Motivation for the Étale Fundamental Group

Kyrie Johnson

April 28, 2021

Abstract

We motivate the need for a different notion of the fundamental group in the realm of algebraic geometry, one which is compatible with the Zariski topology. The precise construction of the algebro-geometric fundamental group – the "étale fundamental group" – is beyond the scope of this short note, so once motivated we precisely describe the way in which Galois theory manifests as a special case of the more general étale theory. Finally, we introduce sheaves, an object essential to defining the étale fundamental group and found throughout algebraic geometry and beyond.

Classical algebraic geometry studies the zero sets of polynomials over algebraically closed fields. It does so by translating geometric notions – such as a graphed set of zeros – into algebraic ones, and then using the tools and techniques of algebra to understand the original geometric objects. Indeed, this algebra-geometry philosophy pervades much of mathematics: another crucial example lies in algebraic topology, where the study of both the fundamental group and the (co)homology groups of topological spaces have proven fruitful tools.

In this paper we focus attention on the fundamental group. Because the fundamental group has proven so useful in the study of many nice topological spaces — e.g. CW complexes – it is natural to wonder if we could use the fundamental group to study the topological spaces of algebraic geometry (classically, these are "varieties"). Unfortunately, the natural topology on a variety V yields open sets which are simply too large to accommodate interesting continuous maps from the interval $[0, 1]$ to V. We get around this obstruction via an algebraic view of covering space theory. Indeed, we may realise the fundamental group as a group of deck transformations, so that for varieties we may define the fundamental group via an algebraic analog of deck transformations. This leads to the notion of the étale fundamental group, an indispensable tool of algebraic geometry.

1 Classical Definitions and Motivation

Throughout this entire paper, k will denote an algebraically closed field; we will regularly remind ourselves that this is the case, as much changes over a non-algebraically closed field. Classical algebraic geometry focuses on the zero sets of polynomials, and it is the topology on these objects which will motivate our study.

Definition 1.1. Fix a positive integer n and an algebraically closed field k. For $I \subset k[x_1, \ldots, x_n]$ an ideal, denote by $\mathbf{V}(I)$ the set

$$\{(v_1, \ldots, v_n) \in k^n : p(v_1, \ldots, v_n) = 0 \text{ for all } p \in I\} \subset k^n.$$

The collection $\{\mathbf{V}(I)\}_{I \text{ an ideal}}$ is the collection of (affine) algebraic subset of k^n. Endow an algebraic set with a topology by declaring all algebraic subsets to be closed; this is the Zariski topology on an algebraic set.

In the preceding definition, the word "affine" refers to the ambient space k^n and the word "algebraic" refers to the roots of polynomial equations. Any algebraic set has a unique decomposition as a union of its irreducible components, so it is natural to restrict attention to the components.

Definition 1.2. An (affine, algebraic) variety V is an irreducible affine algebraic set.

To motivate our eventual study of the étale fundamental group, we first look at how the topological fundamental group breaks down for various classes of spaces, including varieties with the Zariski topology.

Lemma 1.3. *Endow an uncountable space X with the cofinite topology, meaning that a subset is closed if and only if it is finite. Then X is contractible.*

Proof. Because X is uncountable, we may take an injection $f : X \times (0, 1) \hookrightarrow X$; any such injection is automatically continuous by the definitions of the cofinite and product topologies. Then for a fixed $x_0 \in X$ we define a homotopy

$$H(x, t) := \begin{cases} x, & t = 0 \\ f(x, t), & t \in (0, 1) \\ x_0, & t = 1 \end{cases}$$

from the identity map to the constant map (continuity follows from noting that the inverse image of any point in X is closed in the product topology). We conclude that X is contractible. □

In other words, the (topological) fundamental group cannot distinguish between uncountable cofinite topological spaces, of which algebraic geometry has many examples.

Definition 1.4. An algebraic curve is a variety of dimension one. The Zariski topology endows an algebraic curve with the cofinite topology.

Thus, an algebraic curve over an uncountable base field (e.g. $\mathbb{C}$) has trivial fundamental group. Because algebraic curves form a crucial class of classical varieties, we see a gap which requires filling. In fact, every classical variety — not just a special subset of algebraic curves — is contractible with the Zariski topology.

