isomorphic (for example, they have a different number of connected components), and H_α and $H_{\alpha'}$ are therefore not isomorphic as groups. □

Proposition 7.7 tells us that the construction $\alpha\mapsto H_\alpha$ defines a monomorphism from the isomorphism classes of totally isotropic bilinear forms over $\mathbb{Z}/p$ to the isomorphism classes of abstract projective elementary abelian p-groups. It remains to show that this function is an epimorphism, which we do by constructing a group for each form. For later purposes, we pay special attention to the identity component.

Proposition 7.8. *Let H be an abstract projective elementary abelian p-group with associated commutator form $\alpha\colon V\times V\to\mathbb{Z}/p$. Let ϕ :*

$S^1 \to H_0$ be an isomorphism of S^1 with the identity component of H. Then H_α is isomorphic to H via an isomorphism that restricts to ϕ on the identity component of H_α.

Proof. Let V_K, V_E, V_F be the subspaces of V defined just prior to Proposition 7.7. The basis elements of V_E can be lifted to elements of H, which can be chosen to be of order p because S^1 is a divisible group. The lifts commute since the form is trivial on V_E. Mapping basis elements of V_E to their lifts in H gives a monomorphism of groups $V_E \hookrightarrow H$ whose image we call W_E. Likewise, we can choose lifts $V_K \hookrightarrow H$ and $V_F \hookrightarrow H$, whose images are subgroups W_K and W_F of H, respectively.

Recall that as a set, $H_\alpha = S^1 \times V$, and for $v \in V$, we have $v = v_K + v_E + v_F$ as before. Let w_K, w_E, and w_F be the images of v_K, v_E, and v_F under the lifting homomorphisms of the previous paragraph. We extend the given isomorphism $\phi : S^1 \to H_0$ to a function $\Phi \colon H_\alpha \to H$ by

$$\Phi\left(z, v_K + v_E + v_F\right) = \phi(z)\, w_K w_E w_F. \tag{7.9}$$

(Note that we write the group operation additively in V, which is abelian, but multiplicatively in H, which may not be.)

We assert that Φ is a group homomorphism. To see that, suppose we have two elements (z, v) and (z', v') of H_α. If we multiply first in H_α we get $\big(zz'\,\alpha(v_F, v'_E),\ v + v'\big)$, and then application of Φ gives us

$$\phi\left(zz'\right)\ \alpha(v_F, v'_E)(v_K v'_K)(v_E v'_E)(v_F v'_F). \tag{7.10}$$

On the other hand, if we apply Φ first and then multiply, we get $(\phi(z)\, v_K v_E v_F)\,(\phi(z')\, v'_K v'_E v'_F)$, which can be rewritten as

$$\phi\left(zz'\right)\ \left(v_K v'_K\right)\left(v_E v_F v'_E v'_F\right). \tag{7.11}$$

To compare (7.10) to (7.11), we need to relate $v'_E v_F$ and $v_F v'_E$. However, the commutators in H are given exactly by α, so $v_F v'_E = \alpha(v_F, v'_E) v'_E v_F$, which allows us to see that (7.10) and (7.11) are equal. We conclude that Φ is a group homomorphism.

Finally, we need to know that Φ is a bijection. To see that Φ is surjective, observe that if $h \in H$ maps to $v \in V$ where $v = v_K + v_E + v_F$, then h and $w_K w_E w_F$ differ only by some element z of the central S^1. Hence every element of H can we written as $z w_K w_E w_F$ for some z, w_K, w_E, w_F, and Φ is surjective. However, (7.9) tells us that Φ is the isomorphism ϕ on the identity component, S^1. Further, Φ is a surjection of the finite set of components, hence a bijection of components. We conclude that Φ is an isomorphism.

□

Proposition 7.12. *The commutator form gives a one-to-one correspondence between isomorphism classes of abstract projective elementary abelian p-groups and isomorphism classes of totally isotropic bilinear forms over* $\mathbb{Z}/p$.

Proof. Every totally isotropic form α is realized as the commutator form of the group H_α. By Propositions 7.7 and 7.8, if H and H' have isomorphic commutator forms α and α', then

$$H \cong H_\alpha \cong H_{\alpha'} \cong H'.$$

□

Proposition 7.12 allows us to give the following explicit classification of abstract projective elementary abelian p-groups. This classification result can also be found in [Gri91] Theorem 3.1, though in this section we have given an elementary and self-contained discussion and proof.

Proposition 7.13. *Suppose that* H *is an abstract projective elementary abelian* p*-group*

$$1 \to S^1 \to H \to V \to 1.$$

Let $2s$ *be the maximal rank of a symplectic subspace of* V *under the commutator form of* H*, and let* $t = \mathrm{rk}(V) - 2s$*. Then* H *is isomorphic to* $\Gamma_s \times \Delta_t$.

Proof. The commutator form of $\Gamma_s \times \Delta_t$ is isomorphic to that of H, so the result follows from Proposition 7.12. To interpret the proposition when $s = 0$, note that $\Gamma_0 = S^1$, so the proposition says that if $s = 0$, then $H \cong S^1 \times \Delta_t \cong S^1 \times V$. □

Remark 7.14. As pointed out by D. Benson, one could also approach this classification result via group cohomology, using the fact that equivalence classes of extensions as in Definition 7.1 correspond to elements of $H^2(V; S^1)$. An argument using the Bockstein homomorphism shows that the group $H^2(V; S^1) \cong H^3(V, \mathbb{Z})$ can be identified with the exterior square of $H^1(V, \mathbb{Z}/p)$, which is in turn isomorphic to the space of alternating forms $V \times V \to \mathbb{Z}/p$. The standard factor set approach to H^2 can be used to identify such a form with the commutator pairing of the extension (as in [Bro82] or [Wei94]). The factor set associated to an extension is similarly defined and often identically denoted as the commutator pairing, but the two pairings are *not* the same. In particular, a factor set need not be bilinear or totally isotropic.

8. Initial list of problematic subgroups

Throughout this section, assume that $n = mp^k$ where m and p are coprime. Suppose that H is a problematic p-toral subgroup of $U(n)$. We know from the first part of Theorem 5.4 that H must be a projective elementary abelian p-group. If H contains $S^1 \subseteq U(n)$, we know the possible group isomorphism classes of H from Proposition 7.13. The purpose of this section is to use the character criterion of Theorem 5.4 to narrow down the possible conjugacy classes of H in $U(n)$.

In Section 6, we described the projective elementary p-subgroup $\Gamma_k \subset U(p^k)$. In this and subsequent sections, we consider the action of Γ_k on $\mathbb{C}^n$ by a multiple of its "standard" action on $\mathbb{C}^{p^k}$: the group Γ_k acts on $\mathbb{C}^n \cong \mathbb{C}^m \otimes \mathbb{C}^{p^k}$ by acting trivially on $\mathbb{C}^m$ and by the standard action on $\mathbb{C}^{p^k}$. In order to streamline notation, we denote this subgroup of $U(n)$ by Γ_k also, since context will indicate the dimension of the ambient space.

Since $\Gamma_k \subseteq U(n)$ is represented by block diagonal matrices with blocks Γ_k, we immediately obtain the following from Lemma 6.9.

Lemma 8.1. *The character of $\Gamma_k \subseteq U(n)$ is nonzero only on the elements of S^1, where the character is $\chi(s) = ns \in \mathbb{C}$.*

Our goal is to show that if H is a problematic subgroup of $U(n)$ where $n = mp^k$ and m and p are coprime, then H is a subgroup of $\Gamma_k \subset U(n)$. Although H itself may not be finite, we can use its finite subgroups to get information about n using the following result from basic representation theory.

Lemma 8.2. *Suppose that G is a finite subgroup of $U(n)$ and that $\chi_G(y) = 0$ unless $y = e$. Then $|G|$ divides n, and the action of G on $\mathbb{C}^n$ is by $n/|G|$ copies of the regular representation.*

Proof. The number of copies of an irreducible character χ in χ_G is given by the inner product

$$\langle \chi_G, \chi \rangle = \frac{1}{|G|} \sum_{y \in G} \chi_G(y) \overline{\chi(y)}.$$

Take χ to be the character of the one-dimensional trivial representation of G. The only nonzero term in the summation occurs when $y = e$, and since $\chi_G(e) = n$, we find $\langle \chi_G, \chi \rangle = n/|G|$. Since $\langle \chi_G, \chi \rangle$ must be an integer, we find that $|G|$ divides n. To finish, we observe that the character of $n/|G|$ copies of the regular representation is the same as χ_G, which finishes the proof. $\square$

We now have all the ingredients we require to prove the main result of this section.

