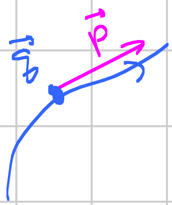
Knot Contact Homology and Augmentation Varieties

Note Title 8/3/2016

Minicourse – Hamilton workshop

Goal: study smooth topology via symplectic topology of cotangent bundles.

Warmup: Hamiltonian formulation of classical mechanics.



Particle in $\mathbb{R}^n$, position $\vec{q}(t)$.

$$-\frac{\partial V}{\partial q} = F = ma = \ddot{q}$$

$\uparrow$ force given by potential energy $V(q)$; $\uparrow_{m=1}$

2nd order diff. eq.

1st order: define $p = \dot{q}$ (momentum); then $\begin{cases} \dot{q}_i = p_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial V}{\partial q_i} = -\frac{\partial H}{\partial q_i} \end{cases}$

Write $H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$ Hamiltonian

In $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$ phase space, particle follows the flow of the Hamiltonian vector field

$$X_H = \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

More modern language: define $\omega = \sum dq_i \wedge dp_i \in \Omega^2(\mathbb{R}^{2n})$; then X_H is determined by

$$\underline{X_H \lrcorner\, \omega = dH}$$

: "dual" to 1-form dH under nondeg. bilinear form ω.

More generally: particle in smooth mfd M, phase space T^*M.
$\exists$ canonical 2-form $\omega \in \Omega^2(T^*M)$ s.t. a particle moving under Hamiltonian $H \in C^\infty(T^*M)$ follows flow of X_H defined above.

in local coords $\underbrace{q_1,\dots,q_n}_{\text{position}}, \underbrace{p_1,\dots,p_n}_{\text{momentum (coords in } dq_1,\dots,dq_n \text{ directions)}}$: $\omega = \sum dq_i \wedge dp_i$.

Fact: this is independent of choice of coords!

in fact: $\omega = -d\lambda$ where $\lambda = \sum p_i dq_i \in \Omega^1(T^*M)$ is indep of coords.
<u>Liouville 1-form</u>.

Coord-free def of λ: given $v \in T_{(q,p)}(T^*M)$:

projection map $\tilde{\pi}$: $TW \to W$, $W = T^*M$

$\pi: T^*M \to M \rightsquigarrow d\pi: T(T^*M) \to TM$

$$\begin{tikzcd}[column sep=small] & v \in T_{(q,p)}T^*M \arrow[dl, "\tilde{\pi}"'] \arrow[dr, "d\pi"] & \\ (q,p) = \tilde{\pi}(v) \in T_q^*M \arrow[rr, dashed, leftrightarrow] & & T_qM \ni d\pi(v) \end{tikzcd}$$

$\lambda(v) = \langle \tilde{\pi}(v), d\pi(v) \rangle \in \mathbb{R}$.

<u>Symplectic mfd</u>: (W^{2n}, ω) with $\omega \in \Omega^2(W)$, $d\omega = 0$, ω^n nowhere 0 (symplectic form).

<u>Contact mfd</u>: (V^{2n-1}, ξ) with ξ locally $= \ker \alpha$, $\alpha \in \Omega^1(V)$, $\alpha \wedge (d\alpha)^{n-1}$ nowhere 0 (contact form).

M smooth: T^*M is a symplectic mfd with symplectic form $\omega = -d\lambda$.
$ST^*M = \{(q,p) \in T^*M \mid \|p\| = 1\}$ is a contact mfd if we choose a metric on M, contact form $\alpha = \lambda|_{ST^*M}$.
(in fact, contact structure is indep of metric).

Q: How much of the symplectic/contact structure remembers of the smooth topology of M?

Fantasy Conjecture: Symplectic type of T^*M encodes smooth type of M: for closed M.

if $(T^*M, \omega) \cong (T^*M', \omega')$ then $M \cong M'$ ← diffeomorphism

(note work of A. Knapp says not true for exotic $\mathbb{R}^4$'s)

Strategy: Apply symplectic/contact invts of T^*M / ST^*M to get smooth invts of M.

Ex Thm (Viterbo, Abbondandolo-Schwarz, Salamon-Weber)
Hamiltonian Floer homology of $T^*M \cong H_*^{sing}(\mathcal{L}M)$.
← free loop space of M

Related work by Cieliebak-Latschev: linearized Contact homology of ST^*M = string homology of M.