Lemma 1.5. *Every irreducible algebraic variety over an uncountable algebraically-closed field is contractible.*

Proof. See [Wof17]. □

Thus, we must look elsewhere for a more useful notion of the fundamental group. A natural starting point is the celebrated classical result of Hilbert which corresponds algebraic sets to ideals over an algebraically closed field.

Theorem 1.6. *(Hilbert's Nullstellensatz) Let k be an algebraically closed field. Then there exists an inclusion-reversing bijective correspondence*

$$\{\textit{algebraic sets } V \subset k^n\} \leftrightarrows \{\textit{radical ideals } I \subset k[x_1, \dots, x_n]\}$$
$$V \mapsto \boldsymbol{I}(V)$$
$$\boldsymbol{V}(I) \leftarrow I$$

where $\boldsymbol{V}(I)$ denotes the zero set of the polynomials in I and $\boldsymbol{I}(V)$ denotes the ideal of polynomials which vanish on V. Moreover, the bijection restricts to a correspondence

$$\{\textit{algebraic varieties in } k^n\} \leftrightarrows \{\textit{prime ideals in } k[x_1, \dots, x_n]\}.$$

We shall see similar, more technical/nuanced statements like the Nullstellensatz later, but the given statements provides some basic insight for now. Indeed, the Nullstellensatz motivates an extremely fruitful topology on the prime ideals of a ring.

Definition 1.7. Let R be a ring. Denote by $\operatorname{Spec} R$ the spectrum of R, that is the set of all prime ideals of R. Endow $\operatorname{Spec} R$ with a topology by defining for an ideal $I \subset R$

$$V_I := \{\mathfrak{p} \supset I : \mathfrak{p} \text{ is prime in } R\}$$

and declaring that the collection $\{V_I\}_{\text{ideals } I}$ is precisely the collection of closed sets. We call the resulting topology the Zariski topology on $\operatorname{Spec} R$.

Note that by passing through Hilbert's Nullstellensatz, the Zariski topology on $\operatorname{Spec} R$ is closely related to the Zariski topology on a variety V. This justifies the use of the same name as well as motivates the study of the spectrum in the first place. Unfortunately, contractibility persists for this Zariski topology as well .

Definition 1.8. Call a topological space X sober if it is oh-so-very serious; equivalently, if every irreducible subset of X is the closure of exactly one point.

Lemma 1.9. *Let R be a ring. Then $\operatorname{Spec} R$ endowed with the Zariski topology is an irreducible sober space. Thus, that every irreducible sober space is contractible implies that $\operatorname{Spec} R$ has trivial fundamental group.*

Proof. We refer the reader to a textbook on commutative algebra for the fact that $\operatorname{Spec} R$ is an irreducible sober space.

Now, let X denote an irreducible sober space and apply the definition of sobriety to obtain a point x_0 such that the closure of x_0 is all of X. Then the homotopy

$$H(x,t) := \begin{cases} x, & t = 0 \\ x_0, & t \in (0,1] \end{cases}$$

demonstrates that the identity map is null-homotopic. □

So far, we have seen two classical structures — algebraic varieties and the spectrum of a ring — which have trivial fundamental groups when endowed with their natural topology. So at last we turn toward an algebraic re-interpretation of the fundamental group which might work in algebraic geometry. For this we will leverage the following important theorem, which says that under some mild conditions we may realise the topological fundamental group as a group of deck transformations.

Theorem 1.10. *Let X be a path connected, locally path connected, and semi-locally simply connected topological space, and fix a point x_0 of X. Then the fundamental group $\pi_1(X, x_0)$ is isomorphic to the group of deck transformations of the covering space $\tilde{X} \to X$, where $\tilde{X}$ denotes the universal cover of X.*

Thus, we define the étale fundamental group of algebraic geometry by identifying analogs of covering spaces, universal covers, and deck transformations. We first illustrate these analogs by describing how Galois theory manifests as a special case, before then going into some of the details of how to construct the algebraic analogs in general.

