Classification of Legendrian Knots and Links

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1   Introduction

The aim of this paper is to use computer program to generate and classify Legendrian knots and links. Modifying the algorithm in [11], we write a program in Java to generate all grid diagrams of up to size 10. Since classification of Legendrian links up to Legendrian isotopy is equivalent to grid diagrams modulo a set of Cromwell moves including translation, commutation and X:NE,X:SW (de)stabilization [17], we can identify two grid diagrams and check whether they represent the same Legendrian link.

We also use various topological, Legendrian and transverse invariants for knots and links to distinguish Legendrian knots and links generated by the algorithm. Since classical Legendrian invariants are not enough for classification, we use several non-classical ones such as graded ruling invariant and linearized contact homology. To refine our result even further, we use several techniques such as the Chekanov-Eliashberg differential graded algebra (DGA) [13] and knot Floer homology [15].

The main result of this paper consists of two atlases, a Legendrian knot atlas (table 1) and an atlas for unoriented Legendrian two-component links (table 3). We also provide a transverse knot atlas (table 2), which can be inferred from the Legendrian knot atlas. In the Legendrian knot atlas, we give information on grid diagram, Thurston-Bennequin number, rotation number, polynomials for graded ruling invariant and linearized contact homology for each Legendrian knot with knot type of arc index up to 9. For each knot type, we also provide conjecture for its mountain range.

The Legendrian knot atlas also illustrates several interesting phenomena such as unusual mountain ranges for 10_{161}, $m(10_{139})$ and $m(12n_{242})$, and transversely non-simple knots for $m(10_{145})$, $m(10_{161})$ and $12n_{591}$. The later group is depicted in their mountain ranges as Legendrian knots whose Thurston-Bennequin number is not equal to the maximal Thurston-Bennequin number of their knot type but are not stabilization of any Legendrian knots. Prior to this paper, such phenomenon was found only in knots with very large number of crossings.

The atlas for unoriented Legendrian two-component links also gives information on grid diagram, Thurston-Bennequin number and rotation number for each unoriented Legendrian link with link type of arc index up to 9. We also give information on whether its two components can be switched via topological isotopy and conjecture on whether they can

be switched via Legendrian isotopy. For each link type, we also provide conjecture for its Thurston-Bennequin polytope [1]. Our result can answer the question posted by Baader and Ishikawa in [1] that whether the tb polytope can always be described by the three linear inequalities.

2   Knots and Links

In mathematics, a knot is a simple *closed* curve in 3-dimensional Euclidean space, $\mathbb{R}^3$. One can simply think of knot as one piece of string with its ends glued together. Formally, we can define a knot as follows.

Definition 1. A *knot* is a smooth embedding $K : S^1 \to \mathbb{R}^3$. Two knots are *equivalent* (or *topological isotopic*) if they are ambient isotopic.

In other word, two knots are equivalent if one can continuously deform to the other without brreaking or intersecting itself. We call an equivalence class of knots a *knot type*. Some simpler knot types are well-studied and have names such as "trefoil knot" and "the figure-eight knot".

We usually represent knots by their immersion on $\mathbb{R}^2$ such that the restriction map $S^1 \to \mathbb{R}^2$ is injective except finite number of points, in which case the map is two-to-one and transverse, i.e., tangent lines should not coincide at the crossing. Such a projection is called *knot diagram*. The two-to-one points are called *crossings*. At each crossing, we need to specify which strand passes over or in front of another strand. We call the strand that passes over *overstrand* and another *understrand*. Distinct knot diagrams can represent knots with the same knot type. We can define *crossing number* of a particular knot type to be the minimum number of crossings of any knot diagrams representing that knot type.

The main problem in knot theory is to classify knots and decide whether two given knots are equivalent. Traditionally, knots are classified by crossing numbers and hence we have a knot table as in Figure 1. Each knot type is labeled as C_i where C is the crossing number and i is an index within the knots of the same crossing number. For example, the trefoil knot is labeled 3_1. The standard knot table with such labeled for all prime knots with less than 11 crossings is called the *Rolfsen knot table*. For knots with 11 crossings or more, we further categorize them by whether a knot is *alternating*, i.e., whether the crossings alternate under, over, under, over, as one travels along the strand. Otherwise, a knot is *non-alternating*. We then labeled an alternating knot by Ca_i and a non-alternating knot by Cn_i. The standard knot table for knots with 11 crossing is called the Hoste-Thistlethwaite table. The largest knot table consists of all knots with up to 16 crossings.

It is not simple to show the equivalence of two given knots by using only the definition. In 1926, Kurt Reidemeister, and independently, in 1927, J.W. Alexander and G.B. Briggs proved that two knot diagrams represent knots with the same knot type if and only if one can be transformed to the other via planar isotopy by a sequence of three kinds of moves,
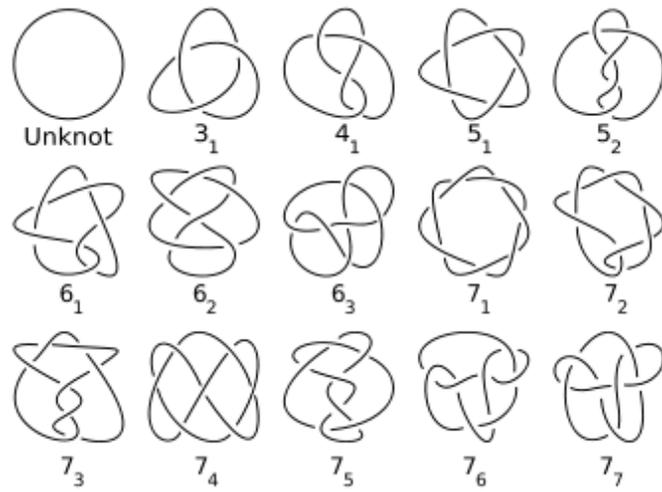
Figure 1: Knot Table [1]

called *Reidemeister moves* as shown in Figure 2.

(a) Type I   (b) Type II

(c) Type III
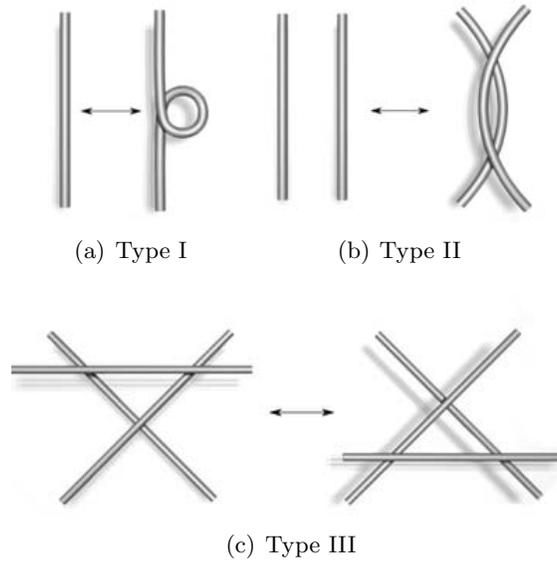
Figure 2: Reidemeister Moves [2]

[1]http://en.wikipedia.org/wiki/Knot_(mathematics)

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Although Reidemeister moves provide more convenient and systematic way to show equivalence of two knot diagrams, they does not give a deterministic algorithm to classify knots. In general, we do not know how many moves we need to use to transform one knot diagram to the other. Hence, it is not possible to use Reidemeister moves to distinguish two knot diagrams that possibly belong to different knot types.

Sometimes we give an orientation to knot, denoted by arrows in knot diagram as in Figure 3. Two knots of the same knot diagram but different orientation may or may not be equivalent. A *mirror image* of a knot, which can be obtained by reversing all crossings, may or may not be equivalent to the original one. The mirror image of a knot K usually denoted by $m(K)$ or $\overline{K}$.
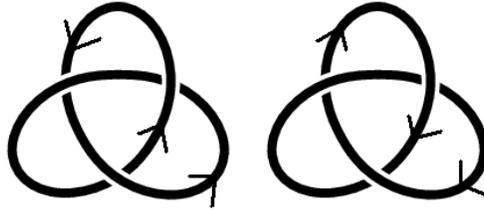
Figure 3: Two Orientations of Trefoil Knot [3]

We can assign some "value" such as integer, polynomial or even algebraic structure, to each knot. If the values is invariant under three types of Reidermester moves, we call such values *knot invariants*. Clearly, crossing number is a knot invariant by definition. Some knot invariants is defined in a way that one can determine such value from knot diagram. We may use these knot invariants to show that two diagrams represent different knot type.

For oriented knot, we call a crossing *positive* if one can turn the direction of the overstrand counterclockwise to match the direction of the understrand with the angle less than a half-turn. Otherwise, we call a crossing *negative* (Figure 4). For each crossing c, we may define

$$\epsilon(c) = \begin{cases} 1 & c \text{ positive} \\ -1 & c \text{ negative} \end{cases}$$

Then the *writhe* of a diagram D is

$$wr(D) = \sum_{\text{crossing } c \text{ in } D} \epsilon(c).$$

Although writhe is not a knot invariant as it is not invariant under Reidemeister I move, it is useful quatity that we will use to compute several invariants later in this paper.

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[2]http://en.wikipedia.org/wiki/Knot_theory

[3]http://www.math.cornell.edu/~mec/2008-2009/HoHonLeung/page2_knots.htm

(a) positive crossing   (b) negative crossing
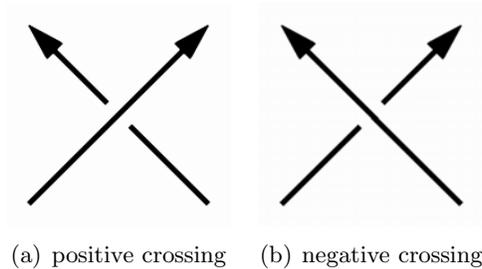
Figure 4: Positive and Negative Crossings[4]

We can also consider a union of disjoint embeddings of S^1 in $\mathbb{R}^3$. We call a collection of disjoint knots a *link*. Two links are equivalent (or topological isotopic) if they are ambient isotopic. Each embedding of S^1 is called a component of a link. A knot is simply a link with one component. Most results for knots apply to links as well. For example, two link diagrams represent links with the same link type if and only if one can be transformed to the other by a sequence of Reidemeister moves. Note that if we consider only one component of a link and omit the rest, it will become a knot. We can also construct a link from a knot to get some extra properties and invariants, which we will discuss later on in this paper. Hence, the study of knots and links are strongly related.

The concept of invariants also applies to links. Several knot invariants are also link invariant. In addition, there are several link invariants that are only useful to links with more than one component. For example, the *linking number* of components L_1 and L_2 of a link L is

$$lk(L_1, L_2) = \frac{1}{2} \sum_{\text{crossing } c \text{ between } L_1 \text{ and } L_2} \epsilon(c).$$

In other word, the linking number is half of the total number of positive crossings between components minus the total number of negative crossings between components. We can see that the linking number of any diagram is always an integer, and the linking number is an invariant of a link with two components.

Several other useful and widely used knot and link invariants are *Kauffman polynomial, Jones polynomial, HOMFLY-PT polynomial* and *Alexander polynomial*. We shall not discuss the meaning of these invariants in detail. Given some representation of knot and *Mathematica* can compute these invariants if the knot is not too large.

Similar to knots, links are traditionally classified by their crossing numbers. Links are also considered whether they are alternating or non-alternating. A link is labeled *Lcai* and *Lcni*, respectively, where c is its crossing number and i is an index within (non-)alternating

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[4]http://en.wikipedia.org/wiki/Writhe
[5]http://en.wikipedia.org/wiki/Linking_number
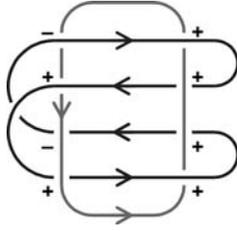
Figure 5: A Two-Component Link with Linking Number Two[5]

knots of the same crossing number (Figure 6). Unlike knots, links are less studied and more complicated. The Thistlethwaite link table consists of all prime links with up to 13 crossings.
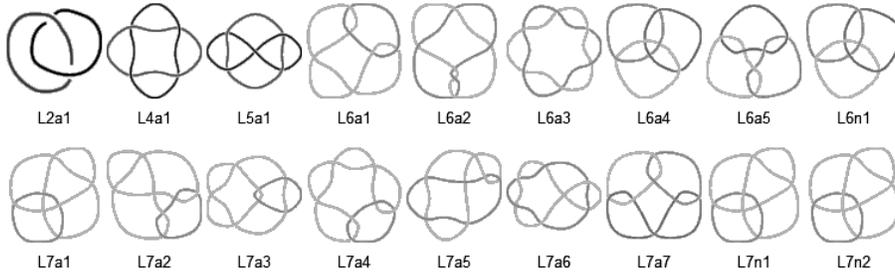
Figure 6: Link Table[6]

3 Legendrian and Transverse Knots

Definition 2. Let L be a knot parametrized by a map $S^1 \to \mathbb{R}^3$ defined by

$$\theta \mapsto (x(\theta), y(\theta), z(\theta)).$$

Then L is a *Legendrian knot* if

$$z'(\theta) - y(\theta)x'(\theta) = 0.$$

There are several ways to represent or project a Legendrian knot in $\mathbb{R}^2$. Throughout this paper, we shall use a *front projection* defined by the map $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ which sends $(x, y, z) \mapsto (x, z)$. Since $z'(\theta) - y(\theta)x'(\theta) = 0$, we can retrieve $y(\theta)$ from $x(\theta)$ and $z(\theta)$ by $y(\theta) = \frac{z'(\theta)}{x'(\theta)} = \frac{dz}{dx}$. We call an image of the projection at $x'(\theta) = 0$ a *cusp*. The image of front projection, called *Legendrian front diagram* or *front diagram*, is a knot diagram with the following properties:

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[6]http://katlas.math.toronto.edu/wiki/The_Thistlethwaite_Link_Table

1. has no vertical tengencies

2. the only non-smooth points are horizontal cusps

3. at each crossing, the slope of the overcrossing strand is smaller (more negative) than the undercrossing strand.

