Abstracts

$\mathbb{A}^1$-Milnor number

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(joint work with Jesse Leo Kass)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞-function with an isolated zero at the origin. Recall that the local degree $\deg_0 f$ of f at zero is defined as

$$\deg_0 f = \deg(\ \partial B(0,\epsilon) \xrightarrow{f/|f|} \partial B(0,1)\ ) \in \mathbb{Z},$$

where $\epsilon > 0$ is chosen sufficiently small. The *Signature Formula* of Eisenbud-Levine/Khimshiashvili [1] [3] gives a formula for $\deg_0 f$ as the signature of the following real symmetric bilinear form. Define $Q_0(f) = \mathbb{R}[[x_1, \ldots, x_n]]/\langle f_1, \ldots, f_n\rangle$ where f_i denotes the ith coordinate projection of f. Let $J = \det(\frac{\partial f_i}{\partial x_j})$. Choose a $\mathbb{R}$-linear function $\varphi : Q_0(f) \to \mathbb{R}$ such that $\varphi(J) > 0$. Define

$$\langle , \rangle_\varphi : Q_0(f) \times Q_0(f) \to \mathbb{R}$$
$$\langle , \rangle_\varphi(g, h) = \varphi(gh).$$

Theorem. *(Eisenbud-Levine/Khimshiashvili Signature Formula)*

$$\deg_0 f = \text{signature } \langle , \rangle_\varphi$$

The complex analogue of their theorem was proven earlier by Palamodov. When f is analytic, and hence has a complexification $f \otimes \mathbb{C}$, Palamodov proved

Theorem. *(Palamodov)*

$$\deg_0 f \otimes \mathbb{C} = \text{rank } \langle , \rangle_\varphi.$$

For an arbitrary field k and a polynomial function f, let

$$Q_0(f) = k[x_1, \ldots, x_n]_{\mathfrak{m}_0}/\langle f_1, \ldots, f_n\rangle,$$

where $\mathfrak{m}_0 = \langle x_1, \ldots, x_n\rangle$, and choose φ to be k-linear such that $\varphi(J) = \dim_k Q_0(f)$. In positive characteristic, assume that $\dim_k Q_0(f)$ is finite and if this dimension is divisible by the characteristic, J is replaced by a distinguished socle element E with $\varphi(E) = 1$. The isomorphism class of $\langle , \rangle_\varphi$ does not depend on the choice of φ.

Eisenbud wrote an AMS Bulletin article about this work [2], and the article ends with some questions. Question 3 [2, p. 763-764] is

> I would propose that the degree of a finite polynomial map $f : k^n \to k^n$, where k is an arbitrary field of characteristic 0 be *defined* to be the equivalence class of the quadratic form $\langle , \rangle_\varphi$ on the local ring of f at 0. . . . There is really no reason to stick to characteristic 0 for all this, . . . The question is, does this idea of degree have some other interpretation (or usefulness), for example in cohomology theory, as in the case of $\mathbb{R}$ or $\mathbb{C}$

We answer this question "yes:" $\langle , \rangle_\varphi$ is the local degree from Morel-Voevodsky's $\mathbb{A}^1$-homotopy theory [5], appearing before $\mathbb{A}^1$-homotopy theory itself.

Theorem 1. *(Kass, W.)*

$$\deg_0^{\mathbb{A}^1} f = \langle , \rangle_\varphi$$

About the left hand side: Morel's degree homomorphism in $\mathbb{A}^1$-homotopy theory over a field k takes an endomorphism of a sphere to an element of the Grothendieck Witt group $\mathrm{GW}(k)$ of k. This group is the group completion of the semi-ring of isomorphism classes of non-degenerate symmetric bilinear forms over k. The above equality is in $\mathrm{GW}(k)$. Morel's construction is compatible with the $\mathbb{Z}$-valued topological degree: when we have an embedding $k \hookrightarrow \mathbb{C}$, the topological degree of the $\mathbb{C}$-points of a map is the rank of the bilinear form $\deg^{\mathbb{A}^1}$; the topological degree of the $\mathbb{R}$-points of a map is the signature. (Note the compatibility with the Signature Formula, Palamodov's Theorem and Theorem 1.)