2 Galois Theory as a Prototype

There exist striking similarities between

- for a pointed topological space (X, x_0), the correspondence of connected covering spaces with subgroups of $\pi_1(X, x_0)$; and
- for a finite Galois field extension K/k, the correspondence of sub-extensions of k with subgroups of $\mathrm{Gal}(K/k)$.

Indeed, both cases correspond a "chain" of mathematical objects with the subgroups of a distinguished invariant group. Moreover, both correspondences are inclusion-reversing and offer a detailed relationship between the properties of an "extension" (meaning either a covering space or a field extension) and the properties of the corresponding subgroup. In this section, we illustrate the similarities of the these two powerful theories by stating them in categorical terms. Ultimately, the categorical version of Galois theory — often called Grothendieck's version of Galois theory — is but a special case of the theory of the étale fundamental group. From this perspective, the similarity between covering space theory and Galois theory is no accident: it falls out of the natural algebraic analogs for covering spaces and deck transformations.

To state Grothendieck's version of Galois theory, we start with a base field k. In "normal" Galois theory, we would proceed to fix a finite Galois extension L of k and then analyse subextensions of L. Instead, we want to simultaneously consider all finite Galois extensions for which we need to fix a separable closure.

Definition 2.1. An algebraic closure of the field k is a field extension k^{alg}/k such that k^{alg} is algebraically closed. A choice of algebraic closure induces a maximal separable subextension

$$k^{sep} := \bigcup_{\substack{L \subset K \\ L/k \text{ separable}}} L$$

of k, called a separable closure of k.

Now, fix a separable closure $\overline{k}$ of k and let L be a finite separable extension of k. Note that, while k and $\overline{k}$ are fixed, we will vary L. Moreover, note that we do not take L as a subset of $\overline{k}$; rather, we will consider how L maps into $\overline{k}$. Indeed, finite Galois theory implies that there exist exactly $[L : k]$ k-algebra homomorphisms from L into $\overline{k}$. We endow the finite set $\mathrm{Hom}_k(L, \overline{k})$ with the structure of a $\mathrm{Gal}(\overline{k}, k)$ set via the group action

$$\begin{aligned} \mathrm{Gal}(\overline{k}, k) \times \mathrm{Hom}_k(L, \overline{k}) &\to \mathrm{Hom}_k(L, \overline{k}) \\ (g, \phi) &\mapsto g \circ \phi. \end{aligned}$$

That is, an element g of the absolute Galois group $\mathrm{Gal}(\overline{k}, k)$ acts on an element of $\mathrm{Hom}_k(L, \overline{k})$ simply by the way in which g shuffles around $\overline{k}$. In particular, that g fixes k pointwise means that $g \circ \phi$ remains a k-algebra homomorphism and so the group action indeed fixes $\mathrm{Hom}_k(L, \overline{k})$ as a set. A little work, along with a recollection of finite Galois theory, then shows that the action of $\mathrm{Gal}(\overline{k}, k)$ on $\mathrm{Hom}_k(L, \overline{k})$ is continuous and transitive. We then have the following result.

Definition 2.2. Call a finite-dimensional k-algebra A ètale over k if A is isomorphic as a k-algebra to a finite product of separable extensions of k.

Theorem 2.3 (Fundamental Theorem of Galois Theory)**.** *Let k be a field and fix a separable closure $\overline{k}$ of k. The functor mapping a finite étale k-algebra A to the finite set $\mathrm{Hom}_k(A, \overline{k})$ gives a dual-equivalence between the category of finite étale k-algebras and the category of finite continuous left G_k-sets. Moreover, separable field extensions correspond to sets with transitive G_k action, and Galois extensions to G_k-sets which are isomorphic to finite quotients of G_k.*

Proof. See theorem 1.5.4 in [Sza09]. □

Let's see how this compares with our correspondence theorem for covering spaces. We'll state the correspondence for covering spaces in terms of the profinite completion $\overline{\pi_1}(X, x_0)$ of the fundamental group because it makes quite explicit the similarities between the two correspondences.