Moreover, any knot diagram satisfying all of the above three properties is a front diagram of a Legendrian knot. The third property allows us to omit the notation indicating which strand is overcrossing at each crossing. We can write front diagrams as in Figure 7.
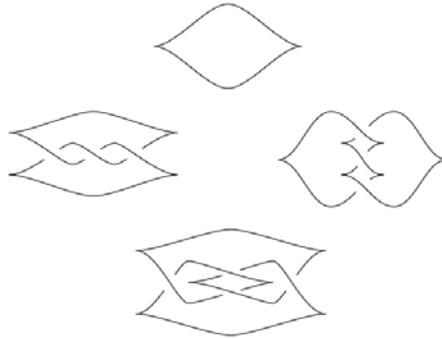
Figure 7: Front Diagrams of unknot, trefoil knot and the figure-eight knot

Similar to topological knot, two Legendrian knots are *equivalent* or *Legendrian isotopic* if they are ambient isotopic in the way that a knot at every state of deformation is Legendrian. Figure 8 shows three types of Legendrian Reidemeister moves on a front diagram which are analogous to Reidemeister moves for topological knots on a knot diagram. The result of Swiatkowski [19] shows that two front diagrams represent equivalent Legendrian knots if and only if they relate by a sequence of Legendrian Reidemeister moves.

Clearly, a Legendrian knot is also a topological knot. One may consider a front diagram as a knot diagram. Legendrian isotopy is also a topological isotopy, but not the other way around. For any topological knot type, there are Legendrian knots representing it. One can construct a Legendrian knot, or, in particular, its front diagram, from a knot diagram by adding cusps and zig-zags as shown in Figure 9.

We can view Legendrian knots as subtypes of topological knots. Each topological knot belongs to exactly one knot type. Hence, topological knot type is an invariant of Legendrian knots. In fact, it is one of three classical Legendrian invariants which are the most fundamental tools to classify Legendrian knots throughout this paper. We shall denote the topological type of a Legendrian knot L by $k(L)$.

Another classical Legendrian invariant is the *Thurston-Bennequin number*, denoted by tb, or $tb(L)$ for the Thurston-Bennequin number of a Legendrian knot L. The Thurston-Bennequin number is defined in several ways, but for our convenience, we shall define it
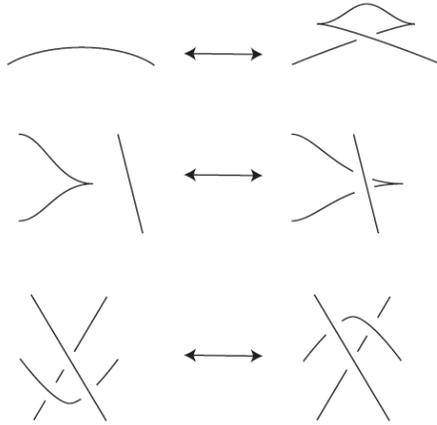
Figure 8: Legendrian Reidemeister Moves [7]

combinatorially from a front diagram. Let $c(D)$ be the number of left cusps in a front diagram D. Note that in any front diagram, the number of left cusps is equal to the number of right cusps. Thus, $c(D)$ is also half of the number of all cusps in D. The Thurston-Bennequin number of a Legendrian knot L with a front diagram D is

$$tb(L) = wr(D) - c(D).$$

The last classical Legendrian invariant is the *rotation number*, denoted by r, or $r(L)$ for the rotation number of an oriented Legendrian knot L. We may define the rotation number combinatorially as follows. For an oriented front diagram D, let $c_\downarrow(D)$ be the total number of downward cusps in D and $c_\uparrow(D)$ be the total number of upward cusps in D (Figure 10). Then the rotation number of L represented by D is

$$r(L) = \frac{1}{2}(c_\downarrow(D) - c_\uparrow(D)).$$

One can check that tb and r are invariant under three types of Legendrian Reidemeister moves and therefore Legendrian invariants. Figure 11 depicts tb and r of several Legendrian knots.

The classical invariants alone are not enough to distinguish several pairs of Legendrian knots, especially the larger ones. Besides the classical invariants, there are several useful nonclassical Legendrian invariants that we shall use to distinguish Legendrian knots and links later on in this paper, namely, *graded ruling invariant* and *linearized contact homology*. We can compute both of them as polynomials using *Mathematica*.

Given a front diagram of a Legendrian knot, we define a *ruling* to be a one-to-one correspondence of left and right cusps, together with a decomposition of the front diagram
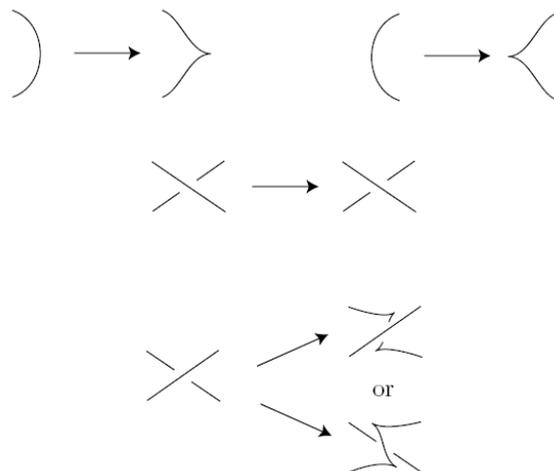
Figure 9: Construction of a front diagram from a knot diagram [7]

downward
cusps

upward
cusps
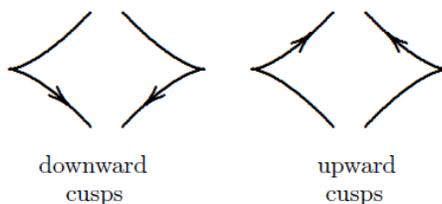
Figure 10: Cusps [6]

to a union of pairs of paths, each of which connects a corresponding pair of left and right cusps, and satisfy the following conditions:

1. all paths are smooth except possibly at crossings and always go from either left to right or right to left

2. the two paths corresponding to the same pair of cusps never intersect except at the cusps

3. at each crossing where two paths intersect and one lies entirely above the other (called such crossing a *switch*), the pair of vertical line segments connecting pairs of paths passing through that crossing, with the same x coordinate as that crossing are either nested or disjoint except at the crossing

Note that a ruling is uniquely determined by its switches. By Chekanov and Pushkar

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(a) $tb = -1$, $r = 0$ (b) $tb = -2$, $r = 1$ (c) $tb = -6$, $r = 1$
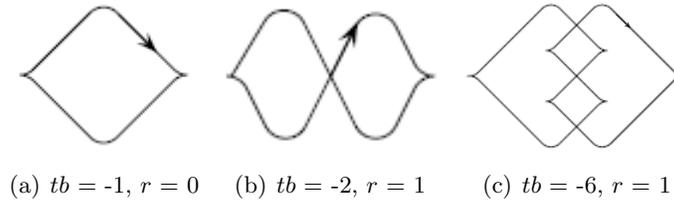
Figure 11: tb and r for some Legendrian knots

[4], the number of rulings is an invariant of Legendrian knot. For each front diagram F, we may define a function $\gamma : \{$rulings of $F\} \to \mathbb{Z}$ by $\gamma(R) = \#(R) - S(R)$, where R is a ruling of F, $\#(R)$ is the total number of left cusps (or right cusps), and $S(R)$ is the total number of switches. Then the map θ is also a Legendrian invariant up to isomorphism of $\{$rulings of $F\}$, called *complete ruling invariant* [7].

We may refine this invariant by considering Maslov degrees. For a Legendrian knot with $r = 0$, we may assign an interger, called *Maslov number*, to each arc (from a left cusp to a right cusp without passing through any other cusps) such that at each cusp, the upper arc has Maslov number 1 greater than the lower arc. At each crossing, we define the *Maslov degree* to be the Maslov number of the strand with more negative slope minus the Maslov number of the strand with more positive slope. The rulings such that every switch has Maslov degree zero are called *graded ruling*. The number of graded ruling and γ restricted to graded rulings are also Legendrian invariants [4]. The later is called *graded ruling invariant*.

Furthermore, we can also restrict the rulings to those with Maslov degree divisible by an integer ρ. For a Legendrian knot with nonzero rotation number, we can consider Maslov number and Maslov degree modulo $2r$, and rulings with every switch has Maslov degree divisible by ρ, where ρ divides $2r$. For each such ρ, we get Legendrian invariants [4], called ρ-*graded ruling invariant*.

Now we shall consider some important moves on a front diagram which do not preserve Legendrian isotopy. *Stabilization* is a process of adding a zig-zag to a front diagram as shown in Figure 12. There are two types of stabilization, *positive stabilization* (S_+) and *negative stabilization* (S_-). If two down cusps are added, then the stabilization is positive. Otherwise, it is negative. Note that

$$tb(S_\pm(L)) = tb(L) - 1$$

and

$$r(S_\pm(L)) = r(L) \pm 1.$$

Thus, $S_\pm(L)$ is not Legendrian equivalent to L.

Theorem 3.1. [10] *If two Legendrian knots are topological isotopic, then after each of them has been stabilized for some number of times they will become Legendrian isotopic.*

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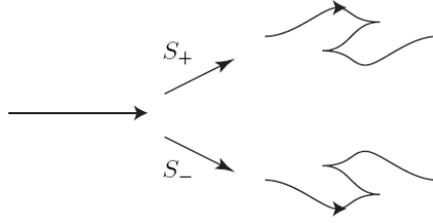
Figure 12: Positive and Negative Stabilization [7]

That means equivalence classes of Legendrian knots under positive and negative stabilization together with Legendrian Reidemeister moves are the same as equivalence classes of topological knots under topological Reidemeister moves.

One may also try to "destabilize" a Legendrian knot. We say L *destabilize* to L' if $L = S_\pm(L')$. However, not every Legendrian knot destabilizes, and it is generally not easy to see whether a Legendrian knot destabilizes.

Now we shall define another type of knots which strongly relates to Legendrian knot.

Definition 3. Let T be a knot parametrized by

$$\phi : S^1 \to \mathbb{R}^3$$

$$\theta \mapsto (x(\theta), y(\theta), z(\theta)).$$

Then L is a *transverse knot* if

$$z'(\theta) - y(\theta)x'(\theta) > 0.$$

Similar to topological knot and Legendrian knot, two transverse knots are *equivalent* or *transverely isotopic* if they are ambient isotopic in the way that a knot at every state of deformation is transverse. Although there is a typical way to study transverse knots without relying on Legendrian knots, for our convenient, we shall study transverse knot through its relationship with Legendrian knot.

Given an oriented Legendrian knot L, there are two transverse knots "close" to L. (For further information on what "close" means, see [7].) The two knots are called the *positive and negative transverse push-offs* of L, determined by whether they follow the same (positive push-off) or different (negative push-off) orientation as the Legendrian knot L. The positive (and negative) transverse push-off is unique up to transverse isotopy, and is denoted by $T_+(L)$ (and $T_-(L)$). There is also a notion of Legendrian approximation of a transverse knot T. Although the Legendrian approximation is not unique up to Legendrian isotopy, it is unique up to Legendrian isotopy and negative stabilization. Thus, the equivalence classes of transverse knots are in one-to-one correspondence with the equivalence classes of Legendrian knots modulo by negative stabilization [6].

The above statement also allows us to represent transverse knot using a front diagram of its Legendrian approximation. Since we know how to calculate tb and r from a front diagram, we may define the *self-linking number* of a transverse knot T with a front diagram for a Legendrian approximation D by

$$sl(T) = wr(D) - c_\downarrow(D).$$

From the definition of tb and r, we can also write

$$sl(T_\pm(L)) = tb(L) \mp r(L).$$

Note that tb and r are not transverse invariants as they are not invariant under negative stabilization. However, their difference, which is the self-linking number, is invariant under negative stabilization. Hence, the self-linking number is a transverse invariant. We may observe that the self-linking number is also a Legendrian invariant, but it is not useful as we can always obtain it from tb and r.

Suppose we have a Legendrian knot L of type $k(L)$ with $tb(L) = N$. We can find a Legendrian knot L' of type $k(L') = k(L)$ with $tb(L') = N-1$ by stabilizing L in either positive or negative direction. Since $r(S_+(L)) \neq r(S_-(L))$, $S_+(L)$ and $S_-(L)$ are not Legendrian isotopic. Similarly, for any $n \in \mathbb{Z}$, let S_i be either positive or negative stabilization for each $i = 1, 2, \ldots, n$ and let S be the composition $S_n S_{n-1} \ldots S_1$. Then a Legendrian knot $L'' = S(L)$ of type $k(L'') = k(L)$ has $tb(L'') = N - n$. One can easily check by Legendrian Reidemester moves that S_+ and S_- commute up to Legendrian isotopy. Thus, there are $n+1$ possible Legendrian knots, distinct up to Legendrian isotopy, as a result of S, completely determined by the number of positive stabilization in S.

In contrast, there is no general way to "increase" $tb(L)$ of a given Legendrian knot L. In fact, the classical result of Bennequin [3] shows that $tb(L)$ is bounded above by some integer. Thus, we can define the *maximal Thurston-Bennequin number*

$$\overline{tb}(K) = \max\{tb(L)|L \text{ Legendrian knot, } k(L) = K\}.$$

By definition, $\overline{tb}$ is a knot invariant. Ng [14] has found $\overline{tb}$ for knots with 11 or fewer crossings. In general, we have

Theorem 3.2. (Khovanov Bound [12]) *Let K be a knot and $Kh_K(a, z)$ is the Khovanov polynomial of K, then*

$$\overline{tb}(K) \leq \min deg_q Kh_K(q, t/q),$$

where Kh_K is the Poincaré polynomial for $\mathfrak{sl}_2$ Khovanov homology (see [12] for more information). The Khovanov polynomial of K can be calculated, given a braid presentation of K, by *Mathematica* package `KnotTheory'` (for more detail, see [2]).

This property of Legendrian knots allow us to write a diagram illustrating the possible tb and r of Legendrian knots of topological type K as shown in Figure 13 for knot type 3_1. Each point indicates a unique Legendrian knot in the position corresponding to its tb and

r. Each edge corresponds to a stabilization where edge with negative (or positive) slope corresponds to negative (or positive) stabilization. As in [7], we shall call this diagram a *mountain range* which associate to a topological knot type. Note that mountain ranges for most knot types are unknown prior to this paper, even for the ones with small number of crossings such as 6_2 and 6_3.
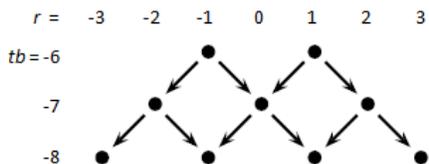
Figure 13: A mountain range for knot type 3_1

For a topological (or Legendrian) knot K, we define a *double* of K to be a link with two components contained in a tubular neighborhood of K such that each component is equivalent to K. We call the double with linking number m an *m-framed* double, denoted by $D_m(K)$. Given a front diagram D, we can construct a double of D by creating a copy of D and shift it slightly upward as shown in Figure 14. For a Legendrian knot L, this construction will give a $tb(L)$-framed double of L, which we shall simply denote $D(L)$.
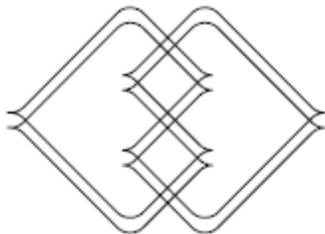
Figure 14: A double of a knot

Double of a knot is a useful tool to compute various invariants of knots and links. We shall see some examples later in this paper.