Theorem 8.3. *Let H be a problematic p-toral subgroup of $U(n)$ that contains S^1, and suppose that $n = mp^k$, where m and p are coprime. Then H is conjugate to a subgroup of $\Gamma_k \subset U(n)$.*

Proof. We know from Theorem 5.4(1) that H is an abstract projective elementary elementary abelian p-group. By Proposition 7.13, H is abstractly isomorphic to $\Gamma_s \times \Delta_t$ for some s and t, so H contains a subgroup $(\mathbb{Z}/p)^{s+t}$ (say, $\mathcal{A}_s \times \Delta_t$). By Theorem 5.4(2), we have the character criterion on H necessary to apply Lemma 8.2, and we conclude that p^{s+t} divides n. Since $n = mp^k$ with m coprime to p, we necessarily have $s + t \leq k$. Hence by Lemma 6.8, we know $\Gamma_s \times \Delta_t \subseteq \Gamma_k$.

To finish the proof, we compare two representations of $\Gamma_s \times \Delta_t$. The first is the composite

$$\Gamma_s \times \Delta_t \hookrightarrow \Gamma_k \subseteq U(n)\,.$$

This map gives an identification of the identity component of the abstract group $\Gamma_s \times \Delta_t$ with the center $S^1 \subseteq U(n)$, and in terms of this identification, the character of the representation is $x \mapsto nx$ on the identity component of $\Gamma_s \times \Delta_t$ and zero else (Lemma 8.1).

To construct the second representation of $\Gamma_s \times \Delta_t$, we construct a map $\Gamma_s \times \Delta_t \to H$. Since H has the same commutator form as $\Gamma_s \times \Delta_t$, by Proposition 7.8 there is an isomorphism $\Gamma_t \times \Delta_t \to H \subseteq U(n)$ that gives the same map on identity components as $\Gamma_s \times \Delta_t \hookrightarrow \Gamma_k \hookrightarrow\subseteq U(n)$. Hence the character for this representation is also zero off the identity component (by Theorem 5.4), and $x \mapsto nx$ on the central S^1.

Thus the two representations of $\Gamma_s \times \Delta_t$ have the same character, and we conclude that they are conjugate. Since the image of one is the subgroup H, and the image of the other is $\Gamma_s \times \Delta_t \subseteq \Gamma_k \subseteq U(n)$, the theorem follows. ∎

Example 8.4. Suppose that p is an odd prime, and let $n = 2p$. Let H be a problematic subgroup of $U(2p)$ acting on $\mathcal{L}_{2p}$. According to Theorem 8.3, the subgroup H is conjugate in $U(2p)$ to a subgroup of Γ_1. Since in addition we assume that H contains the central S^1, there are only three possibilities for H: S^1 itself, Γ_1 acting by two copies of its standard representation, or $S^1 \times \Delta_1 \subset \Gamma_1$.

9. Fixed points and joins

In this section, we begin the work of computing the fixed points of the p-toral subgroups of $U(n)$ that are identified in Theorem 8.3 as potentially problematic. Throughout this section, let $n = mp^k$, and fix an isomorphism $\mathbb{C}^n \cong \mathbb{C}^m \otimes \mathbb{C}^{p^k}$. Let Γ_k act on $\mathbb{C}^n$ by acting trivially on $\mathbb{C}^m$ and by its standard representation (described in Section 6) on $\mathbb{C}^{p^k}$. There is also an action of $U(m)$ on $\mathbb{C}^n$ that commutes with the action of Γ_k, by letting $U(m)$ act by the standard action on $\mathbb{C}^m$ and trivially on $\mathbb{C}^{p^k}$. This action passes to an action of $U(m)$ on the fixed point space $(\mathcal{L}_n)^{\Gamma_k}$.

Our goal in this section and the next is to establish which subgroups of $\Gamma_k \subseteq U(n)$ actually have noncontractible fixed points on $\mathcal{L}_n$. A starting point is provided by the following result of [AL] for $\Gamma_k \subset U(p^k)$ acting on $\mathcal{L}_{p^k}$. We will bootstrap this result to fixed points of $\Gamma_k \subset U(n)$ acting on $\mathcal{L}_n$.

Proposition 9.1 ([AL])**.** *For $k \geq 1$, the fixed point space of Γ_k acting on $\mathcal{L}_{p^k}$ is homotopy equivalent to a wedge of spheres of dimension $k-1$.*

Let $X * Y$ denote the join of the two spaces X and Y. The theorem below establishes a formula for the fixed point space $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ that was suggested to us by G. Arone. Notice that there is no assumption that m and p should be coprime.

Theorem 9.2. *Suppose that $n = mp^k$ with $k \geq 1$. Then there is an equivalence of $U(m)$-spaces*

$$\mathcal{L}_m * \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} \to \left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}.$$

We begin by outlining the proof of Theorem 9.2. The strategy is to identify a $U(m)$-subcomplex Z of $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ such that the nerve of Z has the $U(m)$-equivariant homotopy type of the join in Theorem 9.2. Then we establish that Z is a $U(m)$-equivariant deformation retraction of $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$.

To construct the subcomplex Z, suppose that μ and ν are orthogonal decompositions of $\mathbb{C}^m$ and $\mathbb{C}^{p^k}$, respectively, with $\mathrm{cl}(\mu) = \{v_1, \ldots, v_s\}$ and $\mathrm{cl}(\nu) = \{w_1, \ldots, w_t\}$. We can tensor the components of μ and ν to obtain a decomposition of $\mathbb{C}^m \otimes \mathbb{C}^{p^k}$ that we denote $\mu \otimes \nu$:

$$\mathrm{cl}(\mu \otimes \nu) = \{v_i \otimes w_j : 1 \leq i \leq s \text{ and } 1 \leq j \leq t\}.$$

If ν is weakly fixed by Γ_k, then so is $\mu \otimes \nu$, and if at least one of μ and ν is proper, then $\mu \otimes \nu$ is proper as well.

Definition 9.3. The subposet $Z \subseteq \left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ is the set of objects of the form $\mu \otimes \nu$ where μ is a decomposition of $\mathbb{C}^m$ and ν is a weakly Γ_k-fixed decomposition of $\mathbb{C}^{p^k}$, and at least one of μ and ν is proper.

Remark 9.4. It follows from the definition that Z is stabilized by the action of $U(m)$ on $\mathbb{C}^n \cong \mathbb{C}^m \otimes \mathbb{C}^{p^k}$. In fact, by [Oli94] the centralizer of Γ_k is actually $U(m)$, and thus by Corollary 4.7 the path components of the object space of Z are actually $U(m)$-orbits. The same is true of the morphism space of Z, and indeed, for the space of d-simplices of Z for every d.

To analyze Z, we write it as a union of two subposets, each of which is closed under the action of $U(m)$. Let X (resp. Y) denote the subposet of Z consisting of the decompositions $\lambda = \mu \otimes \nu$ where μ (resp. ν) is a proper decomposition of $\mathbb{C}^m$ (resp. $\mathbb{C}^{p^k}$). The object space of X is stabilized by $U(m)$, and hence (Remark 9.4) is a union of path components of the object space of Z. The same is true of Y, and likewise the morphism spaces of X and Y are unions of path components of the morphism space of Z.

Any refinement in Z of an object in X is also in X, and likewise any refinement in Z of an object in Y is likewise in Y. Hence the nerve of Z is the union of the nerve of X and the nerve of Y. We construct a $U(m)$-equivariant map of diagrams

$$\left(\begin{array}{ccc} \mathcal{L}_m \times \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} & \longrightarrow & \mathcal{L}_m \\ \downarrow & & \\ \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} & & \end{array} \right) \longrightarrow \left(\begin{array}{ccc} X \cap Y & \longrightarrow X \\ \downarrow & \\ Y & \end{array} \right) \tag{9.5}$$

by doing the following.