Theorem 2.4 (Correspondence Theorem for Covering Spaces)**.** *Let X be a path connected, locally path connected, semi-locally simply connected topological space and fix a point x_0 of X. The functor mapping a finite cover $p : Y \to X$ to the finite fibre $p^{-1}(x)$ gives an equivalence between the category of finite covers and the category of finite continuous left $\overline{\pi_1}(X, x_0)$-sets. Moreover, connected covers correspond to sets with transitive $\overline{\pi_1}(X, x_0)$ action, and Galois covers to quotient spaces by open normal subgroups.*

Proof. See corollary 2.3.9 in [Sza09]. □

The continuous action of $\overline{\pi_1}(X, x_0)$ on a fibre $p^{-1}(x)$ is precisely the monodromy action of $\pi_1(X, x_0)$, but passed to the completion in a natural algebraic way. To emphasise the remarkable similarities between the two theorems, we tabulate their relationships below.

Covering Space Object	**Galois Theory Analog**
topological space X	field k
fixed point x_0	fixed separable closure $\overline{k}$
$\overline{\pi_1}(X, x_0)$	$\mathrm{Gal}(\overline{k}/k)$
finite cover p	finite étale k-algebra A
finite continuous $\overline{\pi_1}(X, x_0)$-set	finite continuous $\mathrm{Gal}(\overline{k}/k)$-set
connected cover $\leftrightarrow$ transitive action	separable extension $\leftrightarrow$ transitive action
Galois cover $\leftrightarrow$ "nice" quotient space	Galois extension $\leftrightarrow$ finite quotient

One aspect remains unsatisfactory: a distinct dissimilarity between the two theorems lies in the word "dual". The Galois theory correspondence employs a contravariant functor, while the covering space correspondence employs a covariant functor. We can correct this misalignment by applying yet another contravariant functor in the world of Galois theory, thereby flipping all arrows back to their original direction.

Lemma 2.5. *A ring homomorphism $f : R \to S$ induces a continuous map*

$$\operatorname{Spec} f : \operatorname{Spec} S \to \operatorname{Spec} R$$

so Spec *defines a contravariant functor from the category of commutative rings to the category of "affine schemes". Moreover, the functor gives a dual-equivalence of categories.*

The "affine schemes" of the lemma have a rather geometric nature, so the interpretation of Grothendieck's Galois theory in terms of affine schemes is somehow the most natural analog for covering space theory. We will address the notion of a scheme more precisely in a subsequent section. For now, we offer some motivation for the definitions of the next section.

Theorem 2.6. *Let k be an algebraically closed field. There is a functor $k\lfloor\ \rfloor$ from the category of algebraic sets over k to the category of finitely-generated reduced k-algebras which establishes a dual equivalence of categories.*

The dual equivalence of the theorem is one of the most important tools of affine algebraic geometry. An especially useful aspect of the equivalence lies in localising the finitely-generated k-algebra $k\lfloor V\rfloor$ at maximal ideals for each point of an algebraic set V. In so doing, one associates to each $v \in V$ a ring $k\lfloor V\rfloor_{\mathfrak{m}_v}$ which reflects some of the local structure of v at V. The notion of associating rings to each point gives rise to the modern notion of "locally ringed spaces", a notion which requires the objects of the next section.

3 Toward the Étale Fundamental Group

The full construction of the étale fundamental group is involved and technical. We focus on some of the most important objects in the construction, whose applications span algebraic geometry, number theory, and algebraic topology.

3.1 Sheaves

Remark 3.1. Let X denote a topological space. Then we may form a category with objects given by the open subsets of X and morphisms given by inclusion maps.

Definition 3.2. A presheaf $\mathcal{F}$ of rings on a topological space X is a contravariant functor from the category of open subsets of X to the category of rings. In particular, a presheaf $\mathcal{F}$ associates to each open $U \subset X$ a ring $\mathcal{F}(U)$ and to each inclusion $U \hookrightarrow V$ of open subsets a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ satisfying

- for all open $U \subset X$, we have $\rho_{U,U} = \mathrm{id}_U$; and
- for all $U \subset V \subset W$ open sets, we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

We call an element $s \in \mathcal{F}(U)$ a section of $\mathcal{F}$ over U and refer to the map ρ_{VU} as the restriction map from V to U. The name "restriction" then motivates that for $s \in \mathcal{F}(V)$ and $U \subset V$ we will often use the notation $s|_U := \rho_{VU}$. Finally, in the same way as for rings, we define a presheaf of sets, groups, vector spaces, R-modules, etc. as a functor from (the category of open subsets of) X to the category of sets, groups, vector spaces, R-modules, etc.

