

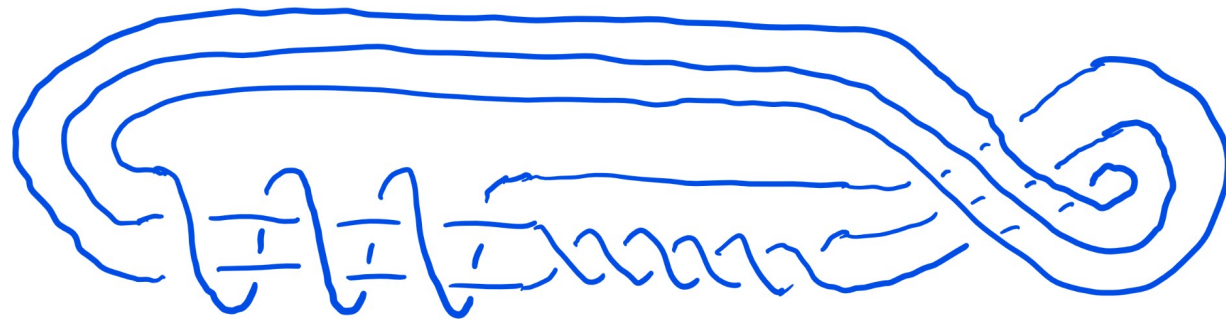
Legendrian links
- and -

Lagrangian Cobordism

Lenny Ng (Duke)

AMS Southeastern Sectional Meeting

March 23, 2024

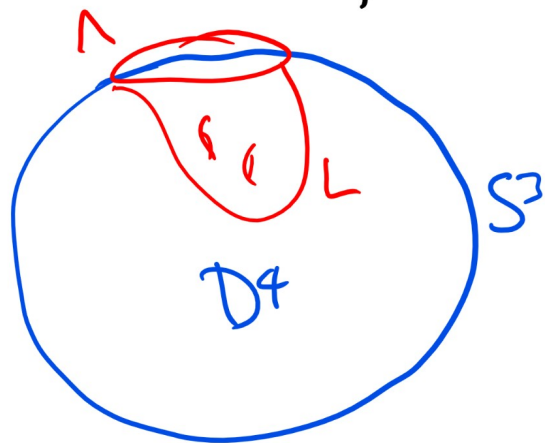


Motivation

Standard symplectic 4-ball $(D^4, \omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$

Boundary: Standard contact 3-sphere $(S^3, \alpha_{std} = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)$

Try to enumerate Lagrangian surfaces in the symplectic 4-ball, with boundary on S^3 .



Let $L \subset D^4$ be a Lagrangian surface $(\omega_{std}|_L \equiv 0)$, with boundary

$$\Lambda := \partial L = L \cap S^3.$$

If L is cylindrical near $\partial D^4 = S^3$ then Λ is Legendrian: $\alpha_{std}|_{\Lambda} \equiv 0$.

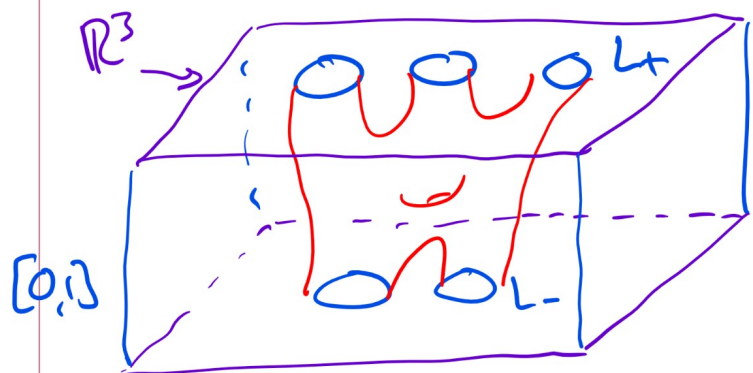
Say L is a filling of the Legendrian link Λ .

What do fillings of a fixed Legendrian Λ look like?

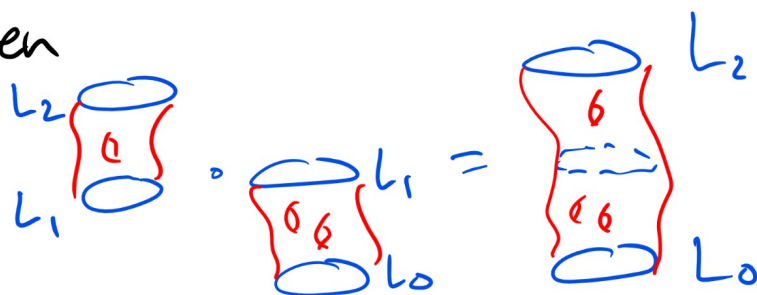
More motivation

Smooth link cobordism category $\underline{Col}$:

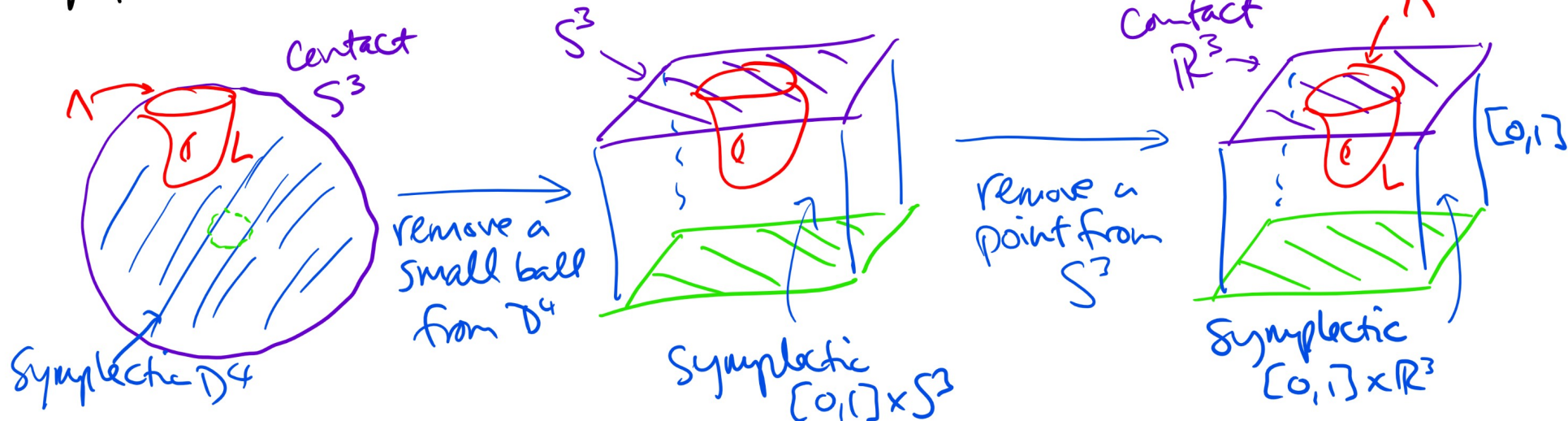
- $Ob \underline{Col} = \{ \text{smooth (oriented?) links in } \mathbb{R}^3 \}$
- $Hom(L_-, L_+) = \{ \text{smooth (oriented?) cobordisms between } L_- \text{ and } L_+ \text{ in } [0,1] \times \mathbb{R}^3 \}$



Composition given by stacking:



Symplectic version from our previous picture:

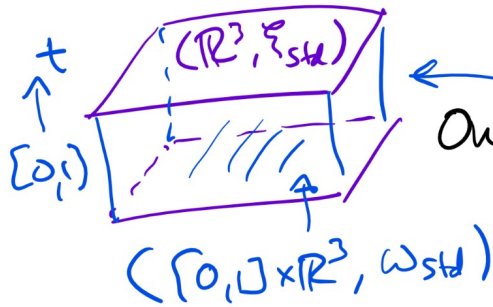


Basic setting

Setting for today:

Our Contact mfd: $\mathbb{R}^3$ with the standard contact structure

$$\xi_{\text{std}} = \ker \alpha_{\text{std}}, \quad \alpha_{\text{std}} = dz - y dx$$



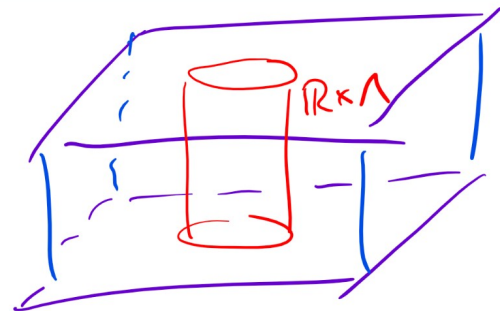
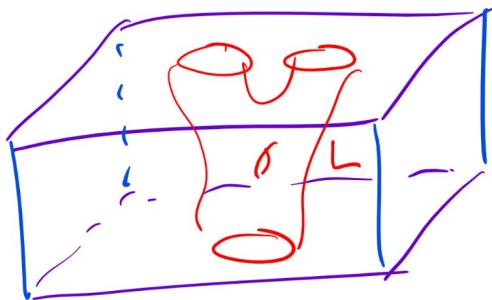
Our Symplectic mfd: $\mathbb{R} \times \mathbb{R}^3$ "Symplectization of $\mathbb{R}^3$ "
(or $[0,1] \times \mathbb{R}^3$) with symplectic form

$$\omega_{\text{std}} = d(e^t \alpha_{\text{std}}).$$

Def A surface $L \subset \mathbb{R} \times \mathbb{R}^3$ is Lagrangian if $\omega_{\text{std}}|_L = d(e^t \alpha_{\text{std}})|_L \equiv 0$.