Theorem 1 is proven by reducing to the étale case, where both sides are computed to be equal. To do the reduction, both sides are shown to be unchanged when f is modified by an n-tuple of polynomials in a sufficiently high power of the maximal ideal. We modify f in this way to be able to extend it to an endomorphism G (satisfying certain conditions) of the sphere $\mathbb{P}^n/\mathbb{P}^{n-1}$ in $\mathbb{A}^1$-homotopy theory. We show that $\langle , \rangle_\varphi$ has certain properties of a local degree, namely that there is a global degree which is a sum of local degrees over points of $G^{-1}(x)$ with $x \in \mathbb{A}^n = \mathbb{P}^n - \mathbb{P}^{n-1}$, making this sum independent of x. We can now check the equality of global degrees using an x so that G is étale at every point of $G^{-1}(x)$. When there is no such rational x, we take an odd-degree field extension, which induces an injection on GW.

As an application, we enrich Milnor's equality between the local degree of the gradient of a complex hypersurface singularity and the number of nodes into which the singularity bifurcates [4]. Classically, this common integer is the Milnor number μ. We enrich this to an equality in $\mathrm{GW}(k)$. Specifically, let k be a field of characteristic not 2, and let $g \in k[x_1, \dots, x_n]$ define a hypersurface with an isolated singularity at 0.

A *node* is a hypersurface singularity isomorphic to $x_1^2 + \ldots + x_n^2$ over k^s where k^s denotes the separable closure of k. Over non-separably closed fields, nodes contain arithmetic information. For example, the isomorphism type of the node of $x_1^2 + ax_2^2 = 0$ depends on the value of a in $k^*/(k^*)^2$. We encode some of this information in a bilinear form. Let $\langle a \rangle$ denote the element of $\mathrm{GW}(k)$ represented by the rank 1 bilinear form $(x, y) \mapsto axy$ for x,y in k. Define the *arithmetic type* of $x_1^2 + ax_2^2 = 0$ to be $\langle a \rangle$ in $\mathrm{GW}(k)$. More generally, for $g = 0$ defining a node at a rational point p, define the arithmetic type to be $\langle H \rangle$, where H is the Hessian $H = \det(\frac{\partial f_i}{\partial x_j}(p))$ evaluated at p. Using descent data, one also defines the arithmetic type when x is not assumed to be rational. (When k is a finite field, we explain the definition later.)

For general $(a_1, ..., a_n) \in \mathbb{A}^n_k(k)$, the family

$$g(x_1, \dots, x_n) + a_1x_1 + \cdots + a_nx_n = t$$

over line with coordinate t contains only nodal fibers as singular fibers, and writing these nodes as $p_i \in X_i$, we have:

Theorem 2. *(Kass, W.) Suppose* $\operatorname{grad} g$ *is finite and has only the origin as an isolated zero. Then*

$$\mu^{\mathbb{A}^1}(g) = \sum \text{arithmetic type}(p_i) \in X_i \tag{1}$$

in $\mathrm{GW}(k)$*, where*

$$\mu^{\mathbb{A}^1} g = \deg_0^{\mathbb{A}^1} \operatorname{grad} g$$

and is called the $\mathbb{A}^1$-Milnor number*.*

Let us now analyze Theorem 2 in the special case where $k = \mathbf{F}_q$ is a finite field of characteristic $p \neq 2$. Describing nodal fibers over a finite field is especially tractable because the structure of a finite field is so simple. The stable isomorphism class of a nondegenerate symmetric bilinear form is determined by its rank and discriminant. Furthermore, the discriminant is an element of $k^*/(k^*)^2$, which is a 2 element group that we write as

$$\mathbf{F}_q^*/(\mathbf{F}_q^*)^2 = \{1, u_q\} \text{ for some } u_q \in \mathbf{F}_q^*.$$

In particular, there are two possibilities for the arithmetic type of a node at the origin:

$$\text{the arithmetic type } \langle 1 \rangle \text{ of } x^2 + y^2 \text{ and} \tag{2}$$

$$\text{the arithmetic type } \langle u_q \rangle \text{ of } x^2 + u_q y^2. \tag{3}$$

However, not every collection of nodes $\{x_i \in X_i\}$ satisfying Equation (1) can be realized as the singular fibers of a family. For the example, the equation $f(x, y) = y^3 + x^4$ of the E_6 singularity over $k = \mathbf{F}_5$ satisfies $\mu^{\mathbf{A}^1}(f) = 3 \cdot \mathbf{H}$. We have $3 \cdot \mathbf{H} = 6 \cdot \langle 1 \rangle$ in $\mathrm{GW}(\mathbf{F}_5)$, but there does not exist an (a, b) such that the associated family has 6 fibers with arithmetic type $\langle 1 \rangle$ because $\mathbf{A}^1_{\mathbf{F}_5}(\mathbf{F}_5)$ only has $5 < 6$ elements.

