AN EXPLICIT SELF-DUALITY

NIKOLAS KUHN, DEVLIN MALLORY, VAIDEHEE THATTE, AND KIRSTEN WICKELGREN

Abstract. We provide an exposition of the canonical self-duality associated to a presentation of a finite, flat, complete intersection over a Noetherian ring, following work of Scheja and Storch.

1. Introduction

Consider a finite, flat ring map $f : \mathrm{A} \to \mathrm{B}$ and assume that A is Noetherian. Coherent duality for proper morphisms provides a functor $f^! : \mathrm{D}(\operatorname{Spec} \mathrm{A}) \to \mathrm{D}(\operatorname{Spec} \mathrm{B})$ on derived categories. The assumptions on f imply that $f^!\mathrm{A}$ is isomorphic to the sheaf on B associated to $\mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{A})$. See for example [Sta18, 0AA2]. If we assume moreover that $f : \operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}$ is a local complete intersection morphism, then $f^!\mathrm{A}$ is locally free [Sta18, 0B6V, 0FNT]. Thus there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{A}) \cong \mathrm{B} \tag{1.0.1}$$

of B-modules under additional hypotheses, for example if we assume that B is local.[1]

There are many choices for the isomorphism (1.0.1). (The set of these isomorphisms form a B^*-torsor.) An explicit presentation of B as

$$\mathrm{B} = \mathrm{A}[x_1, \dots, x_n]/(f_1, \dots, f_n) \tag{1.0.2}$$

singles out a particular choice, which satisfies certain nice properties such as compatibility with base change and the trace. In addition to the advantages of having a canonical choice (e.g. gluing such isomorphisms together), this choice is closely related to the degree map in $\mathbb{A}^1$-homotopy theory due to F. Morel. See Remark 1.1.

In this expository paper, we follow the approach of [SS75] to construct this canonical isomorphism for B a finite, flat A-algebra equipped with a presentation (1.0.2).

The approach is as follows: Consider the ideals

$$(f_1 \otimes 1 - 1 \otimes f_1, \dots, f_n \otimes 1 - 1 \otimes f_n) \subset (x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$$

of $\mathrm{A}[x_1, \dots, x_n] \otimes \mathrm{A}[x_1, \dots, x_n]$. One writes

$$f_j \otimes 1 - 1 \otimes f_j = \sum a_{ij}(x_i \otimes 1 - 1 \otimes x_i).$$

and defines the element $\Delta \in \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}$ as the image of $\det(a_{ij})$ under the morphism $\mathrm{A}[x_1, \dots, x_n] \otimes \mathrm{A}[x_1, \dots, x_n] \to \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}$. This is shown to be independent of the choice of a_{ij}. There is a canonical A-module morphism

$$\chi : \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{A}), \mathrm{B}).$$

Let I denote the kernel of multiplication $\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{B}$, or in other words the image of $(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$. One checks that χ restricts to an isomorphism

$$\chi : \mathrm{Ann}_{\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}}\, \mathrm{I} \to \mathrm{Hom}_{\mathrm{B}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{A}), \mathrm{B})$$

[1] An alternate point of view on the equivalence $f^!\mathrm{A} \simeq \mathrm{B}$ is that a factorization $\mathrm{A} \xrightarrow{p} \mathrm{A}[x_1, \dots, x_n] \xrightarrow{i} \mathrm{B}$ of f into a regular immersion and structure map for $\mathbb{A}^n_{\mathrm{A}}$ allows one to compute $f^!\mathrm{A}$ as $i^!p^!\mathrm{A} \simeq i^!(\mathrm{A}[x_1, \dots, x_n][n]) \simeq \det \mathrm{N}_i^*[-n][n] \simeq \mathrm{B}$, where N_i^* denotes the conormal bundle of the regular immersion $\operatorname{Spec} \mathrm{B} \hookrightarrow \mathbb{A}^n_{\mathrm{A}}$. See for example [Har66, Ideal Theorem p. 6, III, particularly Corollary 7.3].

of B-modules and identifies the annihilator as $\mathrm{Ann}_{\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}}\,\mathrm{I} \cong \Delta$. Finally, one shows that

$$\chi(\Delta) =: \Theta \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{A}), \mathrm{B})$$

provides the desired isomorphism of B-modules $\Theta : \mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{A}) \to \mathrm{B}$ guaranteed by the general theory of coherent duality. This is Theorem 3.4 (or [SS75, Satz 3.3]) and the main result. For the compatibility of Θ with base change and the trace see [SS75, p. 183-184 and Section 4] respectively.