That we may regard a presheaf as the more familiar notion of a functor offers some evidence that presheaves are natural and useful. The following examples develop further intuition and motivation.

Example 3.3. For any topological space X and any non-negative integer n, we have a presheaf $\mathcal{F}_n$ of groups given by $\mathcal{F}_n(U) := H^n(U)$, the n^{th} cohomology group of U. In particular, the restriction map which corresponds to the inclusion $i : U \hookrightarrow V$ is the induced map $\rho_{VU} = i^* : H^n(V) \to H^n(U)$ on cohomology.

Example 3.4. For any topological space X, we have a presheaf of rings given by $\mathcal{F}(U) := H^*(U)$, the cohomology ring of U.

Example 3.5. For any compact n-manifold M, we have a presheaf of groups given by $\mathcal{F}(U) := H_n(M, M - U)$. For $U \subset V$, we have an inclusion $i : (M, M - V) \hookrightarrow (M, M - U)$ of pairs which induces a restriction map $\rho_{VU} := i_* : H_n(M, M - V) \to H_n(M, M - U)$. This is called the orientation (pre)sheaf on the manifold M (because of these homology groups' role in determining the orientability of the manifold M).

Example 3.6. For any topological space X, we have a presheaf of rings given by $\mathcal{F}(U) := \mathcal{C}(U)$, the ring of continuous real-valued functions on U, with $\rho_{VU} : \mathcal{C}(V) \to \mathcal{C}(U)$ given by restriction of functions. It is this (extremely important and rather general) example which motivates the terminology "restriction" in the first place.

Indeed, there are analogous (pre)sheaves — "sheaf" is defined momentarily — given by the natural functions on various topological spaces: the sheaf of smooth functions on a smooth manifold, the sheaf of differential k-forms on a smooth manifold, the sheaf of holomorphic functions on a subset of $\mathbb{C}$, and many more. Likewise, one can replace "real-valued functions" in the example with continuous functions with values in any other topological space.

Informally, a presheaf for a topological space X collects local information into a single structure along with some basic way of restricting that local data from bigger spaces to smaller ones. But there is something unsatisfactory about a presheaf's lack of global data. In the first two examples, in particular, building cohomology groups/rings of X from the cohomology groups/rings of subspaces of X generally requires technical tools like excision and Mayer-Vietoris sequences. The last example, in contrast, involves continuous functions on a topological space, where the pasting lemma tells us that continuous functions on subspaces often knit together to form a continuous function on the larger space. We see then that the "pre-" part of presheaf indicates the existence of some additional structure, hence our next definition.

Definition 3.7. A sheaf $\mathcal{F}$ (of rings, sets, groups, etc.) is a presheaf (of rings, sets, groups, etc.) which, for any open covering $\{U_i\}_{i\in I}$ of an open $U \subset X$, further satisfies

- (the locality condition) if any two sections $s, t \in \mathcal{F}(U)$ have $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$; and
- (the gluing condition) if a system of sections $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ has $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. By the locality condition, when such a section s exists it is unique.

Thus, the notion of a sheaf upgrades that of a presheaf: a sheaf not only collects together local data, but also requires that the local data stitches together in a natural way. We have of course already seen many sheaves: examples 3.5 and 3.6 define the sheaves which motivate their definition in the first place. We offer three more crucial examples, the last of which brings us back to covering space theory.

Example 3.8. For any topological space X and discrete topological space S, we have a sheaf $\mathcal{F}_S$ on X by taking $\mathcal{F}_S(U)$ to be the ring of continuous functions $X \to S$. We call $\mathcal{F}_S$ the constant sheaf on X with value S because the sections of $\mathcal{F}_S$ are functions which are constant on the connected components of U.