Similarly, given a topological knot K, we can define the *maximal self-linking number*

$$\overline{sl}(K) = \max\{sl(T) | T \text{ transverse knot, } k(T) = K\}.$$

Again, by definition, the maximal linking number is a knot invariant. By the result of Bennequin [3], $\overline{sl}(K) < \infty$ for every knot type K.

A topological knot K is called *transversely simple* if every transverse knot of knot type K can be completely classified by self-linking number. In other words, all transverse knots

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of knot type K with the same self-linking number are transverse isotopic. If a topological knot is not transversely simple, it is called *transversely non-simple*.

Several knot types such as unknot, the figure-eight knot and torus knots are proved to be transversely simple [8]. However, it is generally difficult to find an example of transversely non-simple knot. Although there are previously found and proved transversely non-simple knots [15], many of such examples are constructed using braid theory or convex surface theory. Most of them belong to knot types with large crossing numbers and are represented by large grid diagrams.

4 Grid Diagrams

The main goal of this paper is to classify Legendrian knots and links as it has been done on topological ones. In order to do so, we need to be able to find all such knots and links within some constraints. However, a front diagram of a Legendrian link is not suitable to be generated by computer. The three Legendrian Reidemeister moves are also hard to be implemented and recognized. Here, we introduce another way to represent knots and links which is suitable not only for being represented as data structure but also for manipulation and computation of invariants.

Definition 4 ([16]). A *grid diagram* with *grid number* (or *size*) n is an $n \times n$ square grid with n X's and n O's placed in distinct squares, such that each row and each column contains exactly on X and one O.

A grid diagram is determined by the positions of X's and O's in xy-coordinate as followed. Let the bottom-left (or southwest-SW) corner of the diagram be the origin and the top-right (or northeast-NE) corner be (n, n) (Figure 15). We call the box with its NE corner at the position (i, j), the box (i, j). Now, the grid diagram is represented by two permutations of $\{1, \ldots, n\}$, $X = (\pi_1(1), \ldots, \pi_1(n))$ and $O = (\pi_2(1), \ldots, \pi_2(n))$, where X's are at boxes $(i, \pi_1(i))$ and O's are at boxes $(j, \pi_2(j))$.

We construct a knot (link) diagram from a grid diagram by connecting every O to X horizontally and X to O vertcally. At each crossing, a vertical segment goes over a horizontal one (Figure 15). By convention, we obtain a front diagram by rotating a knot diagram $45°$ counterclockwise, smoothing NE and SW corners, and changing NW and SE corners into cusps.

We may define the *arc index* of a knot K, denoted $\alpha(K)$ to be the minimal grid number over all grid diagrams that represent K. By definition, it is a knot invariant. Jin, Kim and Lee [11] enumerated all prime knots with arc index up to 10.

There are 3 types of moves on grid diagram called the *Cromwell moves*, namely, translation, commutation, and stabilization/destabilization. We shall define each type of moves as follows.

Translation consists of vertical and horizontal translation. Vertical translation moves the top most row of the diagram to the bottom of the diagram or vice versa while leaving the
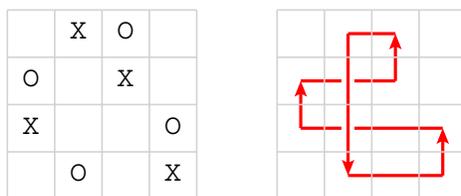
(a) unknot
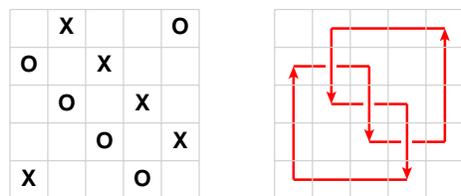
(b) trefoil

Figure 15: Grid Diagrams and Corresponding Knot Diagrams

rest unchanged. Horizontal translation, similarly, moves the leftmost column of the diagram to the rightmost or vice versa. We can consider a class of grid diagrams modulo translation as a diagram drawn on a torus.

Commutation interchanges two adjacent rows or two adjacent columns satisfying the following conditions.

1. To commute two rows (or columns), four X's and O's in those rows (or columns) must lie in different columns (or rows).

2. Line segments connecting X's and O's in the rows (or columns) must be either nested, or disjoint (see Figure 16).

Lastly, *destabilization* replaces a 2×2 subgrid containing two X's and one O's (or two X's and one O's) by a 1×1 subgrid containing O (or X), and results in a grid diagram with one fewer row and one fewer column as in Figure 17. *Stabilization* is simply an inverse process of destabilization. Hence, there are 8 types of (de)stabilization, denoted by X:NW, X:NE, X:SW, X:SE, O:NW, O:NE, O:SW, and O:SE. The first part corresponds to the letter in a 1×1 subgrid, and the second part corresponds to the position of the empty box in 2×2 subgrid. Note that one can write any O (de)stabilization by a composition of translations, commutations and exactly one X (de)stabilization of the diagonally opposite position (e.g. NE is diagonally opposite to SW). Thus, we may consider only X (de)stabilization as a type of Cromwell moves.

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(a) nested   (b) disjoint
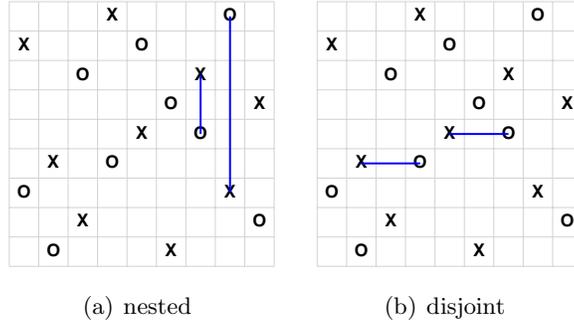
Figure 16: Commutation

$\leftrightarrow$
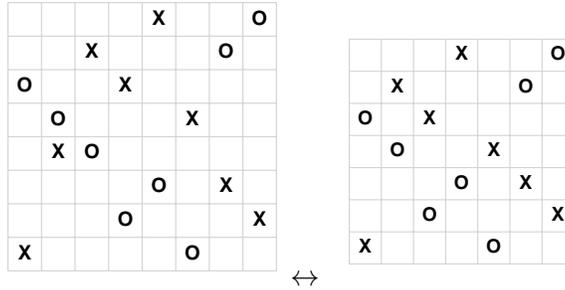
Figure 17: Destabilization

Let $\mathcal{G}$ denote the set of all grid diagrams, $\mathcal{L}$ the set of all oriented Legendrian links up to Legendrian isotopy, $\mathcal{T}$ the set of all oriented transverse links up to transveral isotopy, $\mathcal{K}$ the set of all oriented links up to topological isotopy, and let $\widetilde{\mathcal{G}}$ denote the quotient of $\mathcal{G}$ by translation and commutation. Ng and Thurston have shown in [16] that there are bijections

$$\widetilde{\mathcal{G}}/(X:NE, X:SW) \;\rightarrow\; \mathcal{L}$$
$$\widetilde{\mathcal{G}}/(X:NE, X:SW, X:SE) \;\rightarrow\; \mathcal{T}$$
$$\widetilde{\mathcal{G}}/(X:NE, X:SW, X:SE, X:NW) \;\rightarrow\; \mathcal{K}$$

induced by the above construction of a front diagram from a grid diagram.

Besides three types of Cromwell moves, we shall define a few other significant operations on grid diagrams. We define the symmetry of grid diagram which reflects the diagram about the NW-SE diagonal and interchanges X's and O's to be S_2. Ng and Thurston [16] show that S_2 operation on a grid diagram preserves Legendrian isotopy.

Given a Legendrian link L, we may define a *Legendrian mirror* of L, denoted $\mu(L)$, to

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be a Legendrian link represented by the front diagram which is the reflection of the front diagram of L in horizontal axis. This operation preserves topological type but generally do not preserve Legendrian isotopy. Let $-L$ be a Legendrian link which has the same front diagram as L but with orientation reverse. These two operations can easily be defined on grid diagram by rotating the grid diagram by $180°$ and interchanging X's and O's, respectively.

O			X		
		O		X	
	X		O		
		X			O
X				O	
	O				X

(a) original diagram

X				O	
		O			X
	X		O		
O		X			
	O			X	
			X		O

(b) S_2

X			O		
	O				X
O			X		
		O		X	
	X		O		
		X			O
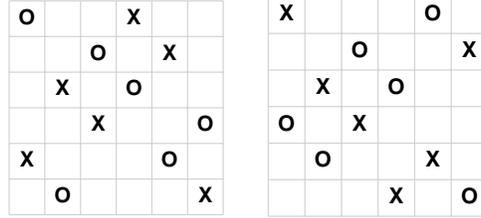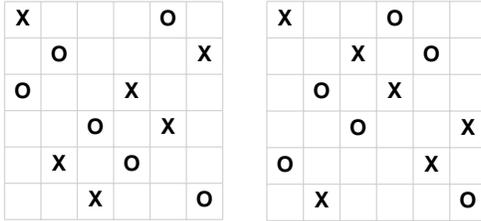
(c) Legendrian mirror

X			O		
		X		O	
	O		X		
			O		X
O				X	
	X				O

(d) orientation reversal

Figure 18: Operation on grid diagrams

In the case of representing unoriented links by grid diagrams, X and O are interchangeable. Hence we may replace all X's and O's by the same symbol, says '1'. We replace all empty grids by '0' and obtain a matrix with only 0's and 1's, called *Cromwell matrix*. A Cromwell matrix has exactly two 1's in each row and each column. Consisting of binary numbers, a Cromwell matrix is the most convenient form of a link to be implemented in programming. Similarly, we can construct a knot (link) diagram of an unoriented knot (link) from a Cromwell matrix by connecting 1's in the same row or the same column such that a vertical segment goes over a horizontal one. We can also turn the diagram $45°$ counter-clockwise to get an unoriented Legendrian front diagram.

We may define translation, commutation, and (de)stabilization on Cromwell matrices analogous to those on grid diagrams but ignoring the difference between X and O. We say two Cromwell matrices are *equivalent* if they relate by a sequence of translations, commutations and $X : NE$, $X : SW$ (de)stabilizations. This definition is well-defined even though we do not distinguish X and O because $O : NE$ and $O : SW$ can be written as $X : SW$ and $X : NE$, respectively. In other word, two Cromwell matrices are equivalent if they represent

two unoriented Legendrian isotopic links.

Note that a stabilization increases the size of a Cromwell matrix by one. Thus, for any Cromwell matrix of size n, there are equivalent Cromwell matrices of size $n+1, n+2, \ldots$. Since we will generate links by their size and we only distinguish them up to Legendrian isotopy, links of the same type will show up again while we generate matrices of larger sizes. To prevent this reoccurence of links of the same type, we introduce a notion of irreducibility, which will reduce redundancy in our algorithm.

Definition 5 ([11])**.** A Cromwell matrix is *reducible* if two 1's in a row or a column are adjacent up to a cyclic permutation of entries.

If a Cromwell matrix is not reducible, we say it is *irreducible*. It is easy to implement an algorithm to check whether a given Cromwell matrix is reducible. A reducible Cromwell matrix represents a grid diagram which can be destabilized. Hence, an $n \times n$ reducible Cromwell matrix does not represent a topological link with arc index n. In other words, if a Cromwell matrix of size n represents a topological link with arc index n, then any Cromwell matrix in the same equivalence class has size at least n. In particular, any Cromwell matrix obtained from such matrix by a sequence of translations and commutations is irreducible. However, it is not necessarily true that if every matrix obtained from a Cromwell matrix of size n by a sequence of translations and commutations is irreducible, then the Cromwell matrix represents a topological link with arc index n.

5   Algorithms

Our classification of Legendrian knots and links will be done for each grid number at a time. From the previous section, there is a 2^n-to-one correspondence between grid diagrams and Cromwell matrices, where n is the number of components ($n = 1$ for knots). Furthermore, the correspondence preserves equivalence class under translation, commutation and X:NE, X:SW (de)stabilization. Thus, we can obtain all irreducible grid diagrams by generating all irreducible Cromwell matrices. Although the uniqueness up to Lengendrian isotopy may not be preserved, comparing as Cromwell matrices greatly reduces work to be done to grid diagrams as there are fewer Cromwell matrices and fewer possible moves.

For each grid number n, the classification is done in three main steps.

1. Generating all Cromwell matrices, unique up to translation and commutation

2. Reducing Cromwell matrices to be unique up to unoriented Lengendrian isotopy

3. Reducing grid diagrams to be unique up to oriented Lengendrian isotopy

5.1 Generating all Cromwell matrices, unique up to translation and commutation

We modify the algorithm used to classify topological knots by Jin, Kim and Lee in [11]. First, we introduce a way to order Cromwell matrices by assigning a distinct integer to each Cromwell matrix.

Definition 6 ([11]). The *norm* of an $n \times n$ Cromwell matrix is the natural number corresponding to the binary number obtained by concatenating its rows.

For example, the norm of

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

is $10010010011010001010001012_2$.

This definition of norm allows us to generate Cromwell matrices in norm-decreasing order. Beginning with an 'empty' matrix with 0 in every entry, we can create a Cromwell matrix by replacing $2n$ 0's with 1's, one at a time. We start from the left 1 in the top row, the right 1, then the left 1 in the second row, and so on. We can reduce number of possible choices of placing 1's by taking translation, commutation and reducibility into account.

As our goal is to generate irreducible Cromwell matrices, we shall not place two 1's next to each other vertically or horizontally. Moreover, we can eliminate the choice of placing in such a way that we can easily get a reducible matrix by translations or commutations. In addition, we can also omit the choice that norm may increase by translations or commutations. Thus, we require that all Cromwell matrices we generate must have a 1 at the leftmost column of the first row. We shall use tree structure to represent each step of placing 1 to an empty matrix (Figure 19). We let the matrix with only one 1 be the root of the tree (depth 0). By the previous constraint, there are $\lceil \frac{n-3}{2} \rceil$ possible positions to place the second 1 without making the matrix reducible and also avoiding redundancy. Hence, there are $\lceil \frac{n-3}{2} \rceil$ nodes in the next level (depth 1) of the tree. The third 1 cannot be placed right below any of the previous ones. The fourth 1 must be as far from the third as the distance between the first two 1's. Otherwise, we can translate them to the first row and get larger norm. By taking these conditions into account, we similarly generate nodes in the fifth, the sixth, and so on, until we reach the last level (depth $2n - 1$). Only the nodes in the last level are complete as Cromwell matrices.