Our arguments largely follow the outline of [SS75], although we make more use of Koszul homology in some proofs than the original did, and provide a self-contained proof of Lemma 2.4; the goal in large part is to provide an English reference for this material. See also [Kun05, Appendices H and I].

Remark 1.1. One motivation for providing an explicit description of this isomorphism is to describe the resulting A-valued bilinear form on B. This form is defined via

$$\langle b, c\rangle \mapsto \Theta^{-1}(b)(c) = \eta(bc) \in \mathrm{A},$$

where $\eta = \Theta^{-1}(1)$. The form $\langle -, -\rangle$ has been used to give a notion of degree [EL77] [Eis78, some remaining questions (3)]. For example, it computes the local $\mathbb{A}^1$-Brouwer degree of Morel [KW19] [BBM+21], and is useful in quadratic enrichments of results in enumerative geometry [Lev20] [KW21] [McK21] [Pau20].

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2. Commutative Algebra Preliminaries

Lemma 2.1. [SS75, 1.2] *Let* A *be a noetherian ring and suppose that* $f_1, \dots, f_n$ *and* $g_1, \dots, g_n$ *are sequences satisfying the following hypotheses:*

(i) $\mathfrak{b} = (g_1, \dots, g_n) \subset \mathfrak{a} = (f_1, \dots f_n)$
(ii) If $\mathfrak{p}$ *is a prime such that* $\mathfrak{a} \subset \mathfrak{p}$*, then the sequence* $f_1, \dots, f_n$ *is a regular sequence in* $\mathrm{A}_{\mathfrak{p}}$*, as is* $g_1, \dots, g_n$*.*

Write $g_i = \sum_{i=1}^{n} a_{ij} f_j$*, and let* (a_{ij}) *be the resulting matrix of coefficients.*

$$\Delta := \det(a_{ij}).$$

Define $\overline{\Delta}$ *to be the image of* Δ *under the map* $\mathrm{A} \to \mathrm{A}/\mathfrak{b}$*. Then:*

(a) The element $\overline{\Delta}$ *is independent of the choices of* a_{ij}*.*
(b) We have an equality (of $\mathrm{A}/\mathfrak{b}$*-ideals):*

$$(\overline{\Delta}) = \mathrm{Fit}_{\mathrm{A}/\mathfrak{b}}(\mathfrak{a}/\mathfrak{b}),$$

where Fit *denotes the 0-th Fitting ideal.*

(c) We have an equality of ideals:

$$(\overline{\Delta}) = \mathrm{Ann}_{\mathrm{A}/\mathfrak{b}}(\mathfrak{a}/\mathfrak{b}),$$

and

$$\mathfrak{a}/\mathfrak{b} = \mathrm{Ann}_{\mathrm{A}/\mathfrak{b}}(\overline{\Delta}).$$

Remark 2.2. We comment on condition (ii). If $(\mathrm{A}, \mathfrak{p})$ is a local ring and $\mathfrak{a} \subset \mathfrak{p}$, then condition (ii) is equivalent to asking that $f_1, \dots, f_n$ and $g_1, \dots, g_n$ are regular sequences. In general, condition (ii) asks only that they are regular sequences after localizing at primes containing $\mathfrak{a}$ (e.g., they may not be regular sequences on A).

Proof. First, we may assume that A is a local ring and each of the f_i's and g_i's are in the maximal ideal $\mathfrak{m}$.

(a): Write $g_i = \sum_{i=1}^{n} b_{ij} f_j$. We want to show that $\det(a_{ij}) - \det(b_{ij})$ is in $\mathfrak{b}$. It suffices to consider the case where $a_{ij} = b_{ij}$ for all j and for $i = 1, \ldots, n-1$, as this allows us to change the presentation of one g_i at a time, and thus all of them. Define

$$c_{ij} = \begin{cases} a_{ij} = b_{ij} & i = 1, \ldots, n-1 \\ a_{ij} - b_{ij} & i = n, \end{cases}$$

By cofactor expansion along the j-th row, we have that

$$\det(a_{ij}) - \det(b_{ij}) = \det(c_{ij}).$$

But now

$$(c_{ij}) \cdot \begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \cdots \\ g_{n-1} \\ 0 \end{pmatrix}$$

By Cramer's rule, for all $k = 1, \ldots, n$ we have that

$$\det(c_{ij}) \cdot f_k \in (g_1, \ldots, g_{n-1}),$$

which means

$$\det(c_{ij}) \cdot \mathfrak{a} \in (g_1, \ldots, g_{n-1}).$$

But $g_n \in \mathfrak{a}$ and hence

$$\det(c_{ij}) \cdot g_n \in (g_1, \ldots, g_{n-1}),$$

which means that $\det(c_{ij}) \in (g_1, \ldots, g_n) = \mathfrak{b}$ since $g_1, \ldots, g_n$ is a regular sequence.