- In the upper right corner, we map a proper decomposition μ of $\mathbb{C}^m$ to the decomposition $\mu \otimes \mathbb{C}^{p^k}$, which is in X.
- In the lower left corner, we map a decomposition ν in $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$ to $\mathbb{C}^m \otimes \nu$, which is in Y.
- In the upper left corner, we map the pair (μ, ν) to $\mu \otimes \nu$, which is in $X \cap Y$.

In (9.5), we would like to relate the homotopy pushout of the left diagram to the strict pushout of the right diagram (which is $X \cup Y$). First we establish commutativity of the map of diagrams in (9.5). We show that at each corner, (9.5) is a homotopy equivalence, and then we show that this statement remains true when we take fixed points under the action of a subgroup $H \subseteq U(m)$. Finally, we show that in the right-hand diagram of (9.5), the map from the homotopy pushout to the strict pushout is a homotopy equivalence and remains so after taking fixed points under $H \subseteq U(m)$.

Lemma 9.6. *The following ladder is $U(m)$-equivariantly homotopy commutative, and the vertical arrows are equivalences of $U(m)$-spaces:*

$$\begin{array}{ccccc} \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} & \longleftarrow & \mathcal{L}_m \times \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} & \longrightarrow & \mathcal{L}_m \\ \simeq\downarrow & & \cong\downarrow & & \simeq\downarrow \\ Y & \longleftarrow & X \cap Y & \longrightarrow & X. \end{array} \tag{9.7}$$

Proof. To establish homotopy commutativity, consider a pair (μ, ν) in $\mathcal{L}_m \times \left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$. Going clockwise around the right-hand square of (9.7) yields the decomposition $\mu \otimes \mathbb{C}^{p^k}$ in X. Going around that square counterclockwise yields the decomposition $\mu \otimes \nu$ in X. There is a natural coarsening $\mu \otimes \nu \to \mu \otimes \mathbb{C}^{p^k}$, and the coarsening morphism is stabilized by $U(m)$. Hence the right-hand square of diagram (9.7) induces a $U(m)$-homotopy commutative diagram of nerves. Similarly, following (μ, ν) clockwise around the left-hand square gives $\mu \otimes \nu$ in Y, and going counterclockwise gives $\mathbb{C}^m \otimes \nu$ in Y, and there is a natural $U(m)$-equivariant homotopy given by the coarsening $\mu \otimes \nu \to \mathbb{C}^m \otimes \nu$.

The vertical equivalences result from similar arguments, as follows.

(1) To see that the right-hand vertical map, $\mathcal{L}_m \to X$, is a homotopy equivalence of $U(m)$-spaces, consider that $\mu \otimes \nu \mapsto \mu \otimes \mathbb{C}^{p^k}$ gives a natural retraction. The coarsening map $\mu \otimes \nu \to \mu \otimes \mathbb{C}^{p^k}$ gives a $U(m)$-equivariant homotopy between the retraction and the identity.

(2) Likewise, we consider the left-hand map, $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k} \to Y$. The map $\mu \otimes \nu \mapsto \mathbb{C}^m \otimes \nu$ gives a natural retraction, and the coarsening map $\mu \otimes \nu \to \mathbb{C}^m \otimes \nu$ is a $U(m)$-equivariant homotopy making it a deformation retraction.

(3) The map $\mathcal{L}_m \times \left(\mathcal{L}_{p^k}\right)^{\Gamma_k} \longrightarrow X \cap Y$ is a $U(m)$-equivariant isomorphism of posets.

□

As a consequence of Lemma 9.6, we know that the map of diagrams in (9.5) induces a map between the homotopy pushouts that is $U(m)$-equivariant and a homotopy equivalence. However, to get equivalences of $U(m)$-spaces, we need to know what happens after taking fixed points of a subgroup $J \subseteq U(m)$. Hence we need to know the relationship between taking fixed points and taking homotopy pushouts. We give an argument that we learned from C. Malkiewich [Mal].

Proposition 9.8. *Let $f : A \longrightarrow B$ be an equivariant map of spaces with an action of a group J, and let* $\mathrm{Cyl}(f)$ *denote the mapping cylinder of f. Let $C = \mathrm{Cyl}\left(f^J\right)$ be the mapping cylinder of the function $A^J \longrightarrow$*

B^J given by restricting f to J-fixed points, and let $D = (\mathrm{Cyl}(f))^J$ be the fixed point space of the action of J on Cyl(f)*. Then the natural map $C \longrightarrow D$ is a homeomorphism.*

Proof. The space C is the quotient space of $\left(A^J \times [0,1]\right) \coprod B^J$ by the relation $(a,1) \simeq f(a)$. The inclusions $A^J \times [0,1] \hookrightarrow A \times [0,1]$ and $B^J \hookrightarrow B$ induce a natural map $C \to \mathrm{Cyl}(f)$, whose image is contained in D, and which is continuous by definition of the quotient topology on C. Further, using the fact that A^J and B^J are closed in A and B, respectively, we can check that $C \to D$ is a closed map. A routine check verifies that $C \to D$ is a bijection. We conclude that $C \to D$ is a homeomorphism. □

Corollary 9.9. *Suppose that*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a homotopy pushout diagram of spaces with an action of a group J, and J-equivariant maps. Then

$$\begin{array}{ccc} A^J & \longrightarrow & B^J \\ \downarrow & & \downarrow \\ C^J & \longrightarrow & D^J \end{array} \tag{9.10}$$

is also a homotopy pushout diagram.

Proof. The corollary is a consequence of Proposition 9.8 once we have replaced B and C with mapping cylinders and D with the double mapping cylinder.

□

Proposition 9.11. *The nerve of Z is $U(m)$-equivariantly homotopy equivalent to $\mathcal{L}_m * \left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$.*

Proof. Lemma 9.6 and Corollary 9.9 tell us that the map of diagrams in (9.5) induces an equivalence of $U(m)$-spaces on homotopy pushouts, i.e., a $U(m)$-equivariant map that is a homotopy equivalence on the fixed point space of any subgroup $J \subseteq U(m)$. The homotopy pushout of the left diagram in (9.5) is $\mathcal{L}_m * \left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$, and we need to relate this space to the strict pushout (not the homotopy pushout) of the right diagram in (9.5), the strict pushout being $X \cup Y = Z$. Hence the proposition follows once we show that the natural map from the homotopy pushout to the strict pushout for the diagram

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

is a homotopy equivalence, and remains so after taking fixed points for any subgroup $J \subseteq U(m)$. This statement follows if we can prove that for all subgroups J of $U(m)$, the space X^J is Reedy cofibrant and $(X \cap Y)^J \to Y^J$ is a Reedy cofibration.

To establish that X itself is Reedy cofibrant, consider the space X_d of d-simplices of X. Let $L_d(X)$ denote the space of degenerate d-simplices of X (i.e., the dth latching object for X). We must show that

$$L_d X \longrightarrow X_d \tag{9.12}$$

is a cofibration of topological spaces. We assert that $L_d X$ is, in fact, a union of path components of X_d. The complex $X \subseteq Z$ is stabilized by $U(m)$, and by Remark 9.4 the path components of the d-simplices of Z are $U(m)$-orbits. Hence X_d is a disjoint union of $U(m)$-orbits, and the same is true of X_{d-1}. The degeneracy maps of X are $U(m)$-equivariant, and it follows that their images are unions of path components. The space $L_d X$ is the union of such images, and is therefore also a union of path components of X_d, establishing that X is Reedy cofibrant.

We need to know that the statements of the previous paragraph remain true when we replace X by X^J, where J is any subgroup of $U(m)$. By Corollary 4.7, the fixed point space of the action of J on a $U(m)$-orbit has path components that are orbits under the action of $C_0(J)$, the identity component of the centralizer of J in $U(m)$. We can now follow exactly the same reasoning as in the previous paragraph to conclude that X^J is Reedy cofibrant.