We describe the configurations of nodes occurring in families associated to the singularities in Table 1 for some small finite fields. Table 1 should be read as follows. The `equation` in the second column is the equation of an isolated plane curve singularity, and over the algebraic closure, that singularity is isomorphic to an ADE singularity, specifically the singularity with the `name` in the first column. The $\mathbb{A}^1$-`Milnor number` of the equation is given in the third column. The `discriminant`, considered as an element of $k^*/(k^*)^2$, is listed in the fourth column. The rank of $\mathbf{A}^1$-Milnor number is the integer appearing in the first column (so e.g. for the D_4 singularity, the rank is 4). In the table, $\mathbf{H} = \langle 1, -1 \rangle$ is the class of the standard hyperbolic space.

Consider the possible nodal fibers of the family $\mathbf{A}^2_k \to \mathbf{A}^1_k$ defined by $f(x, y) + ax + by = t$. Thus suppose that $x_0 \in X_{t_0}$ is a node of the fiber over the closed point $t_0 \in \mathbf{A}^1_k$. As was mentioned earlier, if $x_0 \in X_0$ has residue field equal to k, then the arithmetic type is the value of the Hessian of f at x_0.

Table 1. Some singularities over $k = \mathbf{F}_q$, $q = p^n$ for $p > 5$ with A_4 and otherwise $p > 3$

Name	Equation	$\mathbf{A}^1$-Milnor number	Discriminant
A_2	$y^2 + x^3$	$\mathbf{H}$	-1
A_3	$y^2 + vx^4$, $v \in k^*$	$\langle 2 \cdot v \rangle + \mathbf{H}$	$-2 \cdot v$
A_4	$y^2 + x^5$	$2 \cdot \mathbf{H}$	1
D_4	$x^2y + xy^2$	$\langle -2, 2 \cdot 3 \rangle + \mathbf{H}$	3
E_6	$x^4 + y^3$	$3 \cdot \mathbf{H}$	-1

In general, the definition of the arithmetic type is more subtle. Colloquially, $x_0 \in X_{t_0}$ corresponds to a Galois orbit of nodes (over, say a large field extension), and if the common arithmetic type of these nodes is α, then the arithmetic type of $x_0 \in X_{t_0}$ is the Scharlau trace $\mathrm{Tr}_{L/k}(\alpha)$.

More formally, suppose first that $k(t_0) = k$ but $k(x_0)/k$ is a nontrivial extension, say $k(x_0) = L$. Then $X_{t_0} \otimes_k L$ has finitely many nodes mapping to x_0, say $\widetilde{x}_1, \dots, \widetilde{x}_n \in X_{t_0} \otimes_k L$. Each of these nodes has residue field L, and a node's arithmetic type (over L) is computed as the class of a Hessian. Moreover, the $\widetilde{x}_i$'s are transitively permuted by the Galois group $\mathrm{Gal}(L/k)$, so any two nodes have the same type, say $\alpha \in \mathrm{GW}(L)$. We then have

$$\text{the arithmetic type of } x_0 \in X_{t_0} = \mathrm{Tr}_{L/k}(\alpha).$$

Here $\mathrm{Tr}_{L/k} \colon \mathrm{GW}(L) \to \mathrm{GW}(k)$ is the Scharlau trace.

The most general case is where $k(t_0)$ is a nontrivial extension, say L. In this case, t_0 corresponds to a $\mathrm{Gal}(L/k)$-orbit of fibers $\widetilde{X}_{\widetilde{t}_1}, \dots, \widetilde{X}_{\widetilde{t}_m}$ that are transitively permuted by the Galois group. Each of the points $\widetilde{t}_1, \dots, \widetilde{t}_m$ has residue field L, so the arithmetic type of a node of $\widetilde{X}_{\widetilde{t}_i}$ is defined as in the previous paragraph. Fixing one fiber, say $\widetilde{X}_{\widetilde{t}_1}$, and defining $\alpha \in \mathrm{GW}(L)$ to be the sum of the arithmetic types of the nodes of $\widetilde{X}_{\widetilde{t}_1}$ that map to x_0, we have

