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Statement of Mathematics Research

My mathematics dissertation work lies at the interface of analytic and algebraic number theory, especially Hilbert's twelfth problem (Hilb12) through a lens of special values of L-functions.

**My Work on Hilbert's Twelfth Problem over ATR Fields**

Hilb12 asks for a generalization of the Kronecker-Weber theorem: the roots of unity generate the abelian extensions of $\mathbb{Q}$, but what about for other ground fields? In a series of papers in the 1970's, culminating in [Sta80], Stark conjectured generalizations of the classical class number formula with implications for Hilb12. Indeed, the recent proof of the Brumer-Stark Conjectures by Dasgupta and Kakde in [DK23] has resulted in explicit generators of abelian extensions over totally real fields.

Let $K/\mathbb{Q}$ denote a degree $n \geq 2$ number field with exactly one complex place (also called an almost totally real, or ATR, field). For example, $K = \mathbb{Q}[\theta]$ with $\theta^3 - 2 = 0$ has one real embedding (which corresponds to the real cube root of 2) and one complex embedding (which corresponds to the conjugate pair of complex cube roots of 2). I generalize Ren and Sczech's work over cubic ATR fields in [RS09] to *all* ATR fields, in part by leveraging the work of Cassou-Noguès in [CN79].

For ideals $\mathfrak{f}$ (the conductor) and $\mathfrak{a}$ (an ideal class mod $\mathfrak{f}$), consider the partial zeta function

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, s) = N\mathfrak{a}^{-s} \sum_{\substack{\alpha \in \mathfrak{a} \\ \alpha \equiv 1 \mod \mathfrak{f}}} \frac{1}{N(\alpha)^s}, \qquad \mathrm{Re}(s) > 1.$$

Then $\zeta_{\mathfrak{f}}(\mathfrak{a}, s)$ has an analytical continuation to all of $\mathbb{C}$ and the Stark conjectures over ATR fields relate the value of $\zeta'_{\mathfrak{f}}(\mathfrak{a}, s)$ at $s = 0$ to units in abelian extensions of $K$. "Smoothing" the partial zeta function by an auxiliary ideal $\mathfrak{c}$, denoted $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)$, gives an even simpler form of Stark:

**Conjecture 0.1** (Stark)**.** Suppose $\mathrm{Nm}(\mathfrak{c}) \geq n + 2$. Then there exists a unit $\eta$ inside the abelian extension $K_{\mathfrak{f}}/K$ such that $\eta \equiv 1 \mod \mathfrak{c}\mathcal{O}_{K_{\mathfrak{f}}}$ and

$$\zeta'_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, 0) = \log|\eta^{\sigma_{\mathfrak{a}}}| \quad \text{or, equivalently,} \quad \exp\left(\zeta'_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, 0)\right) = |\eta^{\sigma_{\mathfrak{a}}}|.$$

I use Shintani's method to construct sums $\mathcal{L}$ of log gamma functions (derivatives of Barnes zeta functions at $s = 0$) that analytically encode the algebraic data of $K$, $\mathfrak{f}$, and $\mathfrak{a}$. Each $\mathcal{L}$ corresponds with an infinite place of $K$ and, to ensure well-definedness, I combine them in a weighted sum

$$\mathcal{C} = \mathcal{L}_{\text{complex}} + \mathcal{R} + \sum_{\sigma: K \hookrightarrow \mathbb{R}} c_\sigma \cdot \mathcal{L}_\sigma.$$

**Theorem 0.2** (My main theorem)**.** *Notation as above,* $\zeta'_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}^{-1}, 0) \equiv \mathcal{C} + \overline{\mathcal{C}} \mod 2\pi i\mathbb{Z}$.

In view of Conjecture 0.1, my main theorem leads to a refinement of Stark over ATR fields.

**Conjecture 0.3** (My refinement of Stark)**.** I conjecture the analytic formula $\eta = \exp(\mathcal{C})$.

Exponentiation of the log gamma function yields the multiple gamma function, introduced by Barnes in a series of papers which culminated in [Bar04]. Therefore, my Conjecture 0.3 expresses the Stark unit $\eta$ as essentially a product of (Barnes) multiple gamma functions.

### Opportunities for Undergraduate Student Involvement

The following table summarizes refinements of the Stark conjectures over ATR fields. While Morain built upon [BCG23], I use alternative methods (Shintani's method) like [RS09].

| Authors | Year | Citation | Degrees of Refinement | Degrees of Numerical Examples |
|---|---|---|---|---|
| Ren, Sczech | 2009 | [RS09] | $n = 3$ | $n = 3$ |
| Bergeron, Charollois, Garcia | 2023 | [BCG23] | $n = 3$ | $n = 3$ |
| Morain | 2024 | [Mor24] | $n \geq 3$ | $n = 3, 4, 5$ |
| Black | 2025 | *my dissertation* | $n \geq 2$ | *in progress* |

My Git repository `https://github.com/kairi-black/ATR-Stark` contains (tested) code for computing $\eta$, pending numerical approximations of the log-gamma functions, denoted $\log \gamma_r$. In [Ren03], Ren approximates $\log \gamma_r$ for $r \leq 3$; this suffices for degree $n = 3$, but not $n > 3$. I would love to mentor undergraduates in two possible directions and update "*in progress*" in the table.

First, we could use Ruijsenaar's formulas of [Rui00] to approximate $\log \gamma_r$ for all $r$. Second, we could generalize the approximations of [Ren03] beyond $r \leq 3$. Both directions have intermediate milestones that undergraduates could celebrate and report (e.g. the case of $r = 4$) and have opportunities to check progress against the literature (e.g. my dissertation, or formulas for $r = 1, 2$).

## References


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