For each Cromwell matrix generated in this way, we check and eliminate the matrix if it has one of the following properties:

1. the matrix represents link with incorrect number of components (more than 1 if we only consider knots)

$$\begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ & & & & \\ & & & & \end{bmatrix}$$

$\swarrow \qquad\qquad \searrow$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & & & \\ & & & & \\ & & & & \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ & & & & \\ & & & & \end{bmatrix}$$

$\downarrow \qquad\qquad \searrow$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ & & & & \\ & & & & \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ & & & & \\ & & & & \end{bmatrix}$$

$\downarrow \qquad\qquad \searrow \qquad\qquad \vdots$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & & \\ & & & & \end{bmatrix}$$

$\vdots \qquad\qquad \vdots$
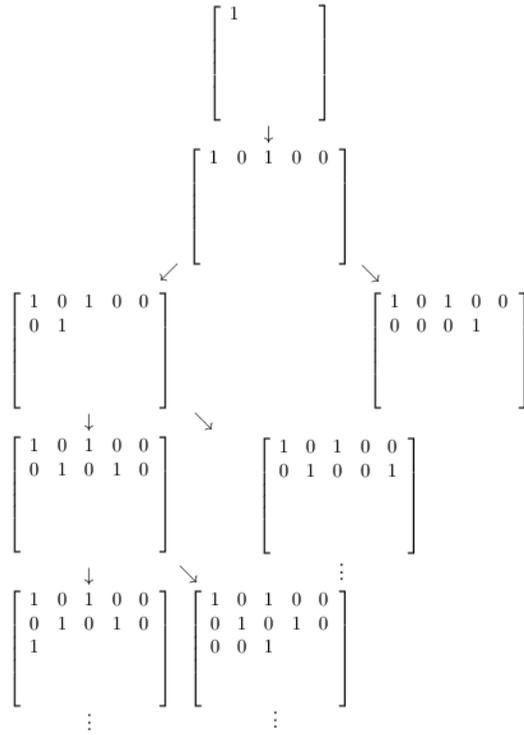
Figure 19: Tree structure for generating Cromwell matrices of size 5

2. translations and/or commutations increase its norm

3. it becomes reducible after a sequences of translations and/or commutations

Cromwell matrices left from this algorithm with be unique up to translations and commutations, and have the highest norm among those in the same equivalence class.

5.2 Reducing Cromwell matrices to be unique up to unoriented Lengendrian isotopy

Now we are ready to classify unoriented Legendrian links generated in the previous step. Recall that any two Legendrian links are equivalent if and only if they relate by translations, commutations and X:NE, X:SW (de)stabilizations. We simply have an algorithm to determine whether two Cromwell matrices represent the same Legendrian links by repeatedly apply the corresponding moves to one matrix until it to become another one. However, this algorithm is indeterministic. That is, if the two matrices represent inequivalent Legendrian

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links, the algorithm does not halt and goes on forever. Even though the two matrices represent the same Legendrian link, we do not know how many moves it may take to transform one to the other. Although there is no simple known deterministic algorithm to determine whether two Legendrian links are equivalent, we can improve this simple algorithm to aid us in making conjectures in the classification.

We can consider this problem as a path searching problem on a graph: let each grid diagram (or Cromwell matrix) be represented by a vertex, and two vertices are connected if they represent grid diagrams that relate by one step of those moves. Since each Cromwell move is reversible, the graph is thus undirected. Hence, the algorithm to determine whether two grid diagrams represent the same Legendrian links is equivalent to the algorithm to determine whether there is a path connecting two vertices on a given graph (with infinite number of vertices). Hence, we may apply any path searching algorithm to answer this question.

Generally, a simple path searching algorithm will start at an initial vertex and search along each edge connected from the initial vertex until reaching the goal. Theoretically, this algorithm should be able to determine whether there exists a path connecting the given two vertices. However,

1. a graph representing all possible grid diagrams is infinitely large as we can always stabilize a grid diagram to get a larger one

2. the graph is not given but need to be generated while searching

3. degree at each vertex is very high.

Since we have limited computing resource, we have to optimize our algorithm in every possible way.

Instead of regular search algorithm, we use bidirectional search algorithm to determine whether there exists a path connecting two vertices. This algorithm is appropriate in our circumstance because the moves are reversible and similar in both directions, and goal state is clear. Note again that this algorithm is undecidable. In the case that two diagrams are not related by any number of moves, i.e. they represent different Legendrian links, this algorithm does not halt. Hence, we need to specify number of steps it will compute. As a result, this algorithm may take very long time to check all possible pairs of grids we have generated if we allow a large number of step.

To reduce the time this algorithm will take to compare a pair of grid diagrams or Cromwell matrices, we use several techniques to optimize the algorithm. Firstly, since translation is the easiest move to be implemented and relatively fast compared to other types of Cromwell moves, every Cromwell matrix (or grid diagram) will be translated to the one with maximal norm among those obtained by translations. This will significantly reduce number of vertices in the graph and allow each step to obtain more variety of other Cromwell matrices (or grid diagrams).

We can also reduce number of steps the algorithm would take to get from one Cromwell matrix (or grid diagram) to another by allowing non-Cromwell moves that preserve Legendrian isotopy. By doing so, although each vertex will have higher degree, number of edges in the graph will be reduced significantly. Thus, for each vertex representing a Cromwell matrix (or grid diagram) D, we generate adjacent vertices as follows:

1. all Cromwell matrices (or grid diagrams) generated by applying one commutation to D, do not exist if D does not contain nested or disjoint rows or columns

2. all Cromwell matrices (or grid diagrams) generated by applying one X:NE, X:SW stabilization at the top left box in D, which we assume to be 1 (for Cromwell matrix) (or O for grid diagram, in which case we will use O:NE and O:SW)

3. all Cromwell matrices (or grid diagrams) generated by applying one X:NE, X:SW destabilization to D, do not exist if D is not reducible

4. a Cromwell matrix (or grid diagram) generated by applying S_2 to D

5. all Cromwell matrices (or grid diagrams) generated by applying the speed up move to D, may not exist if D does not contain the specific pattern

where the speed up move is defined locally as in Figure 20 together with its rotation reverse and 180° rotation. Note that this constitutes a Legendrian isotopy since it only involves X:NE and X:SW (equivalently, O:SW and O:NE) (de)stabilization.
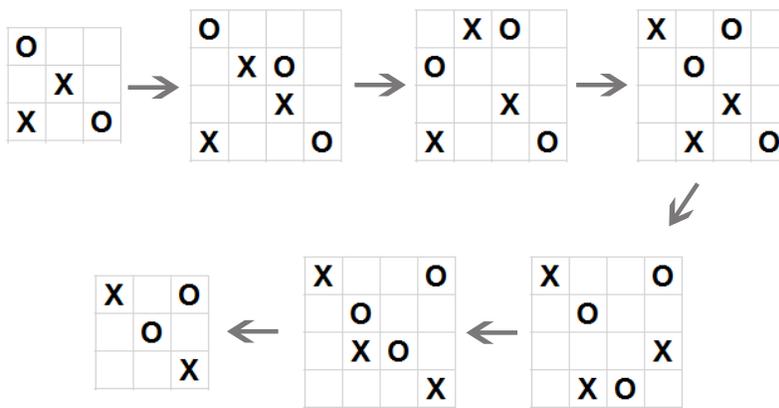
Figure 20: Speed up move

We call any one of the above a *step*. Note that we only apply a stabilization to one box in D since we can move it to any other places using commutations. We choose the top left box because it will always be 1 for Cromwell matrix or O for grid diagram if D has maximal norm up to translation.

Let `IsLegendrianIsotopic`(d, s) be the algorithm to check whether two Cromwell matrices (or grid diagrams) represent the same Legendrian link while allowing no more than d steps and the grid size is never more than s. We can describe `IsLegendrianIsotopic`$(d, s)(D_1, D_2)$ as follows:

1. begin with two sets of Cromwell matrices (or grid diagrams), $A = \{D_1\}$ and $B = \{D_2\}$; also let $A' = \emptyset$ and $B' = \emptyset$

2. for each $D \in A$,

   - move D to A'
   - for each D', a Cromwell matrix (or grid diagram) generated from D in one step, put D' in A if D' is not in A' and size of D' is at most s

3. check whether $A \cap B = \emptyset$; if not, halt and return true

4. repeat the two previous steps with A and B switched

5. repeat the last three steps d times or until this algorithm halts.

Since there are finitely many grid diagrams of size at most s, `IsLegendrianIsotopic`(d, s) must halt for any $d \in \mathbb{N} \cup \{\infty\}$.

If D_1, D_2 are grid diagrams, by allowing $X : SE$ (de)stabilization, this algorithm can be applied to the problem of determining whether they represent the same transverse link. Similarly, by allowing both X:SE and X:NW (de)stabilization, this algorithm can also be applied to the problem of determining whether two Cromwell matrices (or grid diagrams) represent the same topological link.

In order to reduce the number of comparison, we can use other methods to eliminate pairs of diagrams which do not represent the same Legendrian knots. These methods must be a lot faster than directly comparing. Classical invariants such as rotation number (r) and Thurston-Bennequin number (tb) are easy to compute and effective in decreasing number of pairs the actual comparison required since we do not need to check pairs with different r or tb. We only use these two invariant for arc index up to 8. For arc index 9, we also check for topological knot type of each grid diagrams as another invariant since several knot types have the same tb.

Instead of comparing two diagrams at a time, we can also improve our bidirectional search algorithm to check for any number of grid diagrams by beginning with each set containing each diagram we consider and comparing them all at once. For larger arc index, there are many grid diagrams which classical invariants cannot classify. As it takes longer to generate new grid diagrams, comparing many grid diagrams at once can make the process significantly faster.

5.3 Reducing grid diagrams to be unique up to oriented Lengendrian isotopy

The previous step gives us a set of Cromwell matrices that potentially represent distinct unoriented Legendrian knots. However, our goal is to classify oriented Legendrian knots. We then need to assign orientations to each unoriented Legendrian knots represented by Cromwell matrices. For a knot, which is a one-component link, this step is simply to assign the 1 at the left upper corner to be either O or X as the rest will be completely determined by the first one. Generally, for a link with n components, we get 2^n oriented links by assigning O or X to one box for each of the n components. These generated grid diagrams may relate by a sequence of Cromwell moves and thus represent knots that are Legendrian isotopic.

We can repeat the above algorithm on grid diagrams representing Legendrian knots that we cannot simply distinguish by classical invariants. The reason that we apply this algorithm both before and after assigning orientation is to reduce number of comparison that need to be made. Note that if two unoriented Legendrian knots cannot be shown to be the same by the algorithm `IsLegendrianIsotopic`(∞, s), they still cannot be shown to be the same after being assigned orientations. However, this may not be true for `IsLegendrianIsotopic`(d, s) for some finite d because the top left corner of an oriented diagram is always set to be O and the stabilization takes place at that box for knot case. Thus, if the top left corner of an unoriented diagram (a Cromwell matrix) is assgined to be X when given orientation, it will get translated so that the box at top left corner becomes O and therefore alter the position where stabilization will occur.

6 Distinguish Legendrian Knots

Remind that our algorithm cannot show that two Legendrian knots are not Legendrian isotopic. The three classical Legendrian invariants alone are not enough to classify Legendrian knots, even for small ones such as type $m(5_2)$. In this section, we will use several non-classical invariants to distinguish Legendrian knots which share topological type, tb and r.

We use the *Mathematica* notebook `Legendrian invariants.nb` from [18] to compute polynomials for graded ruling invariant and linearized contact homology. In `Legendrian invariant.nb`, the graded ruling invariant of a Legendrian knot L is written as a polynomial $\sum_i a_i z^i$ where a_i is the number of graded rulings R with $\gamma(R) = i$. Note that these two Legendrian invariants are defined only on Legendrian knots with $r = 0$. Thus, they cannot be used to distinguish several pairs of Legendrian knots such as two knots of type 6_3. They are also invariant under orientation reversal and Legendrian mirror, and hence cannot be used to distinguish pairs of Legendrian knots that are orientation reversal and Legendrian mirror of each other.

Despite such limitation, graded ruling invariant and linearized contact homology are very useful and can be used to classify several pairs of Legendrian knots. The Legendrian

knots that can be distinguished using graded ruling invariant are those of type $m(5_2)$, $m(7_2)$, 7_3, 7_4, $m(7_5)$, $m(7_6)$, $m(9_{45})$, 9_{48}, 9_{49}, 10_{128}, 10_{136}, 10_{142} and 10_{160}. By using linearized contact homology, we can further distinguish pairs of type $m(6_1)$ and $m(7_2)$ that cannot be done using graded ruling invariant though they can be distinguished using ρ-graded ruling invariants for some integer ρ.

We can see from table 1 that we can use linearized contact homology to distinguish every pair that we can do by graded ruling invariant. However, it is still useful to mention graded ruling invariant as it is easier to compute by hand, and it is not true in general that any pair of knot distinguished by graded ruling invariant can be distinguished by linearized contact homology.

Furthermore, Ng [13] provides a technique using the Chekanov-Eliashberg differential graded algebra (DGA) which can be used to distinguish between several pairs that cannot be distinguished by any invariants mentioned earlier. In his paper, Ng distinguishes

1. two Legendrian knots of type 6_2 with $(tb, r) = (-7, 0)$ which correspond to the first grid diagram of type 6_2 in the table and its Legendrian mirror

2. two Legendrian knots of type 6_3 with $(tb, r) = (-4, 1)$ which correspond to two grid diagrams of type 6_3 in the table

3. two Legendrian knots of type $m(7_2)$ with $(tb, r) = (1, 0)$ which correspond to the second and the fourth grid diagrams of type $m(7_2)$ in the table

4. two Legendrian knots of type 7_4 with $(tb, r) = (1, 0)$ which correspond to the second and the third grid diagrams of type 7_4 in the table.

The Chekanov-Eliashberg DGA can also be used to distinguish the first and the third grid diagrams of type 7_4, and the third grid diagrams of the same type with its Legendrian mirror.