We claim that the map $X \cap Y \to Y$ is a Reedy cofibration for essentially the same reasons. We must show that for each d, the map from the pushout of the diagram

$$\begin{array}{ccc} L_d(X \cap Y) & \longrightarrow & (X \cap Y)_d \\ \downarrow & & \\ L_d(Y) & & \end{array} \tag{9.13}$$

to Y_d is a cofibration in topological spaces. But X_d, Y_d, and $(X \cap Y)_d$ are disjoint unions of $U(m)$-orbits, as are their subspaces of degenerate simplices. As a consequence, each of the spaces in (9.13) is a union of path components of Y_d, and the maps are inclusions. So the pushout is likewise a union of path components of Y_d, and its inclusion into Y_d is a cofibration.

We also need to know that the map remains a Reedy cofibration after taking J-fixed points for any subgroup J of $U(m)$, and the argument is obtained by combining the previous two paragraphs, since taking J-fixed point spaces results in spaces of d-simplices that are orbits of $C_0(J)$. □

Now that we have constructed the $U(m)$-equivariant subcomplex Z of $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ with the desired homotopy type, we need to prove that there is a deformation retraction of $U(m)$-spaces from $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ to Z. We obtain it in steps, by constructing an interpolating subcategory between Z and $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ using the following definition.

Definition 9.14. Suppose that $H \subseteq U(n)$ and λ is an object of $(\mathcal{L}_n)^H$.

(1) For $v \in \mathrm{cl}(\lambda)$, define the *$H$-isotropy group* of v as $\{g \in H : gv = v\}$.
(2) We say that λ has *uniform H-isotropy* if every $v \in \mathrm{cl}(\lambda)$ has the same H-isotropy group. In this case, we write I_λ for the H-isotropy group of each element of $\mathrm{cl}(\lambda)$.
(3) We say that λ is *H-isotypical* if λ has uniform H-isotropy and I_λ acts isotypically on each component $v \in \mathrm{cl}(\lambda)$.

There is an easy criterion guaranteeing uniform isotropy.

Lemma 9.15. *Suppose $\lambda \in \mathrm{Obj}\,(\mathcal{L}_n)^H$ has the property that for some $v \in \mathrm{cl}(\lambda)$, the H-isotropy subgroup of v is normal in H. If H acts transitively on $\mathrm{cl}(\lambda)$, then λ has uniform H-isotropy.*

Proof. Since the action of H on $\mathrm{cl}(\lambda)$ is transitive, the H-isotropy subgroups of elements of $\mathrm{cl}(\lambda)$ are all conjugate in H to the isotropy group of v, which is assumed to be normal. We conclude that they are all actually the same. □

Corollary 9.16. *All objects in $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$ have uniform isotropy, as do objects in Z.*

Proof. Since Γ_k acts irreducibly on $\mathbb{C}^k$, it necessarily acts transitively on the components of any weakly Γ_k-fixed decomposition of $\mathbb{C}^{p^k}$. An object $\mu \otimes \nu$ in Z has the same Γ_k-isotropy group as ν. □

Since objects in Z have uniform Γ_k-isotropy, whereas objects in $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ may not, we consider an interpolating subcomplex that focuses on uniform isotropy.

Definition 9.17. Let $\operatorname{Unif}\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ be the subposet of $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ given by objects with uniform Γ_k-isotropy.

Proposition 9.18. *The inclusion* $\operatorname{Unif}\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k} \subseteq \left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ *induces a homotopy equivalence of* $U(m)$*-spaces on nerves.*

Proof. As usual, we define a $U(m)$-equivariant deformation retraction. Let λ be a decomposition that is weakly fixed by Γ_k, say $\operatorname{cl}(\lambda) = \{v_1, \ldots, v_j\}$. Let $I_{v_1}, \ldots, I_{v_j}$ be the Γ_k-isotropy subgroups of $v_1, \ldots, v_j$, respectively. Each of them contains S^1 and is therefore normal in Γ_k. Let $I_\lambda \subseteq \Gamma_k$ be the product, $I_\lambda = I_{v_1} \ldots I_{v_j}$. The construction of I_λ is $U(m)$-invariant, since $U(m)$ centralizes Γ_k.

Recall that $\lambda_{\operatorname{st}(I_\lambda)}$ denotes the strongly I_λ-fixed coarsening of λ created by summing components in the same orbit of the action of I_λ on $\operatorname{cl}(\lambda)$ (see Definition 4.2). Consider the assignment

$$\lambda \mapsto \lambda_{\operatorname{st}(I_\lambda)},$$

and observe that it is $U(m)$-equivariant.

First we check that this assignment actually lands in $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$. Because $S^1 \subseteq I_\lambda$, we know $I_\lambda \lhd \Gamma_k$. Lemma 4.4 then tells us that $\lambda_{\operatorname{st}(I_\lambda)}$ is weakly fixed by Γ_k. We also need to check that $\lambda_{\operatorname{st}(I_\lambda)}$ is in fact proper. If not, then I_λ acts transitively on $\operatorname{cl}(\lambda)$, and therefore Γ_k does also. By Lemma 9.15, we have $I_{v_1} = \cdots = I_{v_j}$, so $I_\lambda = I_{v_1}$. But then I_{v_1} acts transitively on $\operatorname{cl}(\lambda)$, which is a contradiction since I_{v_1} fixes v_1 and λ is proper.

We now check that the assignment $\lambda \mapsto \lambda_{\operatorname{st}(I_\lambda)}$ is continuous, by considering the Γ_k-isotropy of decompositions in the same path component of $\operatorname{Obj}(\mathcal{L}_{mp^k})^{\Gamma_k}$ as λ. By Corollary 4.7, the path component of λ consists of elements $c\lambda$ where c is in the identity component of $C_{U\left(mp^k\right)}(\Gamma_k)$. However, if $v \in \operatorname{cl}(\lambda)$ then $I_{cv} = c\,(I_v)\,c^{-1} = I_v$. As a result, $I_\lambda = I_{c\lambda}$. Therefore the same subgroup is being used to coarsen the entire path component of λ, and we already know that this operation is continuous from Lemma 4.8.

Next we verify that the assignment $\lambda \mapsto \lambda_{\operatorname{st}(I_\lambda)}$ respects coarsenings: we must check that if $\mu \leq \lambda$, then $\mu_{\operatorname{st}(I_\mu)} \leq \lambda_{\operatorname{st}(I_\lambda)}$. Suppose that $w \in \operatorname{cl}(\mu)$ and that $w \subseteq v \in \operatorname{cl}(\lambda)$, and further suppose that $\gamma \in I_w$. Since $I_w \subset \Gamma_k$ and Γ_k weakly fixes λ, we have that γv is either equal to v or orthogonal to v, whence $\gamma v = v$. Hence $\gamma \in I_v$, establishing that $w \subseteq v$ implies $I_w \subseteq I_v$. As a result, $I_\mu \subseteq I_\lambda$. We conclude that $\mu_{\operatorname{st}(I_\mu)} \leq \lambda_{\operatorname{st}(I_\mu)} \leq \lambda_{\operatorname{st}(I_\lambda)}$.

The coarsening $\lambda \to \lambda_{\mathrm{st}(I_\lambda)}$ gives a $U(m)$-equivariant homotopy from the identity functor on $\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k}$ to the composition of retraction and inclusion, and the lemma follows. □

At this point, we have

$$Z \subseteq \mathrm{Unif}\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k} \subseteq \big(\mathcal{L}_{mp^k}\big)^{\Gamma_k}$$

and the second inclusion is an equivalence of $U(m)$-spaces. Next, we interpolate again by defining $\mathrm{Isotyp}\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k}$ as the subcomplex of $\mathrm{Unif}\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k}$ of decompositions λ that are I_λ-isotypical, where I_λ is the (uniform) Γ_k-isotropy of λ.