(b): First observe that

$$\mathrm{Fit}_{\mathrm{A}}(\mathfrak{a}/\mathfrak{b}) \mod \mathfrak{b} = \mathrm{Fit}_{\mathrm{A}/\mathfrak{b}}(\mathfrak{a}/\mathfrak{b}).$$

Therefore, to prove the claim, it suffices to prove that

$$\mathrm{Fit}_{\mathrm{A}}(\mathfrak{a}/\mathfrak{b}) = \Delta + \mathrm{I},$$

where $\mathrm{I} \subset \mathfrak{b}$.

To prove this claim, note that the Fitting ideal of the A-module $\mathfrak{a}/\mathfrak{b}$ is computed by a presentation:

$$\mathrm{A}^{\oplus n} \oplus \mathrm{A}^{\oplus \binom{n}{2}} \xrightarrow{\mathrm{T}} \mathrm{A}^{\oplus n} \to \mathfrak{a}/\mathfrak{b} \to 0,$$

where T is given by:

$$(a_{ij}) \times d_2^{\mathrm{Kosz}}.$$

In other words, the matrix of T has the first n-columns are just given by a_{ij} and, the last $\binom{n}{2}$ columns are composed of the usual Koszul relations among the f_i. (Note that the sequence $f_1, \ldots, f_n$ is regular in our local ring, so the corresponding Koszul complex produces a resolution of $\mathfrak{a}$ [Sta18, 062F].)

Now, the Fitting ideal is given by the $n \times n$-minors of the matrix of T. The first minor is Δ. If Δ' is another $n \times n$ minor, then it is the determinant of a matrix T', which is composed of some r columns of (a_{ij}) and $n - r$ columns of d_1^{Kosz}; without loss of generality we may assume T' contains the first r columns of (a_{ij}) (if not, simply reorder the g_i, using that the ring A is local and thus regularity of the sequence of g_i preserved). Applying T' to (f_k) we get

$$(\mathrm{T}') \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We again conclude that $\Delta' f_i = \det(\mathrm{T}')f_i \in \mathfrak{b}$ for each $i = 1, \ldots, n$. Thus,

$$\Delta' \cdot \mathfrak{a} \in (g_1, \ldots, g_{n-1}),$$

and in particular

$$\Delta' \cdot g_n \in (g_1, \ldots, g_{n-1}),$$

which by regularity of the g_i means that $\Delta' \in \mathfrak{b}$ and thus $\mathrm{Fit}_{\mathrm{A}}(\mathfrak{a}/\mathfrak{b}) = \Delta + \mathrm{I}$ with $\mathrm{I} \subset \mathfrak{b}$.

(c): First, we claim that we have an isomorphism:

$$\mathrm{Ann}_{\mathrm{A}/\mathfrak{b}}(\mathfrak{a}/\mathfrak{b}) \cong \mathrm{Tor}_n^{\mathrm{A}}(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{a}).$$

We will abbreviate $\mathrm{Tor}_j^{\mathrm{A}}$ by Tor_j and $\otimes_{\mathrm{A}}$ by $\otimes$ in what follows. To prove this, we deploy the Koszul complex. (As noted above, a regular sequence is Koszul-regular by [Sta18, 062F].) We thus have a quasi-isomorphism.

$$\mathrm{K}_\bullet(f_1, \ldots, f_n) \simeq \mathrm{A}/\mathfrak{a}$$

Therefore the Tor group above is computed as the kernel of $1 \otimes d_n^{\mathrm{Kosz}}$ in the complex $\mathrm{A}/\mathfrak{b} \otimes \mathrm{K}_\bullet(f_1, \ldots, f_n)$:

$$0 \to \mathrm{A}/\mathfrak{b} \xrightarrow{(f_1,\ldots,f_n)} (\mathrm{A}/\mathfrak{b})^{\oplus n}.$$

Indeed, the cohomology of this small complex is the desired annihilator and thus we obtain the desired isomorphism.