If L is cylindrical: $L = \mathbb{R} \times \Lambda$ for some $\Lambda \subset \mathbb{R}^3$, then $\alpha_{\text{std}}|_\Lambda \equiv 0$.

Def A link $\Lambda \subset \mathbb{R}^3$ is Legendrian if $\alpha_{\text{std}}|_\Lambda = (dz - y dx)|_\Lambda \equiv 0$.



Exact Lagrangian cobordisms

Def $\Lambda_+, \Lambda_- \subset \mathbb{R}^3$ oriented Legendrian links.

An exact Lagrangian cobordism from Λ_- to Λ_+

is a Lagrangian $L \subset \mathbb{R} \times \mathbb{R}^3$ st.

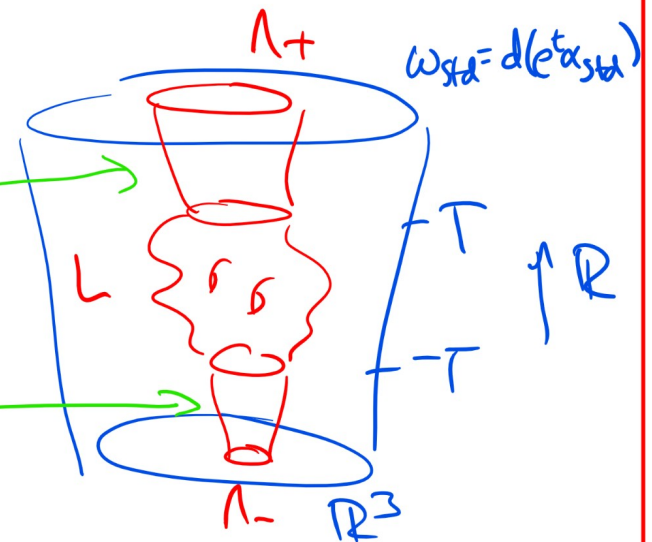
① L is embedded and oriented

② for some T , $L \cap \{t > T\} = (T, \infty) \times \Lambda_+$

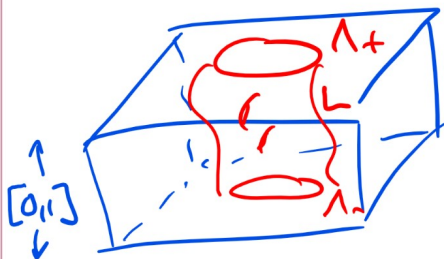
$L \cap \{t < -T\} = (-\infty, -T) \times \Lambda_-$

③ $\exists f: L \rightarrow \mathbb{R}$ st. $df = e^t \alpha_{\text{std}}|_L$

($\rightarrow \omega|_L \equiv 0$) and f is constant on each end.



View this as a cobordism in $[0, 1] \times \mathbb{R}^3$ between Λ_+ and Λ_- :



Then we can stack two exact Lagrangian cobordisms (with matching ends) to get another one:

$$L_0 = \begin{array}{c} \Lambda_1 \\ \text{---} \\ \Lambda_0 \end{array}, L_1 = \begin{array}{c} \Lambda_2 \\ \text{---} \\ \Lambda_1 \end{array} \Rightarrow L_1 \circ L_0 = \begin{array}{c} \Lambda_2 \\ \text{---} \\ \Lambda_1 \\ \text{---} \\ \Lambda_0 \end{array}$$

Lagrangian cobordism category

Def The Lagrangian cobordism category $\underline{LCob}$ is defined by:

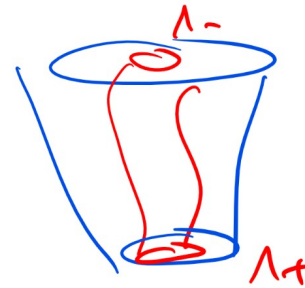
- Ob $\underline{LCob}$ = oriented Legendrian links in $\mathbb{R}^3$
- $\text{Hom}(\Lambda_-, \Lambda_+) = \{ \text{exact Lagrangian cobordisms from } \Lambda_- \text{ to } \Lambda_+ \}$.



Lagrangian cobordisms are directional in general:



Can't be flipped over to



Special case of an exact Lagrangian cobordism:

Def $\Lambda \subset \mathbb{R}^3$ Legendrian. A filling of Λ is an exact Lagrangian cobordism from $\emptyset$ to Λ .



Interlude on Legendrian knots/links

Two ways to depict Legendrian links Λ in $\mathbb{R}^3$: $dz - y dx = 0$ along Λ .