$$\text{the arithmetic type of } x_0 \in X_{t_0} = \mathrm{Tr}_{L/k}(\alpha).$$

For given $k = \mathbf{F}_q$, $f(x, y) \in k[x, y]$, $a, b \in k$, the arithmetic types of the nodal fibers of $f(x, y) + ax + by = t$ can be computed using Gröbner basis techniques. For example, consider the family $x^2y - xy^2 + 2x + y = t$ over $k = \mathbf{F}_{17}$. The singular fibers are the fibers over the points of the closed scheme defined by $k[t] \cap (f(x, y) + ax + by - t, \frac{\partial f}{\partial x} + a, \frac{\partial f}{\partial y} + b)$. A Gröbner basis computation shows that this ideal is generated by $d(t) = t^4 + 14$, an irreducible polynomial. In $L := k[r]/t^4 + 14$, a second Gröbner basis computation shows that X_{t_1} has a node at the point $(4r^3 + 5r, 9r^3)$. The value of the Hessian at this point is $4r^2 = 1$ in $L^*/(L^*)^2$. We conclude that the nodal fibers of the family consists of a Galois orbit of 4 fibers, each with a single node of type $\langle 1 \rangle \in \mathrm{GW}(L)$. Table 2 was generated by similar computations.

The table should be read as follows. The first column describes a `singularity` from Table 1. For a given singularity, the possible singular fibers of a family

$f(x, y) + ax + by = t$ with only `nodal fibers` are listed in the second column. The last column is the `count` of the (a, b)'s that define a family with singular fibers as described by the corresponding entry in the second column. (E.g. for the A_2 singularity over $k = \mathbf{F}_5$, there are 5 elements $(a, b) \in \mathbf{A}^2_k(k)$ s.t. $f(x, y)+ax+by = t$ has 2 nodal fibers, each with a node of type $\langle 1 \rangle$.)

References

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Computing other invariants of topological spaces of dimension three

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The computation of ...

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TABLE 2. Possible singular fibers of a family

Singularity	Nodal fibers	Count
A_2 with $k = \mathbf{F}_5$	1 orbit of 2 fibers of type $\langle u_{p^2} \rangle$	10
	2 fibers of type $\langle u_p \rangle$	5
	2 fibers of type $\langle 1 \rangle$	5
	Total	20
A_2 with $k = \mathbf{F}_7$	1 orbit of 2 fibers of type $\langle 1 \rangle$	21
	1 fiber of type $\langle 1 \rangle$, 1 fiber of type $\langle u_p \rangle$	21
	Total	42
A_3 with $k = \mathbf{F}_5$, $v = 1$	1 fiber of type $\langle 1 \rangle$, 1 orbit of 2 fibers of type $\langle 1 \rangle$	20
	Total	20
A_3 with $k = \mathbf{F}_5$, $v = 2$	1 node of type $\langle u_p \rangle$, 1 orbit of 2 fibers of type $\langle 1 \rangle$	20
	Total	20
A_3 with $k = \mathbf{F}_7$, $v = 1$	1 orbit of 3 fibers of type $\langle u_{p^3} \rangle$	28
	3 nodes of type $\langle u_p \rangle$	14
	Total	42
A_3 with $k = \mathbf{F}_7$, $v = -1$	3 fibers of type $\langle 1 \rangle$	14
	1 orbit of 3 fibers of type $\langle 1 \rangle$	28
	Total	42
A_4 with $k = \mathbf{F}_7$	1 type $\langle 1 \rangle$ fiber, 1 type $\langle u_p \rangle$ fiber, 1 orbit of 2 type $\langle 1 \rangle$ fibers	21
	2 orbits of 2 fibers of type $\langle 1 \rangle$	21
	Total	42
D_4 with $k = \mathbf{F}_5$	1 orbit of 4 fibers of type $\langle 1 \rangle$	12
	1 orbit of 2 fibers of type $\langle 1 \rangle$, 2 fibers of type $\langle 1 \rangle$	2
	2 fibers of type $\langle u_p \rangle$, 1 fiber of type $\mathrm{Tr}_{\mathbf{F}_{p^2}/\mathbf{F}_p}(\langle 1 \rangle)$	6
	1 orbit of 2 fibers of type $\langle 1 \rangle$, 1 fiber of type $\langle 1, 1 \rangle$	4
	Total	24
E_6 with $k = \mathbf{F}_5$	1 fiber of type $\mathrm{Tr}_{\mathbf{F}_{p^2}/\mathbf{F}_p}(\langle u_p \rangle)$, 2 orbits of 2 fibers of type $\langle u_{p^2} \rangle$	4
	2 fibers of type $\langle 1 \rangle$, 2 orbits of 2 fibers of type $\langle 1 \rangle$	4
	2 fibers of type $\langle u_p \rangle$, 2 orbits of 2 fibers of type $\langle 1 \rangle$	4
	3 orbits of 2 fibers of type $\langle u_{p^2} \rangle$	4
	Total	16