Proposition 9.19. *The inclusion* $\mathrm{Isotyp}\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k} \hookrightarrow \mathrm{Unif}\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k}$ *induces a homotopy equivalence of* $U(m)$*-spaces on nerves.*

Proof. Again we define a $U(m)$-equivariant deformation retraction. Let λ be a decomposition in $\mathrm{Unif}\big(\mathcal{L}_{mp^k}\big)^{\Gamma_k}$ with Γ_k-isotropy $I_\lambda \subseteq \Gamma_k$. Now consider the assignment $\lambda \mapsto \lambda_{\mathrm{iso}(I_\lambda)}$, which we assert is continuous. As in the proof of Proposition 9.18, continuity follows from the fact that I_λ is constant on each path component, and isotypical refinement with respect to a subgroup is continuous on each path component (Lemma 4.9). Further, the value of I_λ does not change when we act on λ by an element of $U(m)$, because $U(m)$ centralizes Γ_k. And because $U(m)$ also necessarily centralizes I_λ, the assignment $\lambda \mapsto \lambda_{\mathrm{iso}(I_\lambda)}$ is also $U(m)$-equivariant.

To check that $\lambda \mapsto \lambda_{\mathrm{iso}(I_\lambda)}$ is natural in λ, suppose given $\mu \leq \lambda$ with uniform Γ_k-isotropy I_μ and I_λ, respectively. As in the proof of Proposition 9.18, $I_\mu \subseteq I_\lambda$. We need to check that $\mu_{\mathrm{iso}(I_\mu)}$ is a refinement of $\lambda_{\mathrm{iso}(I_\lambda)}$. Suppose we have $w \in \mathrm{cl}(\mu)$ with $w \subseteq v \in \mathrm{cl}(\lambda)$. We need to prove that for every I_μ-isotypical summand of w, there exists a I_λ-isotypical summand of v that contains it. It is sufficient to show that non-isomorphic irreducible representations of I_λ contained in v cannot contain isomorphic irreducible representations of I_μ.

However, any I_λ-irreducible subspace of v is contained in the restriction of the standard representation of Γ_k to I_λ. By Frobenius reciprocity ([Kna96, Theorem 9.9]), the restriction of the standard representation of Γ_k to I_μ splits as the sum of pairwise nonisomorphic I_μ-irreducibles. Since $I_\mu \subseteq I_\lambda$, we conclude that nonisomorphic representations of I_λ contained in v cannot contain isomorphic representations of I_μ.

Finally, the coarsening morphism $\lambda_{\mathrm{iso}(I_\lambda)} \to \lambda$ provides the necessary $U(m)$-equivariant natural transformation from the composition of retraction and inclusion to the identity functor. □

We continue with a result on decompositions of tensor products. Since the particular properties of Γ_k are not needed, we use a more general statement.

Lemma 9.20. *Suppose that $H \subseteq U(i)$ acts irreducibly on $\mathbb{C}^i$, and let $\mathbb{C}^m$ have trivial H-action. If v is an H-invariant subspace of $\mathbb{C}^m \otimes \mathbb{C}^i$, then there exists a well-defined subspace $w_v \subseteq \mathbb{C}^m$ such that $v = w_v \otimes \mathbb{C}^i$. The assignment $v \mapsto w_v$ is natural in v, is continuous, preserves orthogonality, and is $U(m)$-equivariant.*

Corollary 9.21. *If $n = mp^k$ then there is a $U(m)$-equivariant isomorphism $\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}_{\mathrm{st}} \cong \mathcal{L}_m$.*

Proof of Lemma 9.20. By Schur's Lemma, there is an isomorphism

$$\operatorname{End}_{\mathbb{C}}\left(\mathbb{C}^m\right) \xrightarrow{\cong} \operatorname{End}_H\left(\mathbb{C}^m \otimes \mathbb{C}^i\right)$$

given by the function $f \mapsto f \otimes \mathbb{C}^i$, and the function is $U(m)$-equivariant by inspection.

The idempotent $e_v \in \operatorname{End}\left(\mathbb{C}^m \otimes \mathbb{C}^i\right)$ corresponding to orthogonal projection to v is H-equivariant. Hence e_v has the form $f_v \otimes \mathbb{C}^i$ for some $f_v \in \operatorname{End}_{\mathbb{C}}\left(\mathbb{C}^m\right)$. Since f_v is necessarily an idempotent, it defines a subspace $w_v = \operatorname{im}\left(f_v\right)$, and $v = w_v \otimes \mathbb{C}^i$. The three assignments $v \mapsto e_v \mapsto f_v \mapsto w_v$ are each continuous. Finally, $v_1 \perp v_2$ implies that $w_{v_1} \perp w_{v_2}$ (for example, because the corresponding idempotents compose to zero). □

We now have everything we need to prove Theorem 9.2.

Proof of Theorem 9.2. We have a sequence of poset inclusions,

$$Z \subseteq \operatorname{Isotyp}\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k} \subseteq \operatorname{Unif}\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k} \subseteq \left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$$

and we have already established that the second two inclusions induce homotopy equivalences of $U(m)$-spaces on nerves (Proposition 9.19 and Proposition 9.18, respectively). To finish the proof, we show that $Z \subseteq \operatorname{Isotyp}\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$ is a $U(m)$-equivariant isomorphism.

Suppose that λ is a decomposition of $\mathbb{C}^n \cong \mathbb{C}^m \otimes \mathbb{C}^{p^k}$ that lies in $\operatorname{Isotyp}\left(\mathcal{L}_{mp^k}\right)^{\Gamma_k}$, that is, λ is weakly fixed by Γ_k, has uniform Γ_k-isotropy group I_λ, and is I_λ-isotypic. We will prove that there exists a decomposition μ of $\mathbb{C}^m$ and a weakly Γ_k-fixed decomposition ν of $\mathbb{C}^{p^k}$ such that $\lambda = \mu \otimes \nu$. Note that if λ is proper, then one of μ or ν (but not both) could have a single component.

As in the proof of Proposition 9.18, Frobenius reciprocity tells us that the restriction of the standard representation of Γ_k on $\mathbb{C}^{p^k}$ to the subgroup I_λ splits as a sum of pairwise nonisomorphic I_λ-irreducible subspaces, say $v_1, v_2, \dots, v_r$. These subspaces are orthogonal, and we use them to define the decomposition ν of $\mathbb{C}^{p^k}$ with $\mathrm{cl}(\nu) = \{v_1, v_2, \dots, v_r\}$. Note that ν is weakly fixed by Γ_k since $I_\lambda \triangleleft \Gamma_k$. The elements of $\mathrm{cl}(\nu)$ are permuted transitively by the action of Γ_k because Γ_k acts irreducibly on $\mathbb{C}^{p^k}$.

Recall that by assumption, the components of λ are isotypical representations of I_λ. Fix $v \in \mathrm{cl}(\nu)$, and consider the components of λ that are I_λ-isotypical for the irreducible representation v, say $c_1, \dots, c_s \in \mathrm{cl}(\lambda)$. Each one is an I_λ-subrepresentation of of $\mathbb{C}^m \otimes v$, which is the canonical v-isotypical summand of $\mathbb{C}^m \otimes \mathbb{C}^{p^k}$. Thus by Lemma 9.20, there exist subspaces $w_1, \dots, w_s$ of $\mathbb{C}^m$ such that $c_1 = w_1 \otimes v, \dots, c_s = w_s \otimes v$. We take μ to be the decomposition defined by $\mathrm{cl}(\mu) = \{w_1, \dots, w_s\}$.

Because every component of λ is isotypical for a unique element of $\mathrm{cl}(\nu)$, and Γ_k acts transitively on $\mathrm{cl}(\nu)$, we know that for every $c \in \mathrm{cl}(\lambda)$ there exists $\gamma \in \Gamma_k$ such that $\gamma c \subseteq \mathbb{C}^m \otimes v$. Since $\gamma c \in \mathrm{cl}(\lambda)$, we must have that γc is equal to some $w_i \otimes v$.

Hence the set $\mathrm{cl}(\lambda)$ must be the union of the orbits of $w_1 \otimes v, \dots, w_s \otimes v$ under the action of Γ_k. These orbits are, respectively, $w_1 \otimes \nu, \dots, w_s \otimes \nu$. We conclude that $\lambda = \mu \otimes \nu$. □

10. Proof of the classification theorem

In this section, we prove the classification theorem for problematic p-toral subgroups of $U(n)$, Theorem 1.2. Recall that if m and p are coprime, then any problematic p-toral subgroup of $U(mp^k)$ is subconjugate to Γ_k acting on $\mathbb{C}^{mp^k}$ by m copies of the standard representation of $\Gamma_k \subseteq U(p^k)$ (Theorem 8.3). Furthermore, all subgroups of Γ_k that contain S^1 have the form $\Gamma_s \times \Delta_t$, where $s + t \leq k$ (Proposition 7.13). Hence, we can use Theorem 9.2 to obtain a reduction of the classification theorem as follows.