- front projection to $\mathbb{R}_{xz}^2$. Recover y coordinate by $y = \frac{dz}{dx}$.



- Lagrangian projection to $\mathbb{R}_{xy}^2$. Recover z coordinate by $z = \int y dx$.



NB not all link diagrams correspond to Lagrangian projections of Legendrian links:
ex

An isotopy between two Legendrian links given by a family of embedded Legendrian links is called a Legendrian isotopy.

One invariant under Legendrian isotopy: Thurston-Bennequin number $tb(\Lambda) \in \mathbb{Z}$.

In xy projection, $tb(\Lambda) = \text{writhe} = \# \nearrow - \# \nwarrow$:



Facts/questions about Lagrangian cobordisms

- Chenraime 2010: given a cobordism L from Λ_- to Λ_+ ,



$$tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L)$$

(if Λ_+ is a knot)

$\Rightarrow$ if $\Lambda_{\pm}$ are knots, $tb(\Lambda_+) \geq tb(\Lambda_-)$, equality $\Leftrightarrow L = \text{Concordance}$ (cylinder).

- Not every Legendrian link has a filling.

Rudolph 1993: $tb(\Lambda) \leq 2g_+(N) - 1$

slice genus of smooth knot type of Λ



$\Rightarrow$ if Λ has a filling then $tb(\Lambda) = 2g_+(N) - 1$.

Open question: is the converse true?

- If $\Lambda_{\pm}$ are isotopic Legendrian knots then $\exists$ (exact) Lagrangian concordance in both directions.



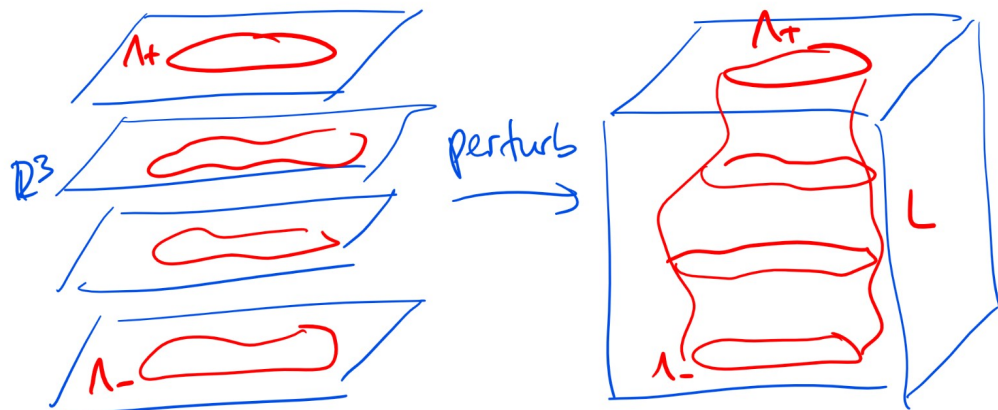
- More generally, say $\Lambda_- \leq \Lambda_+$ if $\exists$ Lagrangian concordance from Λ_- to Λ_+ .

Open question: is $\leq$ antisymmetric? ($\Lambda_- \leq \Lambda_+$ and $\Lambda_+ \leq \Lambda_- \stackrel{?}{\Leftrightarrow} \Lambda_+, \Lambda_-$ are Legendrian isotopic)

Constructing exact Lagrangian cobordisms

Ekhholm-Honda-Kálmán 2012: Can construct exact Lagrangian cobordisms by concatenating elementary cobordisms:

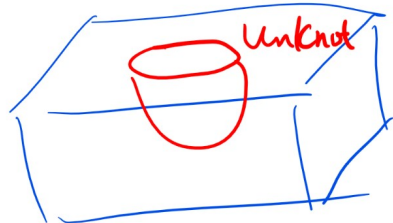
① isotopy cylinders



Deform the trace of an isotopy to get a Lagrangian cylinder:

Λ_+
 $\uparrow L$
 Λ_-

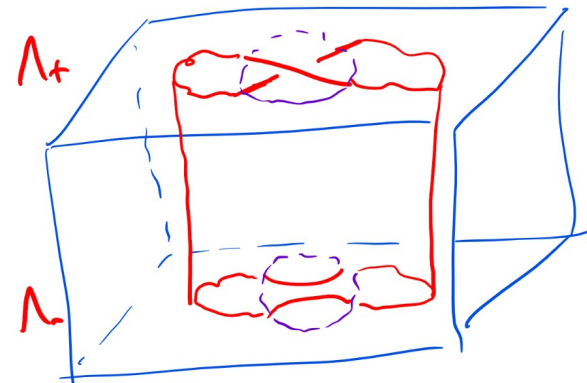
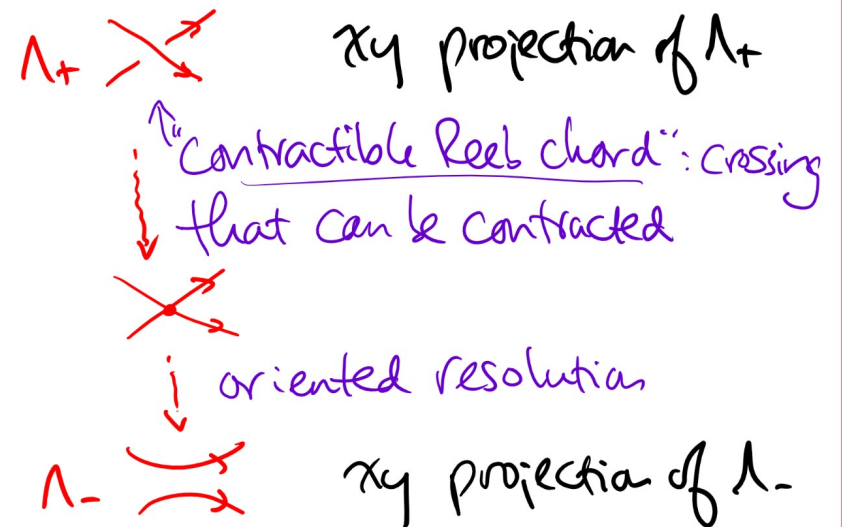
② minimum cobordisms



$\exists$ Lagrangian disk filling for the standard unknot:

∞
 $\uparrow$ 0-handle
 $\otimes$

③ saddle cobordisms



Λ_+
 $\uparrow$ 1-handle
 Λ_-

Constructing exact Lagrangian cobordisms, continued

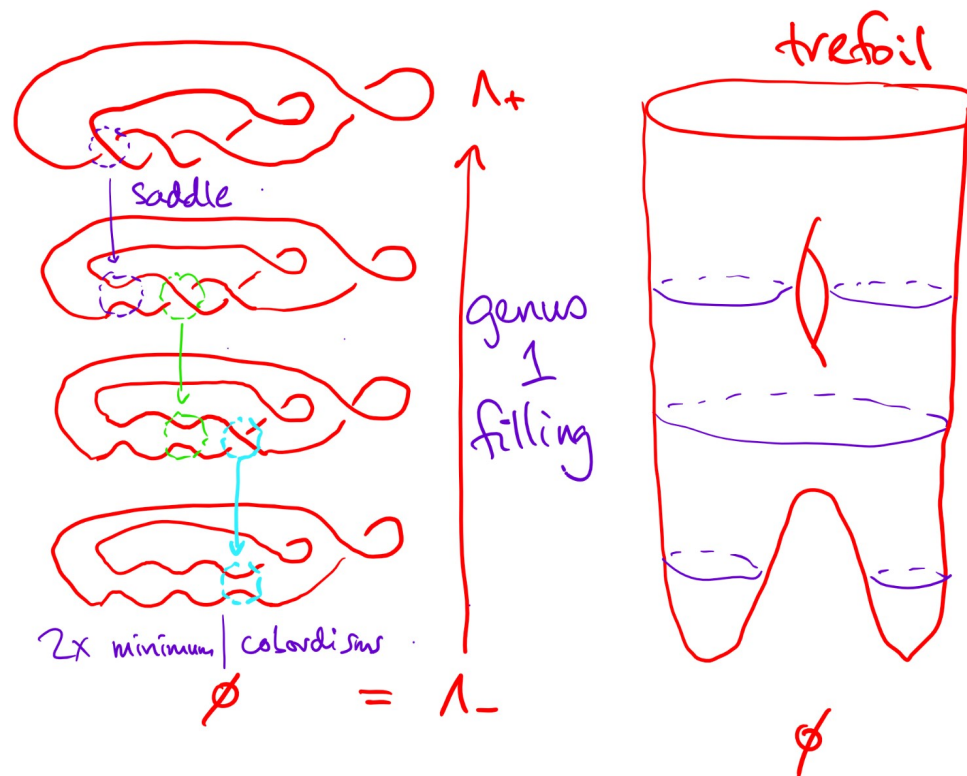
Def An exact Lagrangian cobordism is decomposable if it's the concatenation of elementary cobordisms:

- isotopy cylinders
- minimum cobordisms
- saddle cobordisms
- also add: $\emptyset$ (Sauvageot)



Open question: is every exact Lagrangian cobordism (isotopic to) decomposable?