On the other hand, we claim that $\mathrm{Tor}_n(\mathrm{A}/\mathfrak{a}, \mathrm{A}/\mathfrak{b}) \cong \Delta \cdot \mathrm{A}/\mathfrak{b}$. To see this note that we have a short exact sequence of A-modules:

$$0 \to \mathfrak{a}/\mathfrak{b} \to \mathrm{A}/\mathfrak{b} \to \mathrm{A}/\mathfrak{a} \to 0.$$

We claim that the induced long exact sequence splits into short exact sequences for $j \geqslant 1$

$$0 \to \mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathfrak{a}/\mathfrak{b}) \to \mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{b}) \to \mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{a}) \to 0$$

Indeed, via the Koszul complex for $\mathrm{A}/\mathfrak{b}$, we see that for $j \geqslant 1$:

$$\mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathfrak{a}/\mathfrak{b}) \cong (\mathfrak{a}/\mathfrak{b})^{\binom{n}{j}} \qquad \mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{b}) \cong (\mathrm{A}/\mathfrak{b})^{\binom{n}{j}}, \tag{2.0.1}$$

and the map $\mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathfrak{a}/\mathfrak{b}) \to \mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{b})$ is identified with the direct sum of copies of the injection $\mathfrak{a}/\mathfrak{b} \hookrightarrow \mathrm{A}/\mathfrak{b}$. To conclude, the functoriality of the Koszul complex [Sta18, 0624] yields a morphism of complexes

$$\mathrm{A}/\mathfrak{b} \otimes \mathrm{K}_\bullet(g_1, \ldots, g_n) \to \mathrm{A}/\mathfrak{b} \otimes \mathrm{K}_\bullet(f_1, \ldots, f_n);$$

where the left end is as follows:

$$\begin{array}{ccc} \mathrm{A}/\mathfrak{b} & \xrightarrow{0} & (\mathrm{A}/\mathfrak{b})^{\oplus n} \\ {\scriptstyle\overline{\Delta}}\downarrow & & \downarrow \\ \mathrm{A}/\mathfrak{b} & \xrightarrow{(f_1,\ldots,f_n)} & (\mathrm{A}/\mathfrak{b})^{\oplus n}. \end{array} \tag{2.0.2}$$

Since the map $\mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{b}) \to \mathrm{Tor}_j(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{a})$ is a surjection, we conclude that

$$\mathrm{Tor}_n(\mathrm{A}/\mathfrak{b}, \mathrm{A}/\mathfrak{a}) \cong \mathrm{Im}(\overline{\Delta}) \cong \Delta \cdot \mathrm{A}/\mathfrak{b}$$

as desired.

For the second claim, note that the ideal $\mathrm{Ann}_{\mathrm{A}/\mathfrak{b}}(\overline{\Delta})$ is obtained as the kernel of the left vertical map in (2.0.2), and is thus isomorphic to $\mathrm{Tor}_n(\mathrm{A}/\mathfrak{b}, \mathfrak{a}/\mathfrak{b})$, which we already know is isomorphic to $\mathfrak{a}/\mathfrak{b}$ by (2.0.1).

□

A module M over a ring R is said to be reflexive if the natural map $\mathrm{R} \to \mathrm{Hom}_{\mathrm{R}}(\mathrm{Hom}_{\mathrm{R}}(\mathrm{M}, \mathrm{R}), \mathrm{R})$ is an isomorphism [Sta18, 0AUY]. A form of the following lemma is in the stacks project ([Sta18, 0AVA]), but assumes that A is integral and that A = B. The following is [SS75, 1.3].

Lemma 2.3. *Let* A *be a Noetherian ring and* B *a finite flat* A*-algebra. A finite* B*-module* M *is reflexive if and only if the following conditions hold:*

(i) If $\mathfrak{p} \subset \mathrm{A}$ *is a prime ideal with* $\operatorname{depth} \mathrm{A}_\mathfrak{p} \leqslant 1$*, then* $\mathrm{M}_\mathfrak{p}$ *is a reflexive* $\mathrm{B}_\mathfrak{p}$*-module.*
(ii) If $\mathfrak{p} \subset \mathrm{A}$ *is a prime ideal with* $\operatorname{depth} \mathrm{A}_\mathfrak{p} \geqslant 2$*, then* $\operatorname{depth}_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p}) \geqslant 2$*.*