As we mentioned earlier, we can view Legendrian knot as a subtype of transverse knot. In other words, if two knots are not transverse isotopic, they are not Legendrian isotopic. Ng, Ozsváth and Thurston [15] provide a technique using knot Floer homology, based on the work of Ozsváth, Szabo and Thurston [17], to distinguish several pairs of transverse knots. We can apply the same technique to distinguish some pairs of Legendrian knots in the table. In [15], two transverse knots, corresponding to the first grid diagram and its Legendrian mirror of type $m(10_{132})$ are distinguished. Since the second grid diagram is transversal isotopic to the first grid and Legendrian isotopic to its own mirror, it cannot be Legendrian isotopic to the first grid. Hence, we can conclude that 3 grid diagrams with knot type $m(10_{132})$ represent 3 distinct Legendrian knots. In the same manner, we can distinguish three Legendrian knots with $(tb, r) = (-1, 0)$ of type $m(10_{140})$. Moreover, we can identify transversely non-simple knots of type $m(10_{145})$, $m(10_{161})$ and $12n_{591}$, each of which lies in the second level (from above) in the mountain range of its type.

7 Legendrian Knot Atlas

The result of our algorithm together with invariants and techniques described in previous section gives the Legendrian knot atlas as shown in table 1. From our generated grid diagrams, we use gridlink [5] to draw diagrams, each of which can be rotated $45°$ to get a front diagram. Note that we do not show all grid diagrams of each knot type. However, readers can find all grid diagrams from the table by using Legendrian mirror and/or orientation reversal as described earlier. Note again that these two operations preserve graded ruling invariant and linearized contact homology which also depicted in the table. Since orientation reversal will negate r, we only illustrate Legendrian knots with nonnegative r. Again, readers can find grid diagrams for negative r by reversing orientation. We also provide the mountain range of each knot type in the table, some of which have been proven while the rest are conjecture drawn from the result of our algorithm and the previous section.

For columns labeled $L = -L?$, $L = \mu(L)?$ and $L = -\mu(L)?$, we use ✓ to indecate pairs of grid diagrams which our algorithm can show that they represent the same Legendrian knot. For example, the $tb = 1, r = 0$ Legendrian knot L of type $m(3_1)$ is Legendrian isotopic to its orientation reversal $(-L)$, its Legendrian mirror $(\mu(L))$ and the orientation reversal of its Legendrian mirror $(-\mu(L))$. We use ✗ to indicate pairs of grid diagrams which can be distinguished by some invariants and/or techniques described above. We use ✗? to indicate pairs of grid diagrams which our algorithm cannot show that they represent the same Legendrian knot, but cannot be distinguished by any invariants or techniques described above. The letters subscripted under some pairs of grid diagrams also indicate sets of Legendrian knots that cannot be distinguished by those invariants and techniques.

Since orientation reversal and Legendrian mirror negate rotation number, for grid diagrams with $r \neq 0$, orientation reversal and Legendrian mirror cannot be Legendrian isotopic to the original. We use $-$ to indicate such case. Similarly, since graded ruling invariant and linearized contact homology are not defined on grid diagrams with $r \neq 0$, such boxes are indicated by $-$. Note that $-$ is different from $\emptyset$ which is when $r = 0$ but there are no graded rulings or linearized contact homology.

In moutain range, we use a box containing more than one dot to indicate that there are (conjecturally if other dots are red) more than one Legendrian knot with the same tb and r. Number of black dots in the box corresponds to number of grid diagrams that can be proved to represent distinct Legendrian knots with that particular tb and r. Number of black and red dots together corresponds to number o grid diagrams that our algorithm cannot show that any pairs of them represent the same Legendrian knots. We also omitted axes of tb and r as they can be inferred from tb and r of the grid diagrams.

Our result agrees with mountain ranges of torus knots and the figure eight knot described by Etnyre and Honda in [8], and also agrees with mountain ranges of twist knots described by Etnyre, Ng and Vértesi in [9]. We indicate torus knots by $T(p, q)$ and twist knots by K_m (notation as in [9]) in "Note" column. Moreover, the result gives more conjectures for mountain ranges of all prime knots with arc index up to 9. Since we generated grid diagrams

up to size 10, all grid diagrams of size at most 10 of each knot type are shown in its mountain range. The black dots in the mountain range are the lower bound on what mountain range of each knot type is, i.e., the smallest possible mountain range. However, we conjecture the mountain ranges to consist of both black dots and red dots. It is still possible that the mountain ranges of some knot types are larger than we describe, but any Legendrian knots not shown in the mountain range, if exist, must be represented by grid diagrams of size 11 or higher.

The mountain range of each knot type also gives an information on transverse knots. We see from the previous section that classes of transverse knots are equivalent to classes of Legendrian knots modulo negative stabilization. Hence, we can read of classes of transverse knots of each knot type from its mountain range and relations of Legendrian classes under negative stabilization. This results in a transverse knot atlas as shown in table 2. We use similar notations as in the Legendrian knot atlas described above. Knot types that are shown to be transversely nonsimple are labeled in blue and knot types that we conjecture to be transversely nonsimple but cannot prove are labeled in red.

8   Legendrian Links

In the previous section, we focused the classification entirely on Legendrian knots and we left out all grid diagrams that represent links with more than one component in generating step of our algorithm. Note that, with little modification, the following step of the algorithm also works for links with more than one component. The main differences for links with n components are as follows:

1. there are 2^n different possible orientations for each unoriented link (Cromwell diagram) although some of them may be Legendrian isotopic

2. there are n positions for stabilization, one in each component

3. may need to keep track of two distinct components in the case that they cannot be switched topologically

Since each component of 2-component link can be considered a knot when we omit the other component, one can separately compute tb for each component. Let L be a topological link with two components L_1 and L_2. For each Legendrian link with topological type L, one can compute a pair of integers $(tb1, tb2)$ where tbi is the tb of the component corresponding to L_i. By taking convex hull of all such pairs in $\mathbb{R}^2$, we get a polytope called the *Thurston-Bennequin polytope* or *tb polytope* of L, denoted $\Delta(L)$ as in [1]. Figure 21 gives an example of tb polytope of $m(L4a1)$.

Note that if (a, b) is a pair of integers in $\Delta(L)$, then $(a - k, b - l)$ must be in $\Delta(L)$ for all $k, l \in \mathbb{N} \cup \{0\}$ since we can stabilize each component of L to decrease tb of that component without making any change on the tb of the other. Baader and Ishikawa [1] has shown that
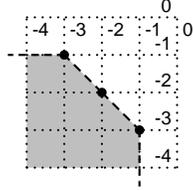
Figure 21: Thurston-Bennequin polytope of $m(L4a1)$

for two-bridge link L with two components L_1 and L_2

$$\Delta(L) = \{(x_1, x_2) \in \mathbb{R} | x_1 \leq -1, x_2 \leq -1, x_1 + x_2 \leq \overline{tb}(L) - 2lk(L_1, L_2)\}.$$

However, the tb polytope is not known in general case.

Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be a Legendrian link with two components $\mathcal{L}_1$ and $\mathcal{L}_2$. Let L be a topological link type of $\mathcal{L}$ with components L_1 and L_2 corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then it follows from the definition that

$$tb(\mathcal{L}) = tb(\mathcal{L}_1) + tb(\mathcal{L}_2) + 2lk(L_1, L_2).$$

Similarly, we may calculate

$$tb(\mathcal{L}_1 \cup D(\mathcal{L}_2)) = tb(\mathcal{L}_1) + 4tb(\mathcal{L}_2) + 4lk(L_1, L_2)$$

from the construction of double of L in previous section. For each value of $tb(\mathcal{L}_2)$, we can find a good bound for $tb(\mathcal{L}_1)$ by using Theorem 3.2 on $L_1 \cup D_{tb(\mathcal{L}_2)}(L_2)$. This is similar to the technique used in [14]. We have

$$tb(\mathcal{L}_1) \leq \overline{tb}(L_1 \cup D_{tb(\mathcal{L}_2)}(L_2)) - 4tb(\mathcal{L}_2) - 4lk(L_1, L_2).$$

We may repeat this calculation but double $\mathcal{L}_1$ instead of $\mathcal{L}_2$. The bound on both parts can be used to give an upper bound on $\Delta(L)$. However, the calculation of the Khovanov polynomial for very large links in `KnotTheory`` package sometimes fails due to running time or memory limitation.

Using our algorithm and invariants in previous sections, we get an atlas for unoriented Legendrian two-component links as shown in table 3. The atlas shows the representation of grid diagrams which can be turned 45° to get unoriented front diagrams. In the case when two components cannot be switched topologically, we also distinguish them by color and size of strands. We also provide our conjecture for tb polytope of each link type. Note that

our result can answer the question posted by Baader and Ishikawa in [1] that whether the tb polytope can always be described by three linear inequalities of the type

$$tb(\mathcal{L}_1) \leq n_1, \ tb(\mathcal{L}_2) \leq n_2, \ tb(\mathcal{L}_1) + tb(\mathcal{L}_2) \leq n_3.$$

Counterexamples illustrated on table 3 are links that cannot be described by three linear inequalities, $m(L7a1)$, $L9n18$, $m(L10n54)$ and $L11n204$, and links that can be described by three linear inequalities but not of the above type, $L9n19$, $L10n24$ and $m(L11n205)$.

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Table 1: Legendrian Knots up to Arc Index 9

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
3_1		$(-6, 1)$	-	-	✓	-	-	$T(2, -3)$, K_1
$m(3_1)$		$(1, 0)$	✓	✓	✓	$2 + z^2$	$2 + t$	$T(2, 3)$, K_{-2}
$4_1 = m(4_1)$		$(-3, 0)$	✓	✓	✓	1	$t^{-1} + 2t$	$K_2 = K_{-3}$
5_1		$(-10, 1)$	-	-	✓	-	-	$T(2, -5)$
		$(-10, 3)$	-	-	✓	-	-	
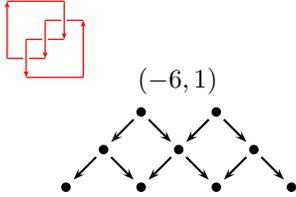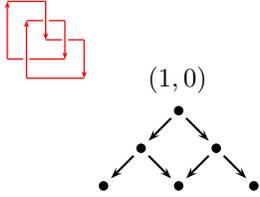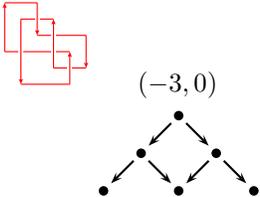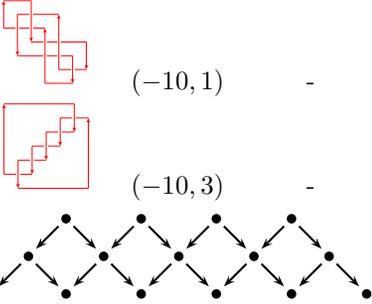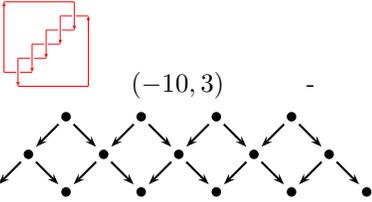
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(5_1)$		$(3, 0)$	✓	✓	✓	$3 + 4z^2 + z^4$	$4 + t$	$T(2,5)$
5_2		$(-8, 1)$	-	-	✓	-	-	K_3
$m(5_2)$		$(1, 0)$	✓	✓	✓	1	$t^{-2} + t + t^2$	K_{-4}
		$(1, 0)$	✓	✓	✓	$1 + z^2$	$2 + t$	
6_1		$(-5, 0)$	✓	✓	✓	1	$2t^{-1} + 3t$	K_4
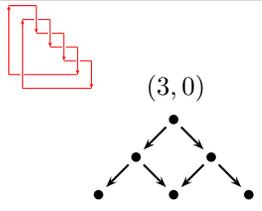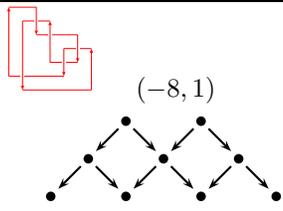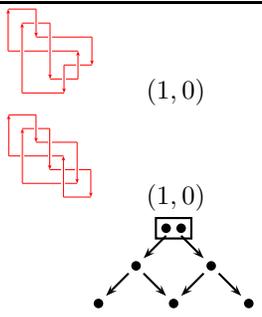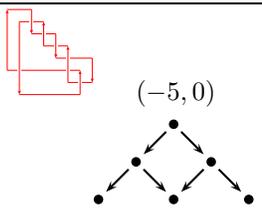
†

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(6_1)$		$(-3, 0)$	✓	✓	✓	1	$t^{-3} + t + t^3$	K_{-5}
		$(-3, 0)$	✓	✓	✓	1	$t^{-1} + 2t$	
6_2		$(-7, 0)$	✓	✗	✗	$\emptyset$	$\emptyset$	
	a	$(-7, 2)$	-	-	✓	-	-	
	a	$(-7, 2)$	-	-	✓	-	-	
$m(6_2)$		$(-1, 0)$	✓	✓	✓	$2 + z^2$	$t^{-1} + 2 + 2t$	
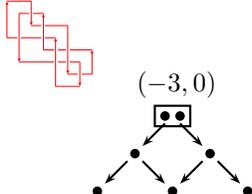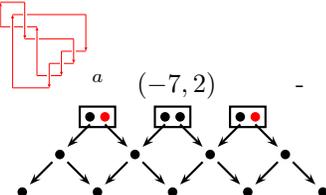
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$6_3 = m(6_3)$		$(-4, 1)$	-	-	✓	-	-	
		$(-4, 1)$	-	-	✓	-	-	
7_1		$(-14, 1)$	-	-	✓	-	-	$T(2, -7)$
		$(-14, 3)$	-	-	✓	-	-	
		$(-14, 5)$	-	-	✓	-	-	
$m(7_1)$		$(5, 0)$	✓	✓	✓	$4 + 10z^2 + 6z^4 + z^6$	$6 + t$	$T(2, 7)$