Ex: decomposable fillings of the trefoil

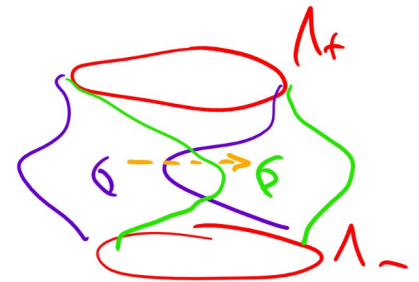


In fact, any of the $3! = 6$ orderings of doing saddle cobordisms at the 3 crossings  gives a filling of the trefoil.

Classifying fillings

Natural notion of equivalence of Lagrangian cobordisms / fillings:

Hamiltonian isotopy: in our setting, same as exact Lagrangian isotopy: isotopy rel ends ($\Lambda_{\pm}$) through exact Lagrangian cobordisms.



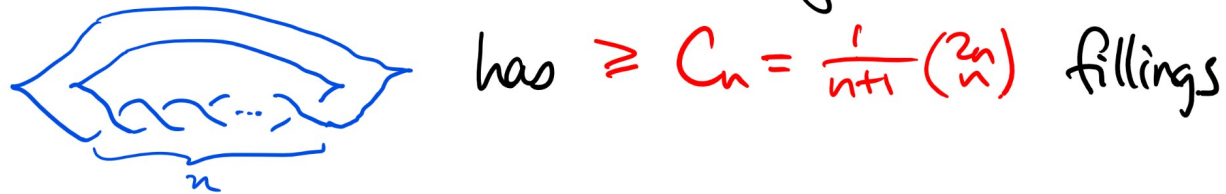
Main question

Given Legendrian $\Lambda \subset \mathbb{R}^3$, how many (orientable, embedded) fillings does Λ have, up to Hamiltonian isotopy?

- many Legendrian links have no fillings: e.g.  or anything non-max-tb
- Eliashberg-Polterovich 1996: the standard Legendrian unknot  has a unique filling
- Ekholm-Honda-Kálmán 2012: the standard Legendrian trefoil  has ≥ 5 fillings: all 6 decomposable fillings are distinct except  $\neq$ 

Classifying fillings, continued

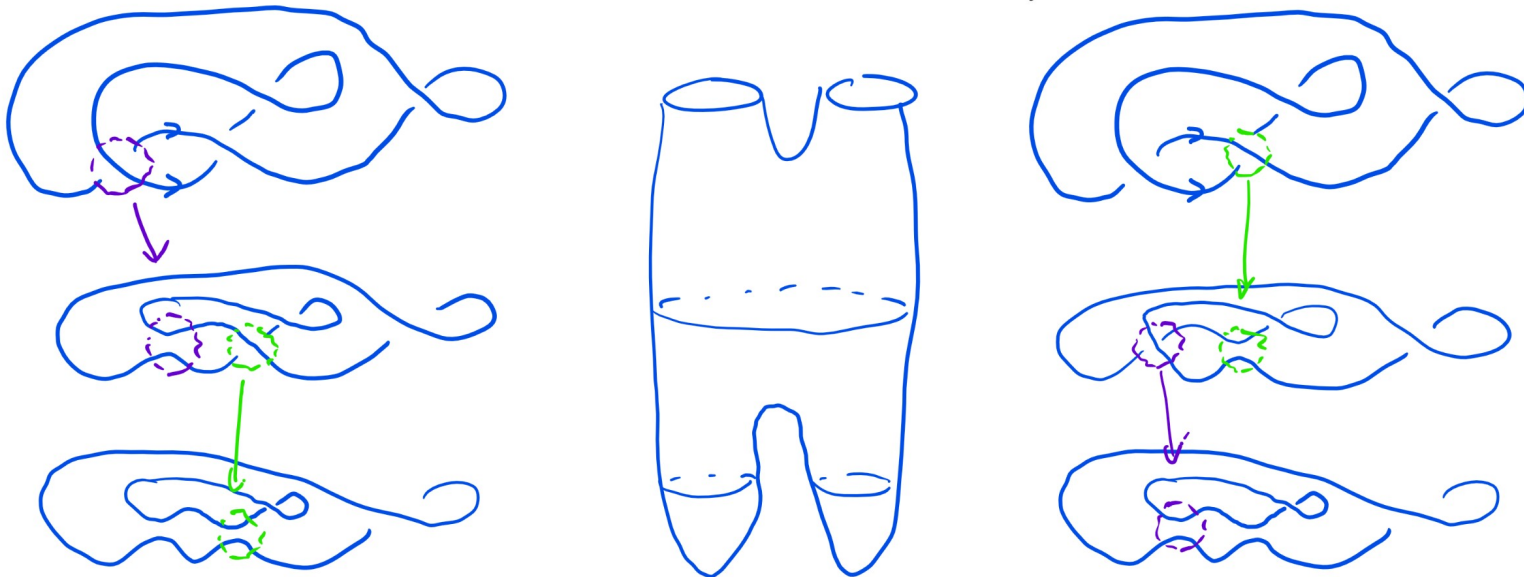
- Pan 2017: the standard Legendrian $(2n)$ torus link



Open question: Completely classify the fillings of any Legendrian link besides $\diamond$ that has at least one filling.