Example 3.9. Fix a topological space X, a point $x \in X$, and an abelian group A. Then we have a sheaf $\mathcal{F}^x$ of abelian groups on X given by

$$\mathcal{F}(U) = \begin{cases} A, & x \in U \\ 0, & x \notin U \end{cases}$$

and restriction maps given by either the identity or the zero map (depending on the location of x relative to U and V). Call $\mathcal{F}^x$ the skyscraper sheaf over x with value A.

Example 3.10. Fix a continuous map $p : Y \to X$ of locally connected topological spaces. We define a sheaf $\mathcal{F}_Y$ of sets on X by setting

$$\mathcal{F}_Y(U) := \{s : U \to Y \text{ such that } p \circ s = \mathrm{id}_U\}$$

and for $U \subset V$ defining $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ in the same way as example 3.6: that is, ρ_{VU} restricts a function $s \in \mathcal{F}(V)$ to U. Recall that a function $s : U \to Y$ such that $p \circ s = \mathrm{id}_U$ is often called a section of the map p over U, so this example motivates the terminology "section" in the definition of a presheaf. Moreover, the same terminology gives $\mathcal{F}_Y$ its name: the sheaf of local sections of the map $p : Y \to X$.

We will use the sheaf of local sections of the map $p : Y \to X$ to determine when p defines a covering space. For p to define a covering space, p must satisfy a sort of locally-discrete condition; for the corresponding sheaf of local sections, this says that $\mathcal{F}$ locally looks like the constant sheaf of example 3.8. To make such a correspondence precise, we require a categorical perspective of (pre)sheaves which will allow us to make sense of what it means for two sheaves to be the same.

Definition 3.11. Recall that we may regard a presheaf of rings (or whatever else) on a topological space X as a functor from the category of subspaces of X to the category of rings. Thus, a morphism of presheaves is a morphism of functors in the "category of categories"; explicitly, a morphism from the presheaf $\mathcal{F}$ to the presheaf $\mathcal{G}$is a collection of ring homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ {\scriptstyle \rho^{\mathcal{F}}_{VU}}\big\downarrow & & \big\downarrow{\scriptstyle \rho^{\mathcal{G}}_{VU}} \\ \mathcal{F}(U) & \xrightarrow[\phi_U]{} & \mathcal{G}(U) \end{array}$$

commutes for all $U \subset V \subset X$. The collection ϕ_U defines an isomorphism of presheaves if ϕ_U is a ring isomorphism for all $U \subset X$. A morphism (respectively, isomorphism) of sheaves is a morphism (respectively, isomorphism) of the underlying presheaves.

Definition 3.12. Let $\mathcal{F}$ define a (pre)sheaf on a topological space X. For $V \subset X$, we define a (pre)sheaf on V by considering only $\mathcal{F}(U)$ for $U \subset V$ and forgetting about the value of $\mathcal{F}$ on open sets of X not contained in V. The resulting (pre)sheaf on V is the restriction of $\mathcal{F}$ to V.

With the notions of isomorphism and restriction in hand, we may make precise what it means for a sheaf to locally look like the constant sheaf.

Definition 3.13. Let $\mathcal{F}$ denote a sheaf on a topological space X. We call $\mathcal{F}$ locally constant if each point of X lies in a neighborhood U such that the restriction of $\mathcal{F}$ to U is isomorphic to the constant sheaf.

And from this discussion emerges the following important results, which connect sheaves to covering space theory.

Theorem 3.14. *Let $p : Y \to X$ denote a map of locally connected topological spaces and let $\mathcal{F}_Y$ denote the sheaf of local sections of p. If $p : Y \to X$ is a covering space, then $\mathcal{F}_Y$ is locally constant. Moreover, $\mathcal{F}_Y$ is itself constant if and only p is the trivial cover.*

Theorem 3.15. *Let X be a path connected, locally path connected, and semi-locally simply connected topological space, and fix a point x_0 in X. Then the category of locally constant sheaves of sets on X is equivalent to the category of finite left $\overline{\pi_1}(X, x_0)$-sets.*

Theorem 3.15 hints at how the notion of sheaves can unify the topology and algebraic notions of covering spaces and Galois theory. To fully develop that relationship, one requires the language of schemes, which are defined locally via sheaves.