Proof. The property of being reflexive is preserved under any localization of B [Sta18, 0EB9], and can be checked locally on B [Sta18, 0AV1]. Therefore reflexivity of M implies (i). Reflexivity implies (ii): Any regular sequence in $\mathrm{A}_\mathfrak{p}$ is a regular sequence on $\mathrm{B}_\mathfrak{p}$ by flatness. Let a_1, a_2 be a length 2 regular sequence on $\mathrm{A}_\mathfrak{p}$. Let N be any $\mathrm{B}_\mathfrak{p}$-module. Then a_1 is a nonzerodivisor on $\operatorname{Hom}_{\mathrm{B}_\mathfrak{p}}(\mathrm{N}, \mathrm{B}_\mathfrak{p})$. The cokernel of multiplication by a_1 is a submodule of $\operatorname{Hom}_{\mathrm{B}_\mathfrak{p}}(\mathrm{N}, \mathrm{B}_\mathfrak{p}/a_1\mathrm{B}_\mathfrak{p})$, on which a_2 is a nonzerodivisor. This shows the claim. (Note: This is almost [Sta18, 0AV5], except that we take $\operatorname{Hom}_\mathrm{B}$ but want the A-depth.)

Conversely, suppose M is not reflexive. We assume for the sake of contradiction that properties (i) and (ii) hold. Since reflexivity can be checked locally, there is some minimal $\mathfrak{p} \subset \mathrm{A}$ among all prime ideals of A for which $\mathrm{M}_\mathfrak{p}$ is a not a reflexive $\mathrm{B}_\mathfrak{p}$-module. Without loss of generality, we may assume that A is local with maximal ideal $\mathfrak{p}$. Since $\mathrm{M}_\mathfrak{p}$ is not reflexive, we must have that $\operatorname{depth} \mathrm{A}_\mathfrak{p} \geqslant 2$ and therefore $\operatorname{depth}_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p}) \geqslant 2$. We consider the exact sequence

$$0 \to \operatorname{Ker} \varphi \to \mathrm{M} \to \operatorname{Hom}_\mathrm{B}(\operatorname{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{B}), \mathrm{B}) \to \operatorname{Coker} \varphi \to 0,$$

where φ is the canonical map to the double-dual. By assumption, φ becomes an isomorphism after localizing at any prime of A different from $\mathfrak{p}$. It follows that $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ have finite length. Since $\operatorname{depth}_\mathrm{A} \mathrm{M} \geqslant 1$, there exists some $x \in \mathrm{A}$ which is a nonzerodivisor on M. But then x is a nonzerodivisor on the finite-length module $\operatorname{Ker} \varphi$, which therefore must vanish. Since $\operatorname{Hom}_\mathrm{B}(\operatorname{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{B}), \mathrm{B})$ is reflexive (as a B-module), it has A-depth $\geqslant 2$ by the forward implication of the lemma. The exact sequence

$$0 \to \mathrm{M} \to \operatorname{Hom}_\mathrm{B}(\operatorname{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{B}), \mathrm{B}) \to \operatorname{Coker} \varphi \to 0,$$

then shows that $\operatorname{depth}_{\mathrm{A}_\mathfrak{p}} \operatorname{Coker} \varphi \geqslant 1$ by the standard behavior of depth in short exact sequences [Sta18, 00LX]. Therefore the cokernel must vanish, which shows that M is reflexive. ☐

Lemma 2.4. [SS75, 1.4] *Let* A *be a Noetherian ring and let* B *be a finite flat* A*-algebra. Let* M *be a finite* B*-module, which is projective as an* A*-module. If* $\operatorname{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{B})$ *is projective as a* B*-module, then* M *is projective as a* B*-module. In particular, if* $\operatorname{Hom}_\mathrm{B}(\mathrm{M}, \mathrm{B})$ *is free, then* M *is free.*

Proof. It is enough to show that M is reflexive. We are therefore reduced to checking the conditions (i) and (ii) of Lemma 2.3. Clearly, (ii) holds, since M is projective over A. It remains to check (i). We may therefore assume that A is a Noetherian local ring with $\operatorname{depth} \mathrm{A} \leqslant 1$, and we want to show that M is projective as a B-module. Since B is finite flat over A, we have $\operatorname{depth} \mathrm{B}_\mathfrak{m} = \operatorname{depth} \mathrm{A}$ for every maximal ideal $\mathfrak{m}$ of B [Sta18, 0337].