Knot Type	Cromwell Diagram	(tb,r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
7_2		$(-10,1)$	-	-	✓	-	-	K_5
$m(7_2)$		$(1,0)$	✓	✓	✓	1	$t^{-4}+t+t^4$	K_{-6}
		$(1,0)$	✓	✓	✓	$1+z^2$	$2+t$	
		$(1,0)$	✗	✗	✓	1	$t^{-2}+t+t^2$	
		$(1,0)$	✓	✓	✓	$1+z^2$	$2+t$	
7_3		$(3,0)$	✓	✓	✓	1	$2t^{-2}+t+2t^2$	
		$(3,0)$	✓	✓	✓	$1+3z^2+z^4$	$4+t$	
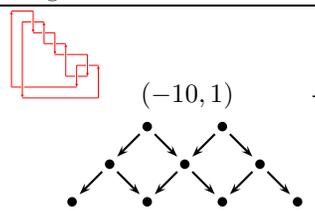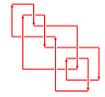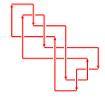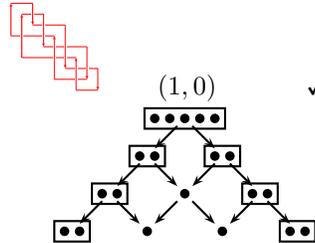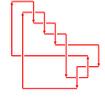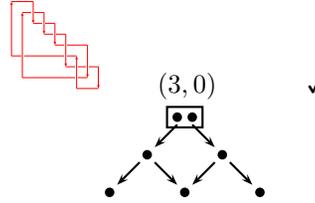
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(7_3)$		$(-12, 1)$	-	-	✓	-	-	
		$(-12, 3)$	-	-	✓	-	-	
7_4	b	$(1, 0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
	b	$(1, 0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
		$(1, 0)$	✓	✗	✗	$\emptyset$	$\emptyset$	
		$(1, 0)$	✓	✓	✓	z^2	$2 + t$	
$m(7_4)$		$(-10, 1)$	-	-	✓	-	-	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
7_5		$(-12, 1)$	-	-	✓	-	-	
		$(-12, 1)$	-	-	✓	-	-	
		$(-12, 1)$	-	-	✓	-	-	
		$(-12, 3)$	-	-	✓	-	-	
$m(7_5)$		$(3, 0)$	✓	✓	✓	$2 + z^2$	$t^{-2} + 2 + t + t^2$	
		$(3, 0)$	✓	✓	✓	$2 + 3z^2 + z^4$	$4 + t$	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
7_6	d	$(-8, 1)$	-	-	✓	-	-	
	d	$(-8, 1)$	-	-	✓	-	-	
	d	$(-8, 1)$	-	-	✗?	-	-	
$m(7_6)$	e	$(-1, 0)$	✓	✓	✓	$1 + z^2$	$t^{-1} + 2 + 2t$	
	e	$(-1, 0)$	✗?	✗?	✓	$1 + z^2$	$t^{-1} + 2 + 2t$	
		$(-1, 0)$	✗?	✗?	✓	1	$t^{-2} + t^{-1} + 2t + t^2$	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
7_7	f	$(-4, 1)$	-	-	✗?	-	-	
	f	$(-4, 1)$	-	-	✓	-	-	
	f	$(-4, 1)$	-	-	✓	-	-	
$m(7_7)$	g	$(-5, 0)$	✓	✓	✓	1	$2t^{-1} + 3t$	
	g	$(-5, 0)$	✓	✓	✓	1	$2t^{-1} + 3t$	
8_{19}		$(5, 0)$	✓	✓	✓	$5 + 10z^2 + 6z^4 + z^6$	$6 + t$	$T(3, 4)$
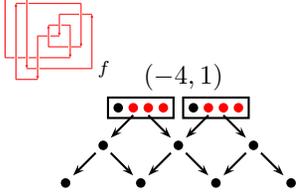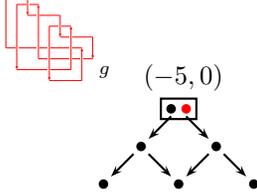
Knot Type	Cromwell Diagram	(tb,r)	$L=-L$?	$L=\mu(L)$?	$L=-\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(8_{19})$		$(-12,1)$	-	-	✓	-	-	$T(3,-4)$
8_{20}		$(-6,1)$	-	-	✓	-	-	
$m(8_{20})$		$(-2,1)$	-	-	✓	-	-	
8_{21}	h	$(-9,0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
	h	$(-9,0)$	✓	✗?	✗?	$\emptyset$	$\emptyset$	
		$(-9,2)$	-	-	✓	-	-	
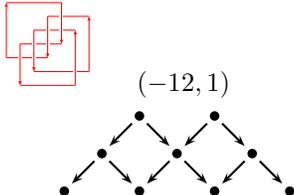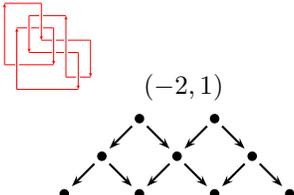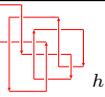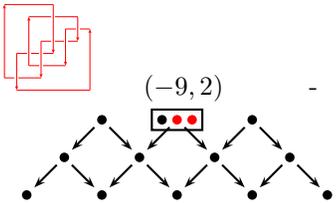
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(8_{21})$		$(1, 0)$	✓	✓	✓	$3 + 2z^2$	$2 + t,\ t^{-1} + 4 + 2t$	
9_{42}		$(-3, 0)$	✓	✗?	✗?	$2 + z^2$	$2t^{-1} + 2 + 3t$	
$m(9_{42})$		$(-5, 0)$	✗?	✗?	✓	$\emptyset$	$\emptyset$	
9_{43}		$(1, 0)$	✗?	✗?	✓	$3 + 4z^2 + z^4$	$t^{-1} + 4 + 2t$	
$m(9_{43})$		$(-10, 1)$	-	-	✗?	-	-	
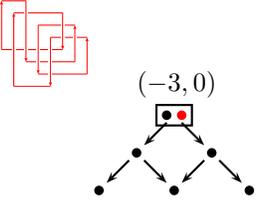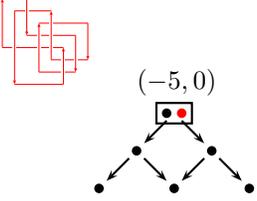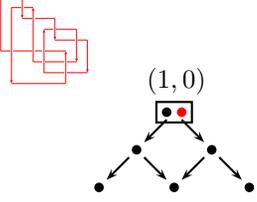
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
9_{44}	i	$(-6, 1)$	-	-	✗?	-	-	
	i	$(-6, 1)$	-	-	✗?	-	-	
	i	$(-6, 1)$	-	-	✗?	-	-	
$m(9_{44})$		$(-3, 0)$	✗?	✓	✗?	1	$t^{-1} + 2t$	
9_{45}	j	$(-10, 1)$	-	-	✗?	-	-	
	j	$(-10, 1)$	-	-	✓	-	-	
	j	$(-10, 1)$	-	-	✗?	-	-	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(9_{45})$		$(1, 0)$	✗?	✗?	✗?	$2 + 2z^2$	$2 + t,\ t^{-1} + 4 + 2t$	
		$(1, 0)$	✓	✗?	✗?	$2 + z^2$	$2 + t,\ t^{-2} + t^{-1} + 2 + 2t + t^2$	
9_{46}		$(-7, 0)$	✓	✓	✓	1	$3t^{-1} + 4t$	
$m(9_{46})$		$(-1, 0)$	✓	✓	✓	2	t	
9_{47}		$(-2, 1)$	-	-	✓	-	-	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(9_{47})$		$(-7, 0)$	✗?	✗?	✓	1	$3t^{-1} + 4t$	
9_{48}	k	$(-1, 0)$	✓	✓	✓	z^2	$t^{-1} + 2 + 2t$	
	k	$(-1, 0)$	✗?	✗?	✓	z^2	$t^{-1} + 2 + 2t$	
	l	$(-1, 0)$	✗?	✗?	✓	$\emptyset$	$\emptyset$	
	l	$(-1, 0)$	✓	✗?	✗?	$\emptyset$	$\emptyset$	
$m(9_{48})$		$(-8, 1)$	-	-	✓	-	-	
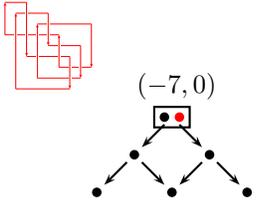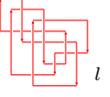
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
9_{49}		$(3, 0)$	✓	✗?	✗?	$\emptyset$	$\emptyset$	
		$(3, 0)$	✓	✓	✓	$2z^2 + z^4$	$4 + t$	
$m(9_{49})$		$(-12, 1)$	-	-	✓	-	-	
10_{124}		$(7, 0)$	✓	✓	✓	$7 + 21z^2 + 21z^4 + 8z^6 + z^8$	$8 + t$	$T(3, 5)$
$m(10_{124})$		$(-15, 2)$	-	-	✓	-	-	$T(3, -5)$

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
10_{128}		$(5, 0)$	✓	✗?	✗?	$2 + z^2$	$2t^{-2} + 2 + t + 2t^2$	
		$(5, 0)$	✗?	✓	✗?	$2 + 6z^2 + 5z^4 + z^6$	$6 + t$	
$m(10_{128})$		$(-14, 1)$	-	-	✓	-	-	
10_{132}		$(-8, 1)$	-	-	✓	-	-	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(10_{132})$		$(-1, 0)$	✗	✗	✓	$\emptyset$	$\emptyset$	
		$(-1, 0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
10_{136}	m	$(-3, 0)$	✗?	✗?	✓	1	$t^{-2} + 2t^{-1} + 3t + t^2$	
	m	$(-3, 0)$	✓	✓	✓	1	$t^{-2} + 2t^{-1} + 3t + t^2$	
	n	$(-3, 0)$	✓	✗?	✗?	$1 + z^2$	$2t^{-1} + 2 + 3t$	
	n	$(-3, 0)$	✓	✗?	✗?	$1 + z^2$	$2t^{-1} + 2 + 3t$	
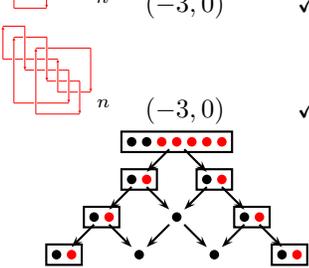
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(10_{136})$		$(-6, 1)$	-	-	✓	-	-	
10_{139}		$(7, 0)$	✓	✓	✓	$6 + 21z^2 + 21z^4 + 8z^6 + z^8$	$8 + t$	
$m(10_{139})^*$		$(-16, 1)$	-	-	✓	-	-	
		$(-17, 4)$	-	-	✓	-	-	
10_{140}		$(-8, 1)$	-	-	✓	-	-	
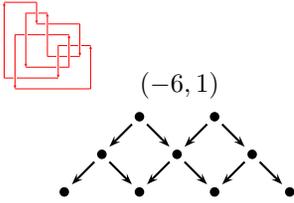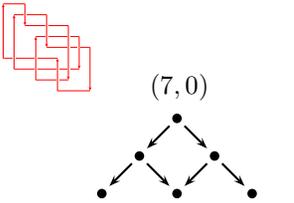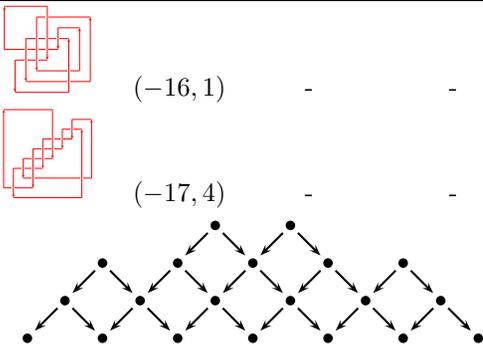
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(10_{140})$		$(-1, 0)$	✗	✗	✓	1	t	
		$(-1, 0)$	✓	✓	✓	1	t	
10_{142}		$(5, 0)$	✓	✓	✓	1	$3t^{-2} + t + 3t^2$	
		$(5, 0)$	✓	✓	✓	$1 + 6z^2 + 5z^4 + z^6$	$6 + t$	
$m(10_{142})$		$(-14, 1)$	-	-	✓	-	-	
10_{145}		$(-12, 1)$	-	-	✓	-	-	
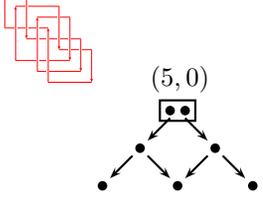
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(10_{145})^*$		$(3, 0)$	✓	✓	✓	$2 + 4z^2 + z^4$	$4 + t$	
		$(2, 1)$	-	-	✓	-	-	
		$(1, 0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
10_{160}		$(1, 0)$	✓	✗?	✗?	1	$2t^{-2} + t^{-1} + 2t + 2t^2$	
		$(1, 0)$	✗?	✗?	✗?	$1 + 3z^2 + z^4$	$t^{-1} + 4 + 2t$	
$m(10_{160})$		$(-10, 1)$	-	-	✓	-	-	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
10^*_{161}		$(-14, 1)$	-	-	✓	-	-	
		$(-15, 4)$	-	-	✓	-	-	
$m(10_{161})^*$		$(5, 0)$	✓	✓	✓	$2 + 9z^2 + 6z^4 + z^6$	$6 + t$	
		$(4, 1)$	-	-	✓	-	-	
		$(3, 0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
$11n_{19}$		$(-8, 1)$	-	-	✓	-	-	

Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(11n_{19})$		$(-1, 0)$	✓	✗?	✗?	$3 + 4z^2 + z^4$	$2t^{-1} + 4 + 3t$	
$11n_{38}$		$(-5, 0)$	✓	✗?	✗?	$2 + z^2$	$3t^{-1} + 2 + 4t$	
$m(11n_{38})$		$(-4, 1)$	-	-	✓	-	-	
$11n_{95}$		$(3, 0)$	✓	✗?	✗?	$3 + 6z^2 + 2z^4$	$4 + t, \; t^{-1} + 6 + 2t$	
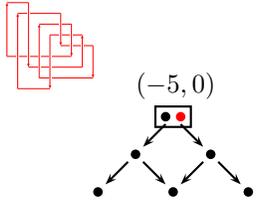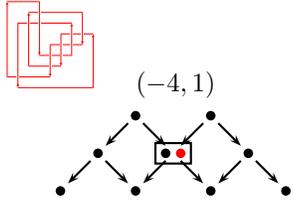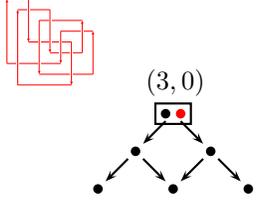
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(11n_{95})$		$(-12, 1)$	-	-	✗?	-	-	
		$(-12, 1)$	-	-	✓	-	-	
$11n_{118}$		$(3, 0)$	✗?	✗?	✓	$4 + 7z^2 + 2z^4$	$4 + t,\ t^{-1} + 6 + 2t$	
$m(11n_{118})$		$(-12, 1)$	-	-	✓	-	-	
$12n_{242}$		$(9, 0)$	✓	✓	✓	$9 + 39z^2 + 57z^4 + 36z^6 + 10z^8 + z^{10}$	$10 + t$	
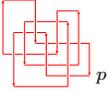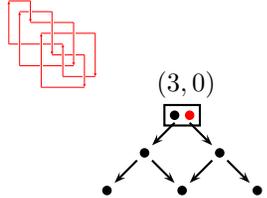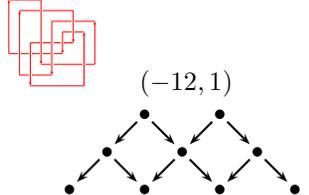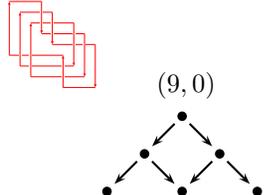
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$m(12n_{242})^*$		$(-18, 1)$	-	-	✓	-	-	
		$(-19, 4)$	-	-	✓	-	-	
$12n_{591}^*$		$(7, 0)$	✓	✓	✓	$4 + 17z^2 + 20z^4 + 8z^6 + z^8$	$8 + t$	
		$(6, 1)$			✓			
		$(5, 0)$	✓	✓	✓	$\emptyset$	$\emptyset$	
$m(12n_{591})$		$(-16, 1)$	-	-	✓	-	-	
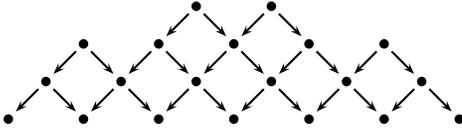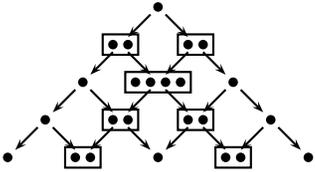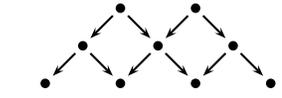
Knot Type	Cromwell Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?	Ruling Invariant	Linearized Contact Homology	Note
$15n_{41185}$		$(11, 0)$	✓	✓	✓	$14 + 70z^2 + 133z^4 + 121z^6$ $+55z^8 + 12z^{10} + z^{12}$	$12 + t$	$T(4, 5)$
$m(15n_{41185})$		$(-20, 1)$	-	-	✓	-	-	$T(4, -5)$
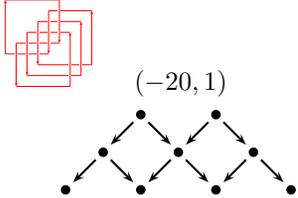
Table 2: Transverse Knots up to Arc Index 9

Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
3_1		-5	✓	$T(2,-3)$, K_1
$m(3_1)$		1	✓	$T(2,3)$, K_{-2}
$4_1 = m(4_1)$		-3	✓	$K_2 = K_{-3}$
5_1		-7	✓	$T(2,-5)$
$m(5_1)$		3	✓	$T(2,5)$
5_2		-7	✓	K_3
$m(5_2)$		1	✓	K_{-4}
6_1		-5	✓	K_4
$m(6_1)$		-3	✓	K_{-5}
6_2		-5	✓	
$m(6_2)$		-1	✓	
$6_3 = m(6_3)$		-3	✓	
7_1		-9	✓	$T(2,-7)$
$m(7_1)$		5	✓	$T(2,7)$

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Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
7_2		-9	$\checkmark$	K_5
$m(7_2)$		1	$\checkmark$	K_{-6}
		1	$\checkmark$	
7_3		3	$\checkmark$	
$m(7_3)$		-9	$\checkmark$	
7_4		1	$\checkmark$	
$m(7_4)$		-9	$\checkmark$	
7_5		-9	$\checkmark$	
$m(7_5)$		3	$\checkmark$	
7_6		-7	$\checkmark$	
$m(7_6)$	a	-1	$\checkmark$	
	a	-1	$\checkmark$	
7_7		-3	$\checkmark$	
$m(7_7)$		-5	$\checkmark$	
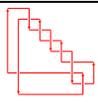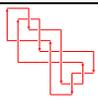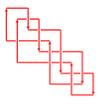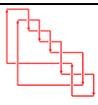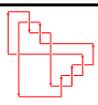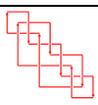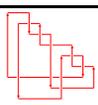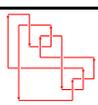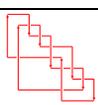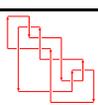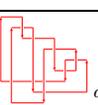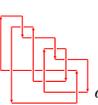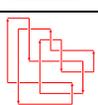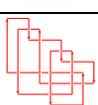
Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
8_{19}		5	$\checkmark$	
$m(8_{19})$		-11	$\checkmark$	$T(3, -4)$
8_{20}		-5	$\checkmark$	
$m(8_{20})$		-1	$\checkmark$	
8_{21}		-7	$\checkmark$	
$m(8_{21})$		1	$\checkmark$	
9_{42}		-3	$\checkmark$	
$m(9_{42})$		-5	$\checkmark$	
9_{43}		1	$\checkmark$	
$m(9_{43})$		-9	$\checkmark$	
9_{44}		-5	$\checkmark$	
		-5	$\boldsymbol{\chi}$?	
$m(9_{44})$		-3	$\checkmark$	
9_{45}		-9	$\checkmark$	

Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
$m(9_{45})$		1	✗?	
9_{46}		-7	✓	
$m(9_{46})$		-1	✓	
9_{47}		-1	✓	
$m(9_{47})$		-7	✓	
9_{48}	c	-1	✓	
	c	-1	✓	
$m(9_{48})$		-7	✓	
9_{49}		3	✓	
$m(9_{49})$		-11	✓	
10_{124}		7	✓	$T(3,5)$
$m(10_{124})$		-13	✓	$T(3,-5)$
10_{128}		5	✗?	
$m(10_{128})$		-13	✓	
10_{132}		-7	✓	

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Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
$m(10_{132})$		-1	✓	
		-1	✓	
10_{136}	d	-3	✓	
	d	-3	✓	
$m(10_{136})$		-5	✓	
10_{139}		7	✓	
$m(10_{139})^*$		-13	✓	
10_{140}		-7	✓	
$m(10_{140})$		-1	✓	
		-1	✓	
10_{142}		5	✓	
$m(10_{142})$		-13	✓	
10_{145}		-11	✓	
$m(10_{145})^*$		3	✓	
		1	✓	

Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
10_{160}		1	✗?	
$m(10_{160})$		-9	✓	
10_{161}		-13	✓	
$m(10_{161})^*$		5	✓	
		3	✓	
$11n_{19}$		-7	✓	
$m(11n_{19})$		-1	✓	
$11n_{38}$		-5	✓	
$m(11n_{38})$		-3	✓	
$11n_{95}$		3	✓	
$m(11n_{95})$		-11	✓	
$11n_{118}$		3	✓	
$m(11n_{118})$		-11	✓	
$12n_{242}$		9	✓	
$m(12n_{242})^*$		-15	✓	

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Knot Type	Cromwell Diagram	sl	$T = -\mu(T)$?	Note
$12n_{591}$		7	✓	
$m(12n_{591})$		-15	✓	
$15n_{41185}$		11	✓	$T(4,5)$
$m(15n_{41185})$		-19	✓	$T(4,-5)$

Table 3: Legendrian Links

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L2a1	✓		$(-1, -1)$	✓	
L4a1	✓		$(-1, -1)$	✓	
m(L4a1)	✓		$(-2, -2)$	✓	
			$(-3, -1)$	-	
L5a1	✓		$(-3, -2)$	-	
			$(-4, -1)$	-	
m(L5a1)	✓		$(-1, -1)$	✓	
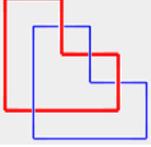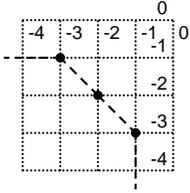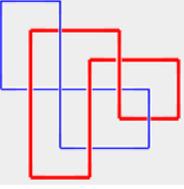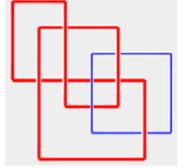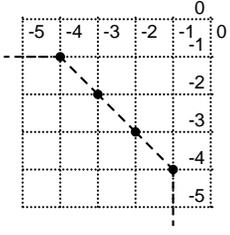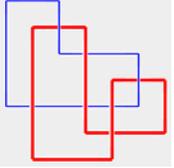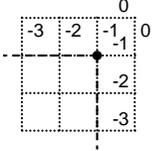
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L6a1	✓		$(-1, -1)$	✓	
			$(-1, -1)$	✓	
m(L6a1)	✓		$(-3, -3)$	✓	
			$(-4, -2)$	-	
			$(-5, -1)$	-	
L6a2	✓		$(-2, -2)$	✓	
			$(-3, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L6a3	✓		$(-3, -3)$	✓	
			$(-3, -3)$	✓	
			$(-4, -2)$	-	
			$(-5, -1)$	-	
m(L6a3)	✓		$(-1, -1)$	✓	
L7a1	✗		$(-1, -2)$	-	
			$(-2, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L7a1)	✗		$(-3, -3)$	-	
			$(-4, -2)$	-	
			$(-5, -1)$	-	
			$(-5, -1)$	-	
			$(-5, -1)$	-	
			$(-2, -5)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L7a2	-		$(-6, -1)$ $(-6, -1)$	- -	
m(L7a2)	-		$(1, -3)$ $(0, -2)$ $(-1, -1)$ $(-1, -1)$	- - - -	
L7a3	-		$(1, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L7a3)	-		$(-6, -3)$	-	
			$(-7, -2)$	-	
			$(-7, -2)$	-	
			$(-7, -2)$	-	
			$(-8, -1)$	-	
			$(-8, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L7a4	✓		$(-1,-1)$	✓	
			$(-1,-1)$	✓	
m(L7a4)	✓		$(-4,-3)$	-	
			$(-5,-2)$	-	
			$(-6,-1)$	-	
L7a5	✓		$(-3,-2)$	-	
			$(-4,-1)$	-	
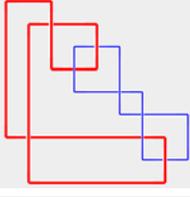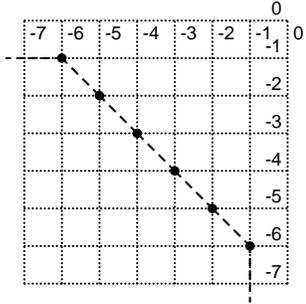
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L7a5)	✓		$(-2, -2)$	✗?[11]	
			$(-2, -2)$	✗?[10]	
			$(-3, -1)$	-	
			$(-3, -1)$	-	
L7a6	✓		$(-1, -1)$	✗	