3.2 Schemes

In the world of algebraic geometry, a "scheme" is the analog of an abstract manifold. Indeed, an abstract manifold is embedding-free way by axiomatically requiring that locally everything appears the same as Euclidean space. In the same way, a scheme looks locally like an affine variety together with some additional structure. That additional structure comes in form of associating rings to local areas by way of sheaves (of rings). The language of schemes then enables one to define the étale fundamental group by way of identifying analogs of covering spaces, as promised. We won't cover the details here, but the table displays the names of key étale analogs. Note the similarity of language to the statements of Galois theory and covering space theory given in theorems 2.3 and 2.4.

Covering Space Theory	**Étale Analog**
covering space	finite étale morphism
category of coverings	category of étale coverings
a group of deck transformations	a finite automorphism group
fundamental group	inverse limit of finite automorphism groups

Crucially, note that one must take an inverse limit in defining the étale fundamental group. This is because there is no finite analog of a universal cover; instead, one must knit together all finite automorphisms group in the inverse limit. This motivates the use of the profinite completion in theorem 2.4 and emphasises the relative simplicity of the covering space theory in the vast étale world.

4 Concluding Remarks

We have motivated the necessity for a different notion of the fundamental group in the realm of algebraic geometry, one which is compatible with the Zariski topology. While these ten compact pages could never cover the full construction, we have precisely described the way in which Galois theory manifests as a special case of the more general étale theory. Moreover, we have introduced sheaves and schemes, two powerful objects in and out of algebraic geometry.

Although we have focused on the étale fundamental group here, related objects merit mention. Early on, we discussed that algebraic curves are contractible when endowed with the Zariski topology. In the case of complex algebraic curves — over the algebraically closed base field $\mathbb{C}$ — the geometry of Euclidean space does indeed provide useful data. Specifically, one can embed complex an algebraic curve in $\mathbb{C}^n$, some n, and then consider the complement of this embedding with the Euclidean topology. Indeed, the so-called Zariski van-Kampen theorem characterises these (often nontrivial) fundamental groups, thereby establishing another algebraic invariant for complex curves.

Finally, we mention that there exists an entire étale theory not only for fundamental groups, but also for cohomology groups. As with the topological fundamental group, for an irreducible variety X with the Zariski topology, the cohomology groups $H^n(X)$ equal zero for $n \geq 1$. Like the étale fundamental group, there exist notions of étale cohomology groups which satisfy analogs of the Eilenberg-Steenrod axioms and Poincaré duality, and which yield rather crucial results in algebraic geometry. In fact, the development étale cohomology remains one of the most complete and fundamental theories underpinning modern number theory, a fact which originally inspired our interest in the étale theories.

References

[Chu] Dexter Chua. The étale fundamental group. Available at https://dec41.user.srcf.net/exp/etale_fundamental_group/.

[DS16] Patrick Da Silva. When is the orientation presheaf a sheaf?, 2016. Available at https://math.stackexchange.com/a/1924609/596046.

[Kun17] Arnab Kundu. The étale fundamental group of an elliptic curve, 2017. Available at http://math.uchicago.edu/~may/REU2017/REUPapers/Kundu.pdf.

[Sco19] Mitchell Scofield. On the fundamental group of plane curve complements. Master's thesis, Boise State University, 2019. Available at https://scholarworks.boisestate.edu/cgi/viewcontent.cgi?article=2644&context=td.

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups*. Cambridge University Press, New York, 2009.

[Urb19] David Urbanik. A brief introduction to schemes and sheaves, 2019. Available at https://www.math.toronto.edu/dburbani/work/schemes.pdf.

[Wof17] Eric Wofsey. What is the homotopy type of the affine space in the zariski topology?, 2017. Available at https://math.stackexchange.com/a/2294516/596046.

[Wof19] Eric Wofsey. Fundamental group of uncountable space with cofinite topology, 2019. Available at https://math.stackexchange.com/a/3098803/596046.