Throughout, we will write $\mathrm{N}^* := \operatorname{Hom}_\mathrm{B}(\mathrm{N}, \mathrm{B})$ for a B-module N. Consider the map

$$\varphi : \mathrm{M} \to \mathrm{M}^{**}.$$

Let $\mathrm{C} := \operatorname{Coker} \varphi$. Taking a presentation of M, we obtain an exact sequence

$$0 \to \mathrm{U} \to \mathrm{F} \to \mathrm{M} \to 0$$

with F free. Consider the dual sequence

$$0 \to \mathrm{M}^* \to \mathrm{F}^* \to \mathrm{U}^*,$$

and let $\mathrm{Q} := \operatorname{Im}(\mathrm{F}^* \to \mathrm{U}^*)$. Since M^* is projective by assumption, Q has projective dimension 0 or 1 as a B-module.

We have the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{F} & \longrightarrow & \mathrm{M} & & & & \\
\big\downarrow{\scriptstyle\sim} & & \big\downarrow & & & & \\
\mathrm{F}^{**} & \longrightarrow & \mathrm{M}^{**} & \longrightarrow & \operatorname{Ext}^1_\mathrm{B}(\mathrm{Q}, \mathrm{B}) & \longrightarrow & 0
\end{array}$$

with exact lower row. Since $\mathrm{F} \to \mathrm{M}$ is a surjection, we see that $\mathrm{C} = \mathrm{Ext}^1_{\mathrm{B}}(\mathrm{Q}, \mathrm{B})$. Suppose $\operatorname{depth} \mathrm{A} = 0$. Apply the Auslander–Buchsbaum formula [Sta18, 090V] to the $\mathrm{B}_{\mathfrak{m}}$-module $\mathrm{Q}_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$. We find that $\mathrm{Q}_{\mathfrak{m}}$ has projective dimension zero, i.e., is projective. Therefore $\mathrm{C}_{\mathfrak{m}} = 0$ and $\mathrm{C} = 0$.

Now suppose that $\operatorname{depth} \mathrm{A} = 1$. Then $\operatorname{depth}_{\mathrm{B}_{\mathfrak{m}}} \mathrm{U}^*_{\mathfrak{m}} \geqslant 1$ by [Sta18, 0AV5], whence

$$\operatorname{depth}_{\mathrm{B}_{\mathfrak{m}}} \mathrm{Q}_{\mathfrak{m}} \geqslant 1$$

by [Sta18, 00LX]. Again by Auslander–Buchsbaum, we find that $\mathrm{Q}_{\mathfrak{m}}$ is projective, and that $\mathrm{C} = 0$.

We have shown that in any case $\mathrm{M} \to \mathrm{M}^{**}$ is surjective. Since M^{**} is projective, this implies $\mathrm{M} \simeq \mathrm{M}^{**} \oplus \mathrm{N}$ for some B-module N. It follows that $\mathrm{N}^* = 0$ and that N is again free as an A-module.

By assumption both M and M^{**} are free over the local ring A. A surjection of finite free A-modules is an isomorphism if they have the same rank. To show two finite free modules have the same rank, we may localize at a minimal prime ideal $\mathfrak{q}$ of A, so that also $\mathrm{B}_{\mathfrak{q}}$ is a zero-dimensional ring. Over the Artinian ring $\mathrm{B}_{\mathfrak{q}}$, $\mathrm{Hom}_{\mathrm{B}_{\mathfrak{q}}}(\mathrm{N}_{\mathfrak{q}}, \mathrm{B}_{\mathfrak{q}}) = 0$ implies $\mathrm{N}_{\mathfrak{q}} = 0$. (To see this, note that we may assume that B is local, with maximal ideal $\mathfrak{m}$. Then $\mathrm{N}_{\mathfrak{q}} \to \mathfrak{m}\mathrm{N}_{\mathfrak{q}}$ is nonzero by Nakayama's lemma. Since $\mathrm{B}_{\mathfrak{q}}$ has finite length, there is a nonzero element annihilated by $\mathfrak{m}$ whence a B-homomorphism $\mathrm{B}/\mathfrak{m} \to \mathrm{B}_{\mathfrak{q}}$.) Thus $\mathrm{M}_{\mathfrak{q}}$ and $\mathrm{M}^{**}_{\mathfrak{q}}$ have the same rank, and therefore $\mathrm{M} \to \mathrm{M}^{**}$ is an isomorphism. □

3. The explicit isomorphism

Recall that a ring map $\mathrm{A} \to \mathrm{B}$ is a *relative global complete intersection* if there exists a presentation $\mathrm{A}[x_1, \dots, x_n]/(f_1, \dots, f_c) \cong \mathrm{B}$, and every nonempty fiber of $\operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}$ has dimension $n - c$ [Sta18, 00SP]. Note that in this case the f_i form a regular sequence [Sta18, 00SV].