More than homotopical info: Can distinguish smooth structures.
Abouzaid '08: if Σ is exotic S^{4k+1} not bounding parallelizable mfd, then $T^*\Sigma \not\cong T^*S^{4k+1}$.
Ekholm-Kragh-Smith '15.

Relative version: conormal bundles

$K \subset M$ embedded submfd (or immersed with $\pitchfork$ self-intersection)

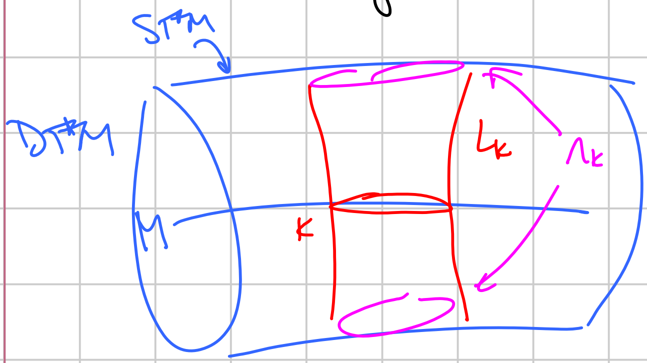
$\rightsquigarrow L_K \subset T^*M$
$$= \{(q,p) \in T^*M \mid q \in K,\ \langle p, T_qM\rangle = 0\}$$ conormal to K.

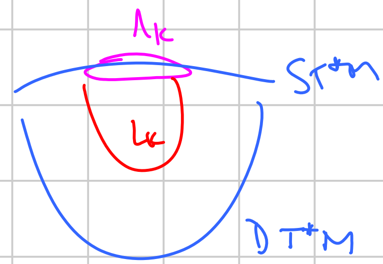
Note $\dim L_K = n = \frac{1}{2}\dim T^*M$.

Exc (a). this is Lagrangian: $\omega|_{L_K} \equiv 0$.

$\rightsquigarrow \Lambda_K = L_K \cap ST^*M \subset ST^*M$ unit conormal to K.

Exc (b) this is Legendrian: (V,α) contact: $\alpha|_\Lambda \equiv 0$ and $\dim \Lambda = \frac{\dim V - 1}{2}$.





If $K \subset M$ changes by smooth isotopy, then $\Lambda_K \subset ST^*M$ changes by Legendrian isotopy.

What does the Legendrian Λ_K remember about K?

Restrict to $M = \mathbb{R}^n$: $ST^*M \cong \mathbb{R}^n \times S^{n-1} \cong J^1(S^{n-1})$

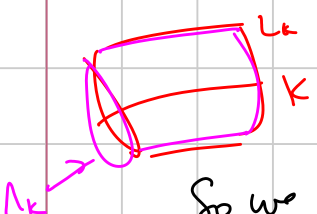
$J^1 N = T^*N \times \mathbb{R}$ is contact with contact form $dz - \alpha_{T^*N}$.

Further restrict to $M = \mathbb{R}^3$, K = knot (or link) in $\mathbb{R}^3$.

$ST^*\mathbb{R}^3 \cong J^1(S^2)$ contact 5-manifold

$\Lambda_K \cong T^2$: think L_K = conormal bundle $\cong$ normal bundle to K

$\Lambda_K = \partial(\text{tubular nbd of } K)$.



So we have a 2-torus in $\mathbb{R}^3 \times S^2$.

Exc Topologically all of these T^2's are isotopic.

Fantasy conj If $\Lambda_{K_1}, \Lambda_{K_2}$ are isotopic as Legendrian submfds, then $K_1 = K_2$ (isotopic as smooth knots).

Now a Theorem! Shende '16, Ekholm-N-Shende '16.

Strategy: Study Λ_K via holomorphic curves / Legendrian contact homology.

$K \subset \mathbb{R}^3$ knot $\rightarrow$ $\Lambda_K \subset ST^*\mathbb{R}^3$ leg. torus

lecture 1

$\downarrow$ Legendrian contact homology

$HC_*(K)$ knot contact homology

augmentation variety
$V_K = \{Aug_K = 0\} \subset (\mathbb{C}^*)^3$

lecture 2

$\downarrow$

topological strings,
mirror symmetry,
HOMFLY-PT

string homology

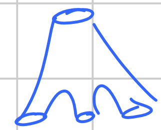
lecture 3

$\downarrow$

Complete knot invariant

Legendrian Contact Homology

Eliashberg-Hofer ....