Proposition 10.1. *Let H be a subgroup of $\Gamma_k \subseteq U(n)$ acting by a multiple of the standard representation of $\Gamma_k \subseteq U(p^k)$. Suppose that $H \cong \Gamma_s \times \Delta_t$, and let $r = n/p^{s+t}$. Then $(\mathcal{L}_n)^H$ is contractible (respectively, mod p acyclic) if and only if the s-fold suspension of $(\mathcal{L}_{rp^t})^{\Delta_t}$ is contractible (respectively, mod p acyclic).*

Proof. If $s = 0$, then we are considering $H \cong S^1 \times \Delta_t$ and $n = rp^t$. Hence $(\mathcal{L}_n)^H = (\mathcal{L}_{rp^t})^{\Delta_t}$, so the proposition is tautologically true.

For $s > 0$, Theorem 9.2 gives an equivalence of $U(rp^t)$-spaces

$$(\mathcal{L}_n)^{\Gamma_s} \simeq \mathcal{L}_{rp^t} * (\mathcal{L}_{p^s})^{\Gamma_s}.$$

Taking fixed points under the action of $\Delta_t \subseteq U(rp^t)$ gives

$$(\mathcal{L}_n)^H \simeq (\mathcal{L}_{rp^t})^{\Delta_t} * (\mathcal{L}_{p^s})^{\Gamma_s}.$$

Recall that $(\mathcal{L}_{p^s})^{\Gamma_s}$ is a wedge of spheres of dimension $s-1$ (Proposition 9.1). Choosing basepoints gives $X * Y \simeq \Sigma(X \wedge Y)$ for any X and Y, so $(\mathcal{L}_n)^H$ has the homotopy type of a wedge of s-fold suspensions of $(\mathcal{L}_{rp^t})^{\Delta_t}$. □

Most of the remainder of this section is devoted to studying fixed points of Δ_t acting on $\mathcal{L}_{rp^t}$, in preparation for assembling the classification theorem at the end of the section. When $r = 1$, we can quote the following result.

Proposition 10.2 ([AL]). *For $t \geq 1$, the fixed point space of Δ_t acting on $\mathcal{L}_{p^t}$ contains, as a retract, a wedge of spheres of dimension $(t-1)$.*

Next we look at the special case of $(\mathcal{L}_{mp^t})^{\Delta_t}$ where m is a positive power of a prime q different from p. In this case, we do not get contractibility, but we do have mod p acyclicity. Because we have two primes in play, we specify them explicitly in the notation for the following proposition. We write $\Delta_t(p) \cong (\mathbb{Z}/p)^t$, and we write $\Gamma_r(q)$ for the mod q irreducible projective elementary abelian q-group of $U(q^r)$,

$$1 \to S^1 \to \Gamma_r(q) \to (\mathbb{Z}/q)^{2r} \to 1.$$

Proposition 10.3. *If $m = q^r$ where q is a prime different from p and $r > 0$, then $(\mathcal{L}_{mp^t})^{\Delta_t(p)}$ is mod p acyclic, but has nontrivial mod q homology, and in particular is not contractible.*

Proof. Mod p acyclicity of $(\mathcal{L}_{mp^t})^{\Delta_t(p)}$ follows from Smith theory, since $\mathcal{L}_{mp^t}$ is a finite complex and is mod p acyclic by Corollary 3.2. To prove the statement about mod q homology of $(\mathcal{L}_{mp^t})^{\Delta_t(p)}$, we use Smith theory again, this time in reverse and for mod q homology, as follows.

We reverse the roles of p and q, and we consider the action of $\Delta_t(p) \times \Gamma_r(q)$ on $\mathbb{C}^{p^t} \otimes \mathbb{C}^{q^r}$. By Theorem 9.2 applied to $\Gamma_r(q)$, with $\Delta_t(p) \subseteq U(p^t)$, we find that

$$(\mathcal{L}_{p^t q^r})^{\Delta_t(p) \times \Gamma_r(q)} \simeq (\mathcal{L}_{p^t})^{\Delta_t(p)} * (\mathcal{L}_{q^r})^{\Gamma_r(q)}.$$

By Proposition 9.1, $(\mathcal{L}_{q^r})^{\Gamma_r(q)}$ is a wedge of spheres. On the other hand, by Proposition 10.2, $(\mathcal{L}_{p^t})^{\Delta_t(p)}$ has a wedge of spheres as a retract.

Hence $(\mathcal{L}_{p^tq^r})^{\Delta_t(p)\times\Gamma_r(q)}$ also has a wedge of spheres as a retract, and therefore has nonzero mod q homology.

But in fact,

$$(\mathcal{L}_{q^rp^t})^{\Gamma_r(q)\times\Delta_t(p)} = \left((\mathcal{L}_{q^rp^t})^{\Delta_t(p)}\right)^{\Gamma_r(q)/S^1}.$$

If $(\mathcal{L}_{q^rp^t})^{\Delta_t(p)}$ were mod q acyclic, then we could apply Smith theory to the finite complex $(\mathcal{L}_{q^rp^t})^{\Delta_t(p)}$ to conclude that $(\mathcal{L}_{q^rp^t})^{\Gamma_r(q)\times\Delta_t(p)}$ would be mod q acyclic also, which we know is not the case. ☐

The bulk of the rest of this section is to study the case $(\mathcal{L}_{mp^t})^{\Delta_t}$ where m is not a power of a prime, with the aim of showing it is contractible. The strategy resembles that of Section 9, in that we replace the category $(\mathcal{L}_{mp^t})^{\Delta_t}$ with the category $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ of Δ_t-fixed decompositions with uniform Δ_t-isotropy, and then we decompose $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ into a union of two categories, the union of whose nerves gives the nerve of $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$.

Definition 10.4.

(1) Let $\mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ denote the subposet of $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ consisting of decompositions λ that have a nontransitive action of Δ_t on $\mathrm{cl}(\lambda)$.

(2) Let $\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ denote the subposet of $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ consisting of decompositions λ such that the action of Δ_t on $\mathrm{cl}(\lambda)$ is nontrivial; that is, λ is not strongly fixed by Δ_t.

As in Section 9, we observe that if λ is an object of $\mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t}$, then all refinements of λ are objects in $\mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ as well, and likewise $\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ contains all refinements of its objects. In addition, any λ with uniform Δ_t-isotropy is in one of these subspaces; hence the nerve of $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ is the union of the nerves of the two subcategories, and we have a pushout diagram that is also a homotopy pushout:

$$\begin{array}{ccc} \mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t} \cap \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t} & \longrightarrow & \mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t} & \longrightarrow & \mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t} \end{array} \quad . \tag{10.5}$$

Lemma 10.6. *The object spaces of the subcategories* $\mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ *and* $\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ *are each unions of path components of the object space of* $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$*.*

Proof. We apply Corollary 4.7. The action of the centralizer of Δ_t on the object space preserves the defining characteristics of $\operatorname{NonTran}(\mathcal{L}_n)^{\Delta_t}$ and $\operatorname{Move}(\mathcal{L}_{mp^t})^{\Delta_t}$. $\square$

The initial step to prove that the nerve of $\operatorname{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ is contractible is to establish that the upper right-hand corner of (10.5) is contractible.