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Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L7a6)	✓		$(-4, -3)$	-	
			$(-4, -3)$	-	
			$(-5, -2)$	-	
			$(-5, -2)$	-	
			$(-6, -1)$	-	
L7n1	-		$(-6, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L7n1)	-		$(1, -1)$	-	
L7n2	-		$(-6, -1)$	-	
m(L7n2)	-		$(1, -1)$	-	
L8n1	-		$(-3, -1)$	-	
			$(-3, -1)$	-	
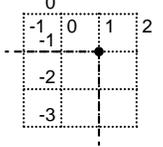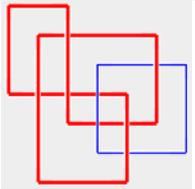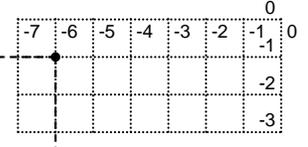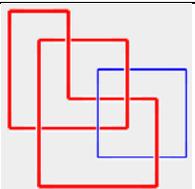
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L8n1)	-		$(-3, -1)$	-	
L8n2	-		$(-3, -1)$	-	
m(L8n2)	-		$(-3, -1)$	-	
L9n1	-		$(-8, -1)$	-	
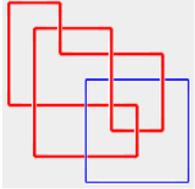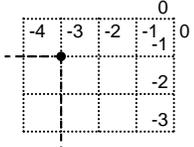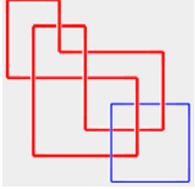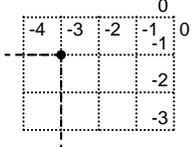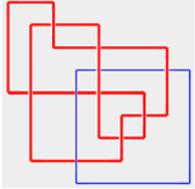
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n1)	-		$(1, -1)$	-	
			$(1, -1)$	-	
L9n2	-		$(-8, -1)$	-	
m(L9n2)	-		$(1, -1)$	-	
			$(1, -1)$	-	
			$(1, -1)$	-	
L9n3	-		$(-8, -1)$	-	
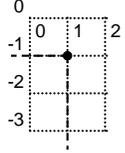
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Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n3)	-		$(1, -1)$	-	
			$(1, -1)$	-	
L9n4	-		$(-10, -1)$	-	
			$(-10, -1)$	-	
m(L9n4)	-		$(3, -1)$	-	
L9n5	-		$(-10, -1)$	-	
			$(-10, -1)$	-	
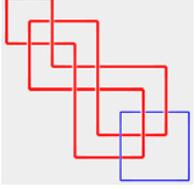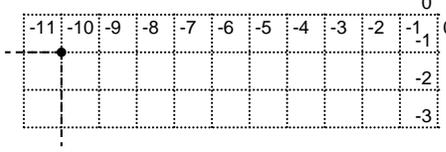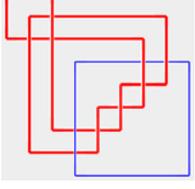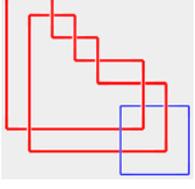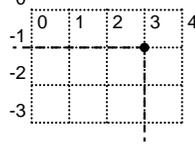
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n5)	-		$(3, -1)$	-	
L9n6	-		$(-10, -1)$	-	
			$(-10, -1)$	-	
			$(-10, -1)$	-	
m(L9n6)	-		$(3, -1)$	-	
L9n7	-		$(-8, -1)$	-	
			$(-8, -1)$	-	
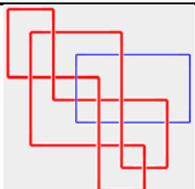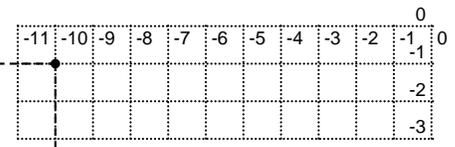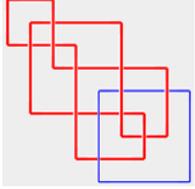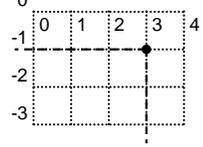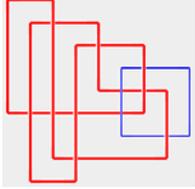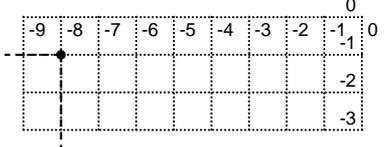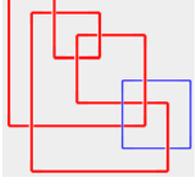
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Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n7)	-		$(1, -1)$ $(1, -1)$	- -	
L9n8	-		$(-8, -1)$ $(-8, -1)$	- -	
m(L9n8)	-		$(1, -1)$ $(1, -1)$	- -	
L9n9	-		$(-8, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n9)	-		$(1, -1)$	-	
L9n10	✗		$(-1, -1)$ $(-1, -1)$	- -	
m(L9n10)	✗		$(-3, -4)$ $(-4, -3)$ $(-4, -3)$ $(-5, -2)$	- - - -	
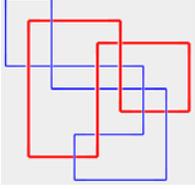
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n10)			$(-5, -2)$	-	
			$(-5, -2)$	-	
			$(-6, -1)$	-	
			$(-6, -1)$	-	
			$(-6, -1)$	-	
			$(-6, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L9n11	✗		$(-3, -2)$	-	?
			$(-4, -1)$	-	
			$(-2, -4)$	-	
m(L9n11)	✗		$(-1, -3)$	-	
			$(-1, -3)$	-	
			$(-2, -2)$	-	
			$(-3, -1)$	-	
			$(-3, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L9n12	-		$(3, -1)$	-	
m(L9n12)	-		$(-10, -1)$	-	
L9n13	✓		$(-3, -3)$	✗?[10]	
			$(-4, -2)$	-	
			$(-5, -1)$	-	
			$(-5, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n13)	✓		$(-2, -1)$	-	
L9n14	-		$(1, -3)$	-	
			$(0, -2)$	-	
			$(-1, -1)$	-	
			$(-1, -1)$	-	
m(L9n14)	-		$(-6, -1)$	-	
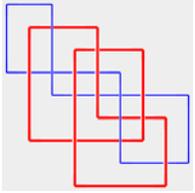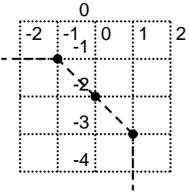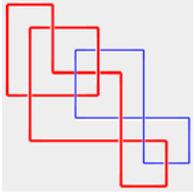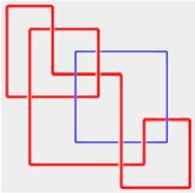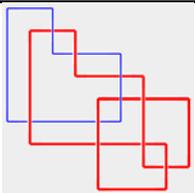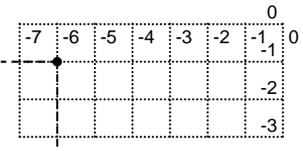
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L9n15	-		$(-7, -1)$	-	
			$(-6, -2)$	-	
m(L9n15)	-		$(1, -1)$	-	
L9n16	-		$(1, -3)$	-	
			$(0, -2)$	-	
			$(-1, -1)$	-	
m(L9n16)	-		$(-6, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L9n17	-		$(-6, -3)$	-	
			$(-7, -2)$	-	
			$(-7, -2)$	-	
			$(-7, -2)$	-	
			$(-7, -2)$	-	
			$(-7, -2)$	-	
			$(-8, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L9n17)	-		$(1, -1)$	-	
			$(1, -1)$	-	
L9n18	✓		$(-3, -3)$	✓	
			$(-5, -2)$	-	
			$(-7, -1)$	-	
m(L9n18)	✓		$(-1, -1)$	✓	
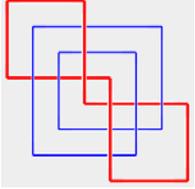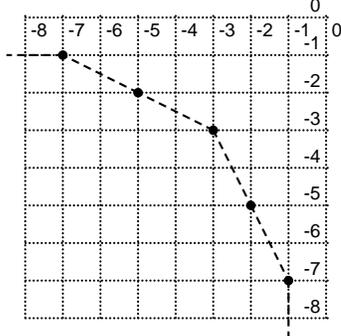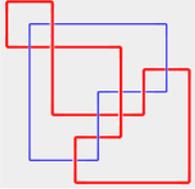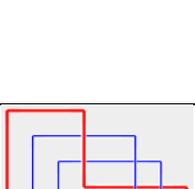
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L9n19	✗		$(-1, -3)$	-	
			$(-3, -2)$	-	
			$(-5, -1)$	-	
L10n23	-		$(3, -1)$	-	
m(L10n23)	-		$(-10, -1)$	-	
L10n24	✗		$(-3, -1)$	-	
			$(-2, -3)$	-	
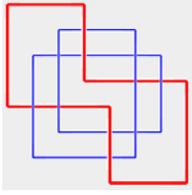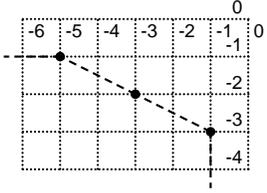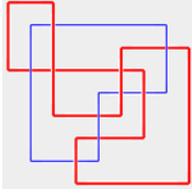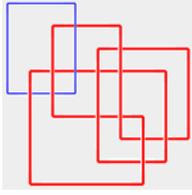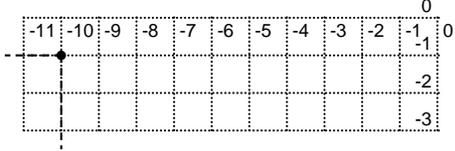
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L10n24)	✗		$(-1, -4)$	-	
			$(-1, -4)$	-	
			$(-2, -3)$	-	
			$(-3, -2)$	-	
			$(-3, -2)$	-	
			$(-4, -1)$	-	
L10n34	-		$(1, -1)$	-	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L10n34)	-		$(-6, -3)$	-	
			$(-7, -2)$	-	
			$(-8, -1)$	-	
L10n42	-		$(1, -1)$	-	
			$(1, -1)$	-	
m(L10n42)	-		$(-6, -3)$	-	
			$(-7, -2)$	-	
			$(-8, -1)$	-	
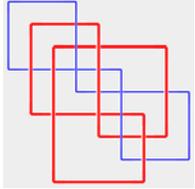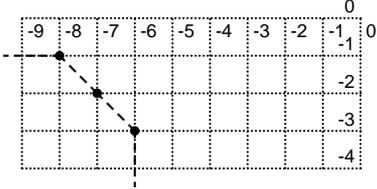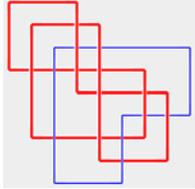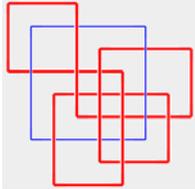
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Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L10n44	✓		$(-4, -3)$	-	
			$(-4, -3)$	-	
			$(-5, -2)$	-	
			$(-5, -2)$	-	
			$(-6, -1)$	-	
			$(-6, -1)$	-	
m(L10n44)	✓		$(-1, -1)$	✗?[10]	

Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L10n45	-		$(-3, -2)$	-	
			$(-4, -1)$	-	
m(L10n45)	-		$(-3, -1)$	-	
			$(-3, -1)$	-	
L10n46	-		$(-3, -1)$	-	
L10n54	✓		$(-1, -1)$	✗?[10]	
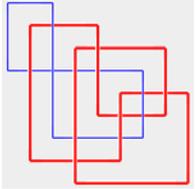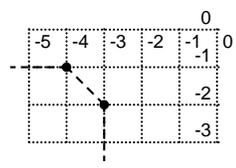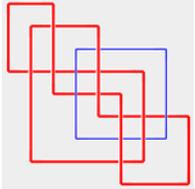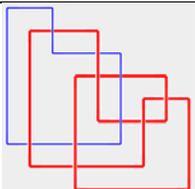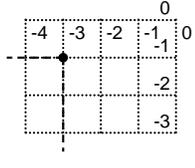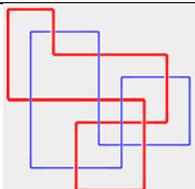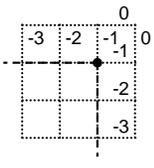
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Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L10n54)	✓		$(-4, -3)$	-	
			$(-6, -2)$	-	
L10n56	✗		$(-4, -3)$	-	?
			$(-6, -2)$	-	
			$(-3, -5)$	-	
m(L10n56)	✗		$(-1, -1)$	-	
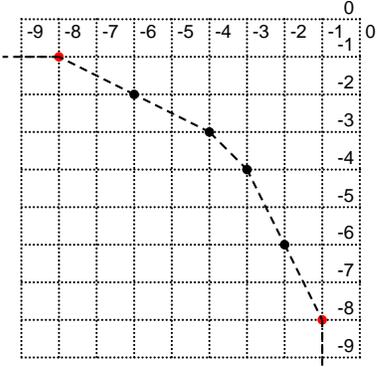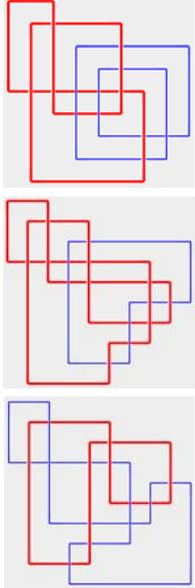
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L10n57	✗		$(-4, -1)$	-	?
			$(-3, -3)$	-	
m(L10n57)	✗		$(-3, -1)$	-	?
L11n119	-		$(3, -1)$	-	
m(L11n119)	-		$(-10, -1)$	-	
L11n132	-		$(-10, -1)$	-	
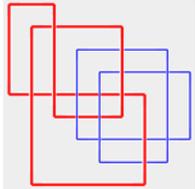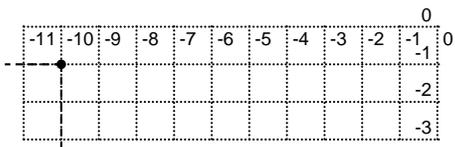
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L11n132)	-		$(3, -1)$	-	
			$(3, -1)$	-	
L11n133	-		$(-10, -1)$	-	
m(L11n133)	-		$(3, -1)$	-	
L11n139	-		$(1, -1)$	-	
			$(1, -1)$	-	
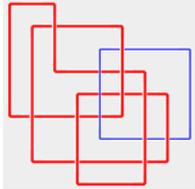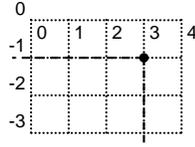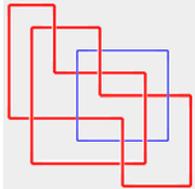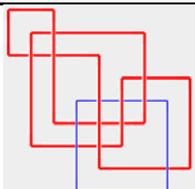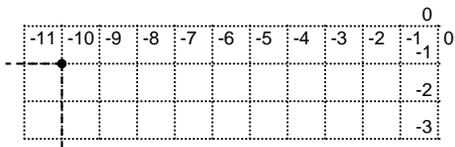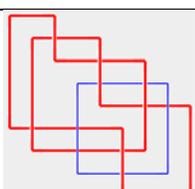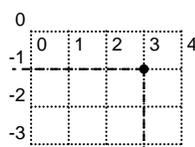
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
m(L11n139)	-		$(-8, -1)$	-	
L11n140	-		$(-8, -1)$	-	
m(L11n140)	-		$(1, -1)$	-	
			$(1, -1)$	-	
			$(1, -1)$	-	
			$(1, -1)$	-	
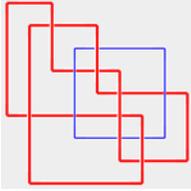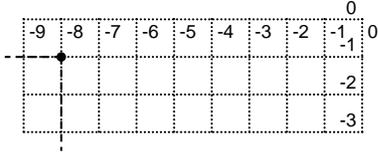
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L11n148	-		$(1, -1)$	-	
m(L11n148)	-		$(-8, -1)$	-	
L11n204	-		$(-6, -3)$	-	
			$(-7, -2)$	-	
			$(-9, -1)$	-	
m(L11n204)	-		$(1, -1)$	-	
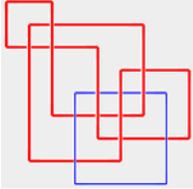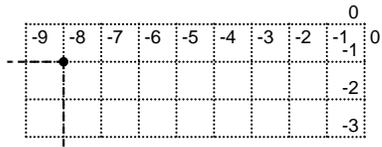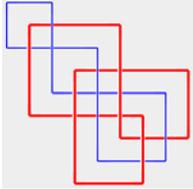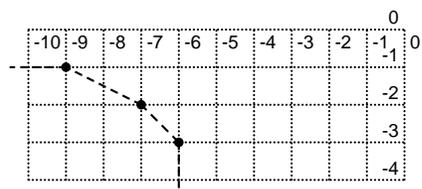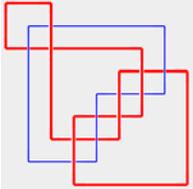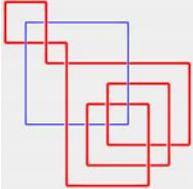
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L11n205	-		$(-6, -1)$	-	
m(L11n205)	-		$(1, -3)$	-	
L11n218	-		$(1, -3)$	-	?
			$(-1, -2)$	-	
m(L11n218)	-		$(-6, -1)$	-	
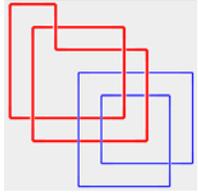
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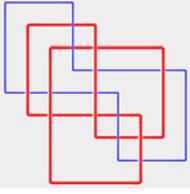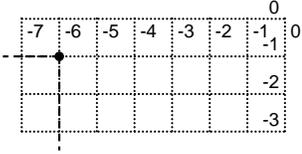
Link Type	Topological Switch?	Cromwell Diagram	$(tb1, tb2)$	Legendrian Switch?	TB Polytope
L11n219	-		$(1, -3)$	-	?
			$(-1, -2)$	-	
m(L11n219)	-		$(-6, -1)$	-	
L11n226	-		$(-6, -3)$	-	?
			$(-8, -2)$	-	
m(L11n226)	-		$(1, -1)$	-	