- Relative version of contact homology (grid by Reeb orbits counting 
- "1st order piece" of Symplectic Field Theory (EGH)

$\Lambda \subset V$ Legendrian $\subset$ contact. LCH is the homology of a differential graded algebra $(\mathcal{A}, \partial)$: $HC_*(V, \Lambda) = H_*(\mathcal{A}, \partial)$.

To define this:

The Reeb vector field for (V, α) is R_α defined by $R_\alpha \lrcorner d\alpha = 0$, $\alpha(R_\alpha) = 1$.

A Reeb orbit is a closed flow for R_α;

$\Lambda \subset V \Rightarrow$ a Reeb chord is a flowline for R_α beginning + ending on Λ.

$\mathcal{A}$ is generated by Reeb chords of Λ. In general this is a module over the closed contact homology of V, grid by Reeb orbits.

Simplifying assumption: $V = J^1(M) = T^*M \times \mathbb{R}$. $\alpha = dz - \lambda$

$R_\alpha = \partial/\partial z$, no closed Reeb orbits.

So: let $\Lambda \subset \overset{\text{"}V\text{"}}{J^1(M)}$ be Legendrian, $\overset{\text{finitely many}}{\text{Reeb chords}}$ $a_1, \dots, a_n$.

- Algebra: $\mathcal{A}$ = free noncommutative unital algebra over $R = \mathbb{Z} H_2(V, \Lambda)$ generated by Reeb chords: $R\langle a_1, \dots, a_n \rangle$.

  Generators over R look like: $a_{i_1} \cdots a_{i_k}$ $k \geq 0$

  ($k=0$: 1 = empty word).

- **Grading:** $|a_i| = CZ(a_i) - 1$

  $|r| = 0, \ r \in R$. extend by product rule $|xy| = |x| + |y|$

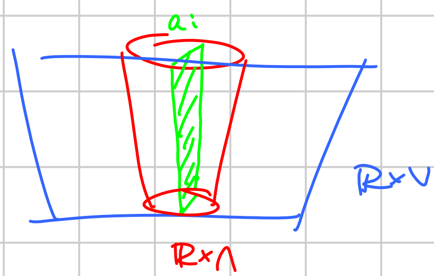
- **Differential:** given $a_i, a_{j_1}, \ldots, a_{j_k}$ $(k \geq 0)$, define

  $\mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k})$ = moduli space of J-hol disks in the symplectization of V.

Symplectization $= (\mathbb{R} \times V, \ \omega = d(e^t \alpha))$

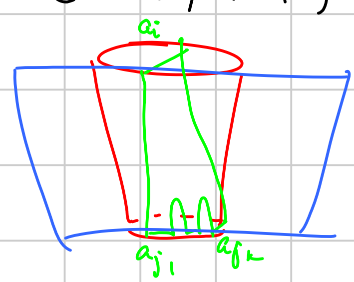
J = almost cx str on $\mathbb{R} \times V$ with $J: \xi \to \xi$, $J(\frac{\partial}{\partial t}) = R_\alpha$, J translation invt.

Note $\mathbb{R} \times a_i$ is a holomorphic strip:



$\mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k}) = \{ J\text{-hol maps}$

$$\Delta: (D^2 - \{p^+, p_1^-, \ldots, p_k^-\}, \partial) \longrightarrow (\mathbb{R} \times V, \mathbb{R} \times \Lambda)$$



asymptotic to a_i at $+\infty$ near p^+

a_{j_ℓ} at $-\infty$ near p_ℓ^-. $\}$

Note $\mathbb{R}$ acts on this moduli space by translation.

Index formula:

$$\dim \mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k}) = |a_i| - |a_{j_1}| - \cdots - |a_{j_k}|.$$

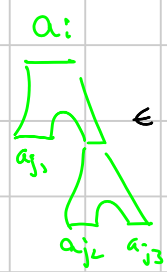
$$\boxed{\partial(a_i) = \sum_{\substack{\dim \mathcal{M}(a_i; a_{j_1}, \dots, a_{j_k})/\mathbb{R} = 0 \\ k \geq 0}} \; \sum_{\Delta \in \mathcal{M}/\mathbb{R}} (\mathrm{sgn})\, e^{[\Delta]} a_{j_1} \cdots a_{j_k}}$$

homology class of Δ in $H_2(V, \Lambda)$ (note: need to close up Reeb chords)

Extend ∂ to $\mathcal{A}$ by Leibniz $\partial(xy) = (\partial x)y + (-1)^{|x|} x(\partial y)$, $\partial r = 0$.