Lemma 10.7. *Suppose that Δ_t is nontrivial and acts on $\mathbb{C}^m \otimes \mathbb{C}^{p^t}$ as the tensor product of the trivial representation and the regular representation. Then $\operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}$ has contractible nerve.*

Proof. As usual, consider the inclusions

$$(\mathcal{L}_{mp^t})^{\Delta_t}_{\text{iso}} \hookrightarrow (\mathcal{L}_{mp^t})^{\Delta_t}_{\text{st}} \hookrightarrow \operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}. \tag{10.8}$$

The first map has the retraction functor $\lambda \mapsto \lambda_{\text{iso}(\Delta_t)}$. The second map has the retraction functor $\lambda \mapsto \lambda_{\text{st}(\Delta_t)}$. (Note that $\lambda_{\text{st}(\Delta_t)}$ is necessarily proper by the definition of $\operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}$.) Hence both inclusions of (10.8) induce equivalences of nerves. Finally, the action of Δ_t on $\mathbb{C}^{mp^t}$ is a multiple of the regular representation, and is therefore not isotypic. Hence $(\mathcal{L}_{mp^t})^{\Delta_t}_{\text{iso}}$ has a terminal object, namely the canonical Δ_t-isotypic decomposition of $\mathbb{C}^{mp^t}$, and has contractible nerve. $\square$

Since we have shown that the upper right corner of diagram (10.5) has contractible nerve, to establish that $\operatorname{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ has contractible nerve it is sufficient to show that the left-hand vertical map of (10.5) induces a homotopy equivalence on nerves. We will apply a topological version of Quillen's Theorem A for categories internal to spaces, as stated and proved in [Lib11, Theorem 5.8]. To check that the conditions of the cited theorem are satisfied, the first step is to look at the overcategories for objects in the lower left-hand corner of (10.5).

Proposition 10.9. *Let λ be an object of $\operatorname{Move}(\mathcal{L}_{mp^t})^{\Delta_t}$, and assume that $m > 1$ is not a power of a prime. Let $\mathcal{I}$ denote the intersection*

$$\mathcal{I} = \operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t} \cap \operatorname{Move}(\mathcal{L}_{mp^t})^{\Delta_t}.$$

Then the overcategory $\mathcal{I} \downarrow \lambda$ has contractible nerve.

Proof. The category $\mathcal{I} \downarrow \lambda$ is the poset of refinements (not necessarily strict) of λ that happen to be in $\operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}$. In other words, $\mathcal{I} \downarrow \lambda$ contains objects $\mu \to \lambda$ such that the action of Δ_t on $\operatorname{cl}(\mu)$ is not transitive. If λ is in $\operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}$, then the identity morphism of λ is a terminal object of $\mathcal{I} \downarrow \lambda$; therefore the category $\mathcal{I} \downarrow \lambda$ has contractible nerve.

So suppose that λ is in $\operatorname{Move}(\mathcal{L}_{mp^t})^{\Delta_t}$, but not in $\operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}$. In particular, λ has uniform Δ_t-isotropy I_λ properly contained in Δ_t, and $\operatorname{cl}(\lambda)$ has a transitive action of Δ_t. To specify a refinement μ of λ that lies in $\operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t}$, it is sufficient to choose one component $v \in \operatorname{cl}(\lambda)$, and to specify an orthogonal decomposition of v, call it μ_v, such that

- μ_v is (weakly) stabilized by the action of I_λ on v,
- I_λ acts non-transitively on $\operatorname{cl}(\mu_v)$ (hence μ_v is proper), and
- components of μ_v have uniform I_λ-isotropy.

The rest of μ is determined by transitivity of the Δ_t-action on $\operatorname{cl}(\lambda)$. If we denote the dimension of v by r, then the above shows that

$$\mathcal{I} \downarrow \lambda \cong \operatorname{NonTran}(\mathcal{L}_r)^{I_\lambda}. \tag{10.10}$$

There are two cases: $I_\lambda = 0$, and I_λ is nontrivial. If $I_\lambda = 0$, then $\operatorname{NonTran}(\mathcal{L}_r)^{I_\lambda} \cong \mathcal{L}_r$. Because r is a multiple of $m > 1$, which is not the power of a prime, we know that $\mathcal{L}_r$ is nonempty and has contractible nerve by Corollary 3.2. On the other hand, if I_λ is not trivial, then $\operatorname{NonTran}(\mathcal{L}_r)^{I_\lambda}$ has contractible nerve by Lemma 10.7. ☐

The next condition to check in order to apply [Lib11, Theorem 5.8] is the Reedy cofibrancy of an associated simplicial space. Here, we need to consider an intermediate object between a topological category and its nerve, namely the *simplicial nerve*, which is a simplicial diagram of spaces obtained by taking the levelwise nerve of the original topological category. The nerve is then obtained by applying a realization functor, which gives a space. The category of simplicial spaces can be given a Reedy model structure which we do not need here, but the properties of cofibrant objects in that model structure are sufficiently nice that they will be helpful in the results that follow.

Lemma 10.11. *The categories* $\mathcal{I} = \operatorname{NonTran}(\mathcal{L}_{mp^t})^{\Delta_t} \cap \operatorname{Move}(\mathcal{L}_{mp^t})^{\Delta_t}$ *and* $\operatorname{Move}(\mathcal{L}_{mp^t})^{\Delta_t}$ *have simplicial nerves that are Reedy cofibrant simplicial spaces.*

Proof. We apply the criterion for Reedy cofibrancy in [Lib11, Proposition A.2.2]. This criterion says that for a simplicial space Y to be Reedy cofibrant, it is sufficient that Y have an action of a compact Lie group G such that in each simplicial dimension, Y is a disjoint union of G-orbits, so that Y/G is discrete. Here, we take $G = C_0(\Delta_t)$, the identity component of the centralizer of Δ_t in $U(mp^t)$. We observe the following.

- In order for a decomposition λ to be in $\mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t}$, the action of Δ_t on $\mathrm{cl}(\lambda)$ must be nontransitive, and λ must have uniform Δ_t-isotropy.
- In order for a decomposition λ to be in $\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$, the action of Δ_t on $\mathrm{cl}(\lambda)$ must be nontrivial, and λ must have uniform Δ_t-isotropy.

If λ satisfies either (or both) of these conditions and $c \in C_0(\Delta_t)$, then $c\lambda$ satisfies the same condition(s) as λ.

But Corollary 4.7 tells us that the path components of both the object and morphism spaces of $(\mathcal{L}_{mp^t})^{\Delta_t}$ are orbits of $C_0(\Delta_t)$. Applying the observations above tells us that both $\mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t} \cap \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ and $\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$ are unions of path components of $(\mathcal{L}_{mp^t})^{\Delta_t}$, each of which is an orbit of $C_0(\Delta_t)$. We conclude that after taking the quotient by the action of $C_0(\Delta_t)$, we have a discrete space in each simplicial dimension of the simplicial nerves, and [Lib11, Proposition A.2.2] now gives the desired result. □

Proposition 10.12. *Suppose $m > 1$ is not a power of a prime. The inclusion of topological categories*

$$\mathcal{I} = \mathrm{NonTran}\,(\mathcal{L}_{mp^t})^{\Delta_t} \cap \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t} \longrightarrow \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}$$

induces a homotopy equivalence of nerves.

Proof. We prove this result by applying Libman's version of Quillen's Theorem A [Lib11, Theorem 5.8] to the inclusion of opposite categories

$$j\colon \mathcal{I}^{op} \longrightarrow \left[\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}\right]^{op}.$$

Thus maps are now refinements of decompositions, i.e., if μ is a refinement of λ, we have a map $\lambda \to \mu$ in the opposite category.

To apply the cited theorem, we need to know that the nerves of $\mathcal{I}^{op}$ and $\left[\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}\right]^{op}$ are Reedy cofibrant; since Reedy cofibrancy is preserved by taking opposites, we have this condition by Lemma 10.11.

Next, we need to know that for any $\lambda \in \left[\mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t}\right]^{op}$, the nerve of the undercategory $\lambda \downarrow j$ is Reedy cofibrant; the proof can be handled just as that of Lemma 10.11. Specifically, the object space of $\lambda \downarrow j$ consists of $\mu \subseteq \lambda$, where μ is in $\mathcal{I}$. Let K_λ denote the $U(n)$-isotropy group of λ; then the group $C_0(\Delta_t) \cap K_\lambda$ acts transitively on each path component of $\lambda \downarrow j$, and taking the quotient by this action gives a discrete space.

Further, we need to know that for any $\lambda \in \left[\text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t}\right]^{op}$, the nerve of the undercategory $\lambda \downarrow j$ is contractible. But this undercategory is the same as the overcategory $\mathcal{I} \downarrow \lambda$, which was proved to be contractible in Proposition 10.9.