We note that a global complete intersection is flat [Sta18, 00SW], and thus syntomic. We will be interested in the situation where $\mathrm{A} \to \mathrm{B}$ is furthermore assumed to be a *finite* flat global complete intersection.

Construction 3.1. Suppose that $\mathrm{A} \to \mathrm{B}$ is a finite flat global complete intersection. Choose a presentation

$$\mathrm{A}[x_1, \dots, x_n] \xrightarrow{\pi} \mathrm{B} \cong \mathrm{A}[x_1, \dots, x_n]/(f_1, \dots, f_n).$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{A}[x_1, \dots, x_n] \otimes_{\mathrm{A}} \mathrm{A}[x_1, \dots, x_n] & \xrightarrow{m_1} & \mathrm{A}[x_1, \dots, x_n] \\ \Big\downarrow{\scriptstyle \pi\otimes\pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} & \xrightarrow{m} & \mathrm{B}, \end{array} \tag{3.0.1}$$

with m_1, m the obvious multiplication maps. We note that the elements

$$\{f_j \otimes 1 - 1 \otimes f_j\}_{j=1,\dots,n}$$

are all in $\ker(m_1)$, which is generated by the $x_i \otimes 1 - 1 \otimes x_i$ for $i = 1, \dots, n$, whence we have a relation

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^{n} a_{ij}(x_i \otimes 1 - 1 \otimes x_i).$$

Define $\Delta := (\pi \otimes \pi)(\det(a_{ij})) \in \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}$. Define also $\mathrm{I} := \ker m$.

Proposition 3.2. *The following properties of* Δ *hold:*

(a) *The element* Δ *is independent of the choice of* a_{ij}.
(b) *We have an equality of* $\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}$*-ideals:*

$$(\Delta) = \mathrm{Fit}_{\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}}\, \mathrm{I},$$

(c) we have an equality of ideals

$$(\Delta) = \mathrm{Ann}_{\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}}\,\mathrm{I} \qquad \mathrm{Ann}_{\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}}(\Delta) = \mathrm{I}.$$

Proof. Consider the ring map

$$\pi\otimes 1 : \mathrm{A}[x_1,\dots,x_n]\otimes_{\mathrm{A}}\mathrm{A}[x_1,\dots,x_n] \to \mathrm{B}\otimes_{\mathrm{A}}\mathrm{A}[x_1,\dots,x_n] \cong \mathrm{B}[x_1,\dots,x_n].$$

Since

$$f_i\otimes 1 - 1\otimes f_i = \sum_{i=1}^{n} a_{ij}(x_i\otimes 1 - 1\otimes x_i)$$

in $\mathrm{A}[x_1,\dots,x_n]\otimes_{\mathrm{A}}\mathrm{A}[x_1,\dots,x_n]$, we have that

$$-1\otimes f_i = \sum_{i=1}^{n} a_{ij}(\pi(x_i)\otimes 1 - 1\otimes x_i)$$

in $\mathrm{B}\otimes_{\mathrm{A}}\mathrm{A}[x_1,\dots,x_n]$.

Note that Δ is the image of $\det(a_{ij})$ under the obvious morphism $\mathrm{B}\otimes_{\mathrm{A}}\mathrm{A}[x_1,\dots,x_n] \to \mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}$, and that if $\mathfrak{a}$ is the ideal generated by the $\pi(x_i)\otimes 1 - 1\otimes x_i$ and $\mathfrak{b}$ the ideal generated by the $(-1\otimes f_i)$, then I is $\mathfrak{a}/\mathfrak{b}$. The desired properties will then follow immediately from applying Lemma 2.1 to $\mathfrak{b} = (-1\otimes f_i) \subset (\pi(x_i)\otimes 1 - 1\otimes x_i) = \mathfrak{a}$, once we show that the conditions of the Lemma are satisfied. It suffices to show that each is a regular sequence.

We claim that $\{-1\otimes f_j\} \subset \mathrm{B}\otimes_{\mathrm{A}}\mathrm{A}[x_1,\dots,x_n]$ is a regular sequence. Indeed, since relative global complete intersections are flat [Sta18, 00SW] and regular sequences are preserved under flat morphisms, this follows by regularity of the f_i in $\mathrm{A}[x_1,\dots,x_n]$ and flatness of $\mathrm{A}\to\mathrm{B}$. It is immediate also that $(\pi(x_i) - x_i)$ forms a regular sequence in $\mathrm{B}[x_1,\dots,x_n]$ as well (the $\pi(x_i)$ are just elements b_i of B, and $(x_i - b_i)$ is always a regular sequence in $\mathrm{B}[x_1,\dots,x_n]$).