<u>Thm</u> (Ekholm-Etnyre-Sullivan, building on work of Eliashberg-Hofer, Chekanov)
$V = J^1(M)$, everything suitably generic.

1. ∂ decreases degree by 1
2. $\partial^2 = 0$
3. $H_*(\mathcal{A}, \partial) =: LCH_*(V, \Lambda)$ is an <u>invariant</u> of Λ up to Legendrian isotopy: <u>Legendrian Contact homology</u>.

Idea for 2: $\partial^2 a_i = a_{j_1} a_{j_2} a_{j_3} + \cdots$
comes from something like 

$\in \partial($1-d moduli space 

$)$

Other end of moduli space is also counted in $\partial^2 a_i$: e.g. could look like 

$(\mathcal{A}, \partial)$ is more "complicated" than some other Floer complexes, because it isn't linear: not fin-dim as an R-module.

<u>But</u> it has a key advantage: multiplication. also $H_*(\mathcal{A}, \partial)$.
(we'll use this heavily)

Knot Contact Homology

Now: $K \subset \mathbb{R}^3 \rightsquigarrow \Lambda_K$ unit conormal $\subset ST^*\mathbb{R}^3$.

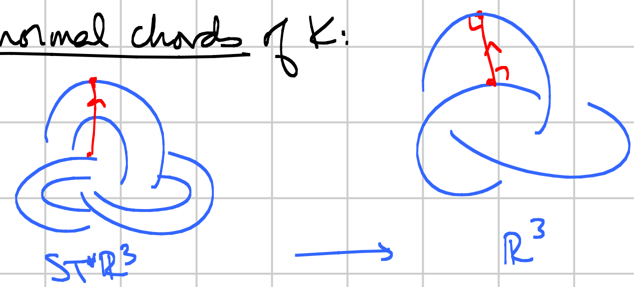
Convenient to think dually in $ST\mathbb{R}^3$: $\Lambda_K \leftrightarrow$ unit normal bundle

$\leftrightarrow \partial(\text{nbd}(K))$

$\cong T^2$.

Exc
- $ST^*\mathbb{R}^3$ is contactomorphic to $J^1(S^2)$ (both type $S^2 \times \mathbb{R}^3$).
- Reeb flow on $ST^*\mathbb{R}^3 \longleftrightarrow$ geodesic flow on $ST\mathbb{R}^3$

so Reeb chords are <u>binormal chords</u> of K:



Def The <u>knot contact homology</u> of K is

$$HC_*(K) := LCH_*(ST^*\mathbb{R}^3, \Lambda_K) = H_*(\mathcal{A}_K, \partial_K).$$

$(\mathcal{A}_K, \partial_K) =$ Chekanov-Eliashberg DGA

$\mathcal{A}_K$ generated by binormal chords over

$R = \mathbb{Z}\, H_2(ST^*\mathbb{R}^3, \Lambda_K)$

$\cong \mathbb{Z}[Q^{\pm 1}, \lambda^{\pm 1}, \mu^{\pm 1}]$.

$H_2(S^2 \times \mathbb{R}^3, T^2) \cong H_2(S^2) \oplus H_1(T^2)$

$\cong \mathbb{Z} \oplus \mathbb{Z}^2$

$\cong \langle Q, \lambda, \mu \rangle$

generates $[S^2]$ long., mer. of K

note: λ depends on choice of framing on K: choose Seifert framing.

(Recall: $\Delta : (D, \partial D) \to (\mathbb{R} \times ST^*\mathbb{R}^3, \mathbb{R} \times \Lambda_K)$

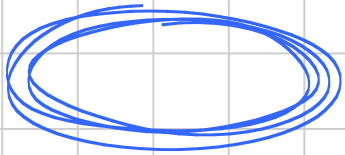
appears in differential with coefficient

$Q^a \lambda^b \mu^c$, $[\Delta] = a[S^2] \in H_2(ST^*\mathbb{R}^3, \Lambda_K)$

$[\partial \Delta] = b[\ell] + c[m] \in H_1(\Lambda_K)$.)

Combinatorial model

For K in general position, difficult to enumerate binormal chords, let alone differential. However: isotop K to be braided around an oval unknot:



in all cases $1 \le i,j \le n$

Can set things up so that:

- if n-strand braid, Reeb chords in degree $\begin{cases} 0 & (a_{ij},\ i \neq j) \\ 1 & (b_{ij