Finally, we need to check that the inclusion j is absolutely tame, a technical condition which we now recall from [Lib11, Definition 5.5]. Associated to the inclusion j is a bisimplicial space $\mathcal{X}(j)$ whose space $\mathcal{X}_{s,r}(j)$ of (s,r)-bisimplices consists of chains of refinements and coarsenings

$$\{\lambda_s \to \cdots \to \lambda_0 \leftarrow \mu_0 \leftarrow \cdots \leftarrow \mu_r\},$$

where $\lambda_i \in \text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t}$ and $\mu_j \in \mathcal{I}$. There is a projection map

$$\pi_{s,r} \colon \mathcal{X}_{s,r}(j) \to \text{Nerve}_s(\text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t})$$

which forgets the chain of μ's. We say that j is *absolutely tame* if $\pi_{s,r}$ is a Serre fibration for all $s, r \geq 0$ [Lib11, Definition 5.7]. We will verify this condition.

A connected component of the object space of $\text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t}$, which is also the space of zero simplices $\text{Nerve}_0(\text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t})$, is a $C_0(\Delta_t)$-orbit; more precisely, the connected component of a decomposition λ_0 in $\text{Nerve}_0(\text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t})$ is $C_0(\Delta_t) / (C_0(\Delta_t) \cap K_{\lambda_0})$, where K_{λ_0} is the $U(n)$-isotropy group of λ_0. Consequently, we can determine that the connected component of $\lambda_\bullet = \{\lambda_s \to \cdots \to \lambda_0\} \in \text{Nerve}_s(\text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t})$ is $C_0(\Delta_t) / (C_0(\Delta_t) \cap K_{\lambda_\bullet})$, where

$$K_{\lambda_\bullet} = K_{\lambda_0} \cap \cdots \cap K_{\lambda_s}.$$

Similarly, the connected component of

$$(\lambda_\bullet, \mu_\bullet) = \{\lambda_s \to \cdots \to \lambda_0 \leftarrow \mu_0 \leftarrow \cdots \leftarrow \mu_t\} \in \mathcal{X}_{s,t}(j)$$

is

$$C_0(\Delta_t) \,/\, (C_0(\Delta_t) \cap K_{\lambda_\bullet} \cap K_{\mu_\bullet}).$$

Restricted to a connected component of $\mathcal{X}_{s,t}(j)$, the map $\pi_{s,t}$ is the quotient induced by the inclusion of subgroups

$$C_0(\Delta_t) \cap K_{\lambda_\bullet} \cap K_{\mu_\bullet} \subseteq C_0(V) \cap K_{\lambda_\bullet},$$

and that quotient is a Serre fibration.

We have established that all conditions of [Lib11, Theorem 5.8] are satisfied for the map j, so we conclude that it induces a homotopy equivalence on nerves. Thus the inclusion (before taking opposites of the categories)

$$\mathcal{I} \to \text{Move}\left(\mathcal{L}_{mp^t}\right)^{\Delta_t}$$

also induces a homotopy equivalence on nerves, as claimed. □

The work above showing that we can use Quillen's Theorem A allows us to establish the following key result.

Theorem 10.13. *Let $m > 1$, and let Δ_t act on $\mathbb{C}^{mp^t}$ by m copies of the regular representation. If m is not a power of a prime, then $(\mathcal{L}_{mp^t})^{\Delta_t}$ is contractible.*

Proof. If $t = 0$ then the result follows from Corollary 3.2, so we can assume that $t > 0$ and Δ_t is nontrivial.

As in Section 9, we proceed by decomposing the category of interest. Let $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ denote the subposet of $(\mathcal{L}_{mp^t})^{\Delta_t}$ consisting of objects with uniform Δ_t-isotropy. Exactly as in Proposition 9.18, the inclusion $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t} \hookrightarrow (\mathcal{L}_{mp^t})^{\Delta_t}$ induces a homotopy equivalence on nerves. We will show that $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ has a contractible nerve.

The object space of $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ is a union of path components of $\mathrm{Obj}\,(\mathcal{L}_{mp^t})^{\Delta_t}$, a fact which follows from Corollary 4.7, since the action of the centralizer of Δ_t preserves the property of having uniform Δ_t-isotropy. Therefore, the pushout diagram

$$\begin{array}{ccc} \mathrm{NonTran}\,(\mathcal{L}_n)^{\Delta_t} \cap \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t} & \longrightarrow & \mathrm{NonTran}\,(\mathcal{L}_n)^{\Delta_t} \\ \downarrow & & \downarrow \\ \mathrm{Move}\,(\mathcal{L}_{mp^t})^{\Delta_t} & \longrightarrow & \mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t} \end{array} \tag{10.14}$$

gives rise to a homotopy pushout diagram after applying the nerve functor.

The upper right-hand corner has contractible nerve, by Lemma 10.7. The left-hand vertical map induces a homotopy equivalence on nerves, by Proposition 10.12. Thus $\mathrm{Unif}(\mathcal{L}_{mp^t})^{\Delta_t}$ must also be contractible, as needed. □

This result brings us at last to the assembly of the proof of the classification theorem. Recall that a coistropic subgroup of Γ_k is one that has the form $\Gamma_s \times \Delta_t$ where $s + t = k$.

Theorem 1.2. Suppose that $n = mp^k$, where m is coprime to p, and suppose that $H \subseteq U(n)$ is a p-toral subgroup of $U(n)$ that contains S^1. Let Γ_k act on $\mathbb{C}^n$ by m copies of the standard representation of Γ_k on $\mathbb{C}^{p^k}$.

(1) If $m = 1$, then H is problematic if and only if H is conjugate to a subgroup of Γ_k.
(2) If $m > 1$, then H is problematic if and only if m is a power of a prime different from p and H is conjugate to a coisotropic subgroup of Γ_k.

Proof. Suppose that H is problematic. By Theorem 8.3, we may assume that H is subconjugate to a subgroup of Γ_k acting on $\mathbb{C}^m \otimes \mathbb{C}^{p^k}$ by the standard action on $\mathbb{C}^{p^k}$ and the trivial action on $\mathbb{C}^m$. Hence $H \cong \Gamma_s \times \Delta_t$ for $s+t \leq k$.

To prove the converse for $m = 1$, we must show that all subgroups of Γ_k are in fact problematic. If $H = \Gamma_k$, then $\left(\mathcal{L}_{p^k}\right)^{\Gamma_k}$ is a wedge of spheres (Proposition 9.1), and hence has nontrivial mod p homology. Since we have assumed that $S^1 \subseteq H \subseteq \Gamma_k$, the quotient Γ_k/H is a finite p-group, so we can apply Smith theory to

$$\left(\left(\mathcal{L}_{p^k}\right)^H\right)^{\Gamma_k/H} = \left(\mathcal{L}_{p^k}\right)^{\Gamma_k}.$$

We conclude that $\left(\mathcal{L}_{p^k}\right)^H$ likewise has nontrivial mod p homology, and therefore is not contractible. Therefore H is problematic.

Now suppose $m > 1$ and $H \cong \Gamma_s \times \Delta_t \subseteq \Gamma_k$ is problematic. Let $r = n/p^{s+t}$. We apply Proposition 10.1 to conclude that because $\left(\mathcal{L}_n\right)^H$ is not contractible, then the s-fold suspension of $\left(\mathcal{L}_{rp^t}\right)^{\Delta_t}$ is not contractible. Hence $\left(\mathcal{L}_{rp^t}\right)^{\Delta_t}$ is likewise not contractible. The contrapositive of Theorem 10.13 implies that r is a power of a prime, and since $m \mid r$ and m is coprime to p, this means that $m = r = q^i$ for $i > 1$ and q a prime different from p. In particular, since $n = rp^{s+t} = mp^k$, we have $s + t = k$, so H is coisotropic.

In summary, if $\left(\mathcal{L}_n\right)^H$ is noncontractible, then

- $\left(\mathcal{L}_{rp^t}\right)^{\Delta_t}$ is not contractible, and
- $r = q^i$ for $i > 0$ and q a prime different from p.

To finish, we need to know that $\left(\mathcal{L}_{q^i p^t}\right)^{\Delta_t}$ is not contractible, a result provided by Proposition 10.3.

□

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