Thus, the proposition follows by Lemma 2.1. □

Now, retain our setup from Construction 3.1. There is a canonical map of A-modules

$$\chi : \mathrm{B}\otimes_{\mathrm{A}}\mathrm{B} \to \mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A}),\mathrm{B}) \qquad \chi(b\otimes c) = (\varphi\mapsto\varphi(b)c).$$

Both $\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}$ and $\mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A}),\mathrm{B})$ each carry two natural B-module structures:

(1) B acts on $\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}$ as multiplication on either the left or right factor (i.e., either $a(b\otimes c) = ab\otimes c$ or $a(b\otimes c) = b\otimes ac$).
(2) B acts on $\mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A}),\mathrm{B})$ as either pre- or post-composing a homomorphism by multiplication (i.e., either $a\varphi : \psi\mapsto\varphi(a\psi)$ or $a\varphi : \psi\mapsto a\varphi(\psi)$).

Lemma 3.3. *χ induces a B-module isomorphism* $\mathrm{Ann}_{\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}}\,\mathrm{I} \cong \mathrm{Hom}_{\mathrm{B}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A}),\mathrm{B})$.

Proof. We note first that this map is an isomorphism of A-modules, for which it suffices to check that it's bijective: Since B is a projective A-module we have that B is canonically isomorphic to $\mathrm{B}^{\vee\vee}$ (where we denote by $^\vee$ the A-module dual), so that we have isomorphisms of A-modules

$$\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B} \cong (\mathrm{B}^\vee)^\vee\otimes_{\mathrm{A}}\mathrm{B} \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{B}^\vee,\mathrm{B}) = \mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A}),\mathrm{B});$$

one can check that χ is simply the composition of these canonical isomorphisms.

It's immediately checked that the morphism χ is in fact a B-bimodule homomorphism for the B-module structures of $\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}$ and $\mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A}),\mathrm{B})$ given by right multiplication and post-composition.

Now, we note the following:

(1) The largest submodule of $\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}$ where the two B-module structures agree is $\mathrm{Ann}_{\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}}\,\mathrm{I}$: this follows since an element $r\in\mathrm{B}\otimes_{\mathrm{A}}\mathrm{B}$ is annihilated by all $a\otimes 1 - 1\otimes a$ exactly when $(a\otimes 1)r = (1\otimes a)r$ for all a, which occurs exactly when the action of every a on r is the same under the two B-module structures.

(2) The largest submodule of $\mathrm{Hom_A}(\mathrm{Hom_A}(B, A), B)$ where the two B-module structures agree is
$$\mathrm{Hom_B}(\mathrm{Hom_A}(B, A), B) \subset \mathrm{Hom_A}(\mathrm{Hom_A}(B, A), B);$$
this is clear since the condition of pre- and post-multiplying by elements of B being the same is exactly B-linearity.

Putting this together, we have that χ induces an isomorphism *of* B*-modules*
$$\chi : \mathrm{Ann}_{B\otimes_A B}\, I \to \mathrm{Hom_B}(\mathrm{Hom_A}(B, A), B),$$
which was our desired claim. □

Theorem 3.4. *The map* $\chi(\Delta) : \mathrm{Hom_A}(B, A) \to B$ *is an isomorphism of* B*-modules.*

Proof. Applying Lemma 3.2(c) we have that $\mathrm{Ann}_{B\otimes_A B}\, I = \Delta(B \otimes_A B)$, and further that $\mathrm{Ann}_{B\otimes_A B}\, \Delta(B \otimes_A B) = I$. Thus, we have that
$$\mathrm{Ann}_{B\otimes_A B}\, I = \Delta(B \otimes_A B) \cong \Delta(B \otimes_A B)/\, \mathrm{Ann}_{B\otimes_A B}\, \Delta = \Delta(B \otimes_A B)/I \cong m(\Delta)B.$$
Applying Lemma 3.3, we have then that $\mathrm{Hom_B}(\mathrm{Hom_A}(B, A), B)$ is a free B-module with basis $\chi(\Delta)$. Applying Lemma 2.4, this implies that $\mathrm{Hom_A}(B, A)$ is a free B-module of rank 1. We must then have that the B-module homomorphism $\chi(\Delta) : \mathrm{Hom_A}(B, A) \to B$ is an isomorphism, as desired. □

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