This has at least **2** distinct fillings:

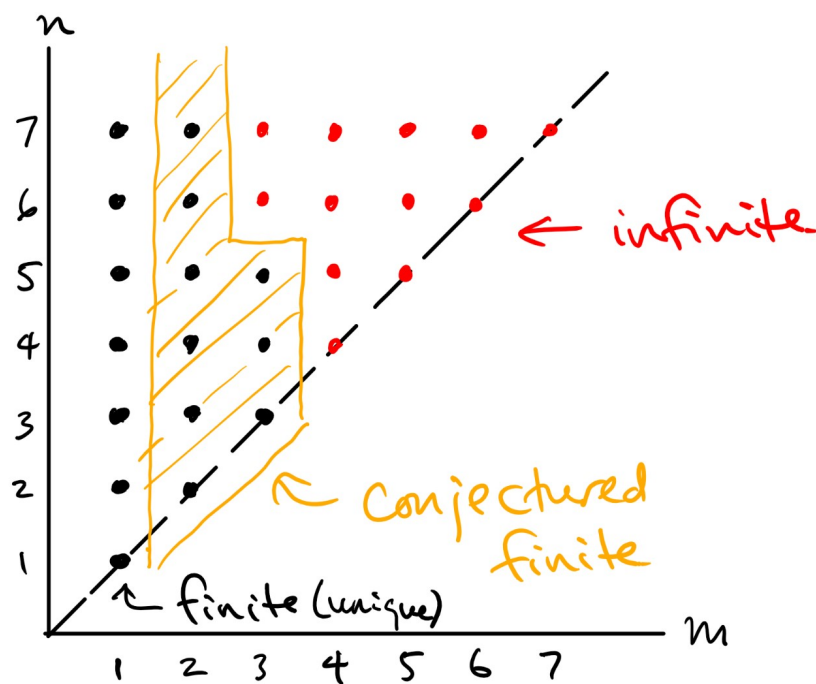
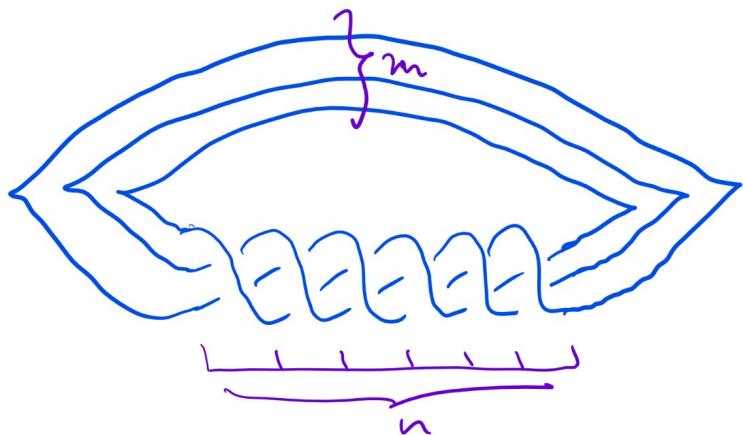


Question:
are these
the only
2 fillings
of the
Hopf link?

Infinitely many fillings

Thm (Casals-Gao 2020)

The Legendrian torus link $T(m, n)$ has infinitely many fillings for $n \geq m \geq 4$ and $m=3, n \geq 6$.



Subsequently: many more families of Legendrian links with infinitely many fillings.

Casals-Zaslav 2020, Gao-Shen-Weng 2020, Casals-N 2021, Capovilla-Searle - Casals 2023, ...

Constructing infinitely many fillings

Def For a Legendrian link Λ , the Lagrangian concordance monoid $\mathcal{M}(\Lambda)$ is:
 $\mathcal{M}(\Lambda) = \{(\text{exact}) \text{ Lagrangian concordances } \Lambda \rightarrow \Lambda\} / \text{Hamiltonian isotopy}.$

$\mathcal{M}(\Lambda)$ acts on $\{\text{fillings of } \Lambda\}$ by concatenation:

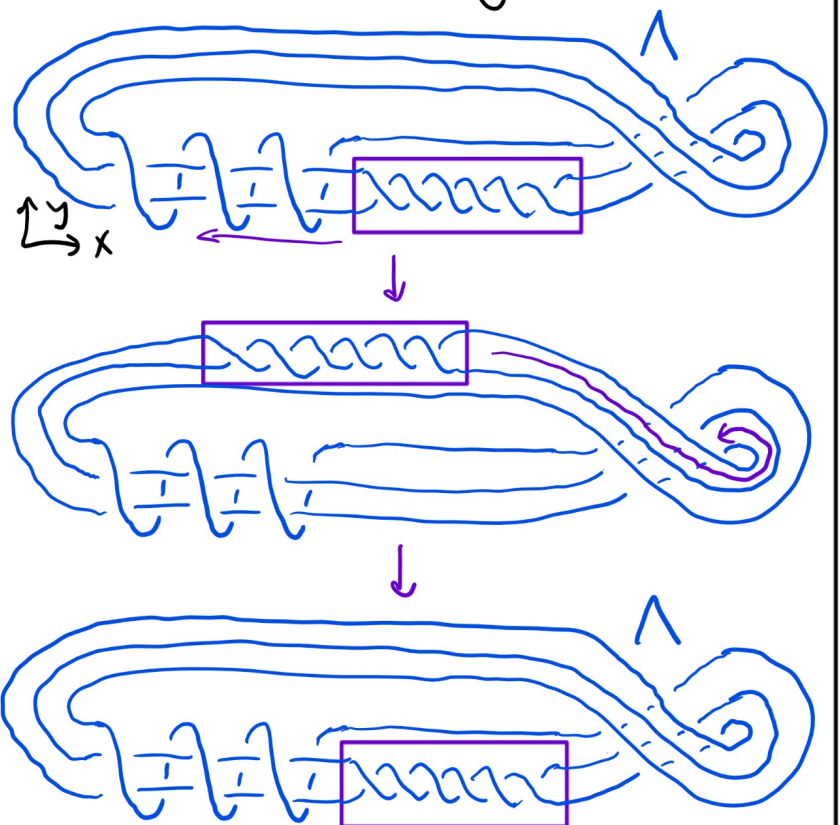
$$\begin{array}{c} \text{red cylinder with } \mathcal{L} \\ \text{blue cylinder with } L \end{array} = \begin{array}{c} \text{red cylinder with } \mathcal{L} \\ \text{blue cylinder with } L \end{array}.$$

Idea: find a filling L of Λ and a self-concordance $\mathcal{L}$ of Λ such that $\{\mathcal{L}^n \cdot L \mid n \geq 0\}$ are distinct for all n .

$$\left\{ \begin{array}{c} \text{blue cylinder with } L \\ \text{red cylinder with } \mathcal{L} \\ \text{red cylinder with } \mathcal{L} \\ \text{red cylinder with } \mathcal{L} \end{array} \right\}$$

Good candidates for the concordance $\mathcal{L}$ are given by cylinders for self-isotopies of Λ .

Ex: self-isotopy of $\Lambda = T(3,6)$



Techniques for distinguishing fillings of Λ

- Floer theory: each filling L of Λ produces an augmentation ϵ_L of the Chekanov-Eliashberg differential graded algebra (A_Λ, ∂) , unchanged by Hamiltonian isotopy.

Distinct augmentations $\Rightarrow$ distinct fillings.

[Ekholm-Honda-Kálmán 2012, Pan 2017, Casals-N 2020, Capovilla-Searle-Casals 2023]

- Legendrian weaves: combinatorial technology by Casals-Zaslav 2020 for understanding fillings

[Casals-Zaslav 2020, Hughes 2023, ...]

- Cluster theory: when Λ is the rainbow closure of a positive braid, Casals-Weng 2022 (related: Shende-Treumann-Williams-Zaslow 2018), the moduli space of sheaves $Sh_1(\Lambda)$ has the structure of a cluster variety, and different fillings give different cluster charts.



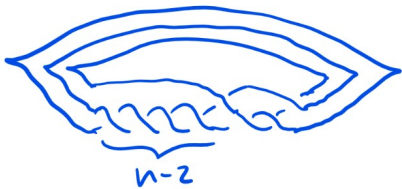
[Casals-Gao 2020, Gao-Shen-Weng 2020, Capovilla-Searle-Hughes-Weng 2023, ...]

Clusters and fillings

If $\Lambda(\beta)$ = rainbow closure of a positive braid β :

Conjecture (Casals 2020) There are exactly as many fillings of $\Lambda(\beta)$ as cluster seeds for the cluster algebra associated to β .

Finite type cluster algebras:

- A_n  : $\frac{1}{n+2} \binom{2n+2}{n+1}$ fillings? (Partial result: Pan 2017)
- D_n  : $\frac{3n-2}{n} \binom{2n-2}{n-1}$ fillings? (Partial result: An-Bae-lee 2021, Hughes 2023)
- E_6  , E_7  , E_8  :
833 , 4160 , 25080 fillings?

All other rainbow closures of positive braids:

infinitely many clusters & infinitely many fillings. (Gao-Shen-Weng 2020)

Number of fillings of the Legendrian (m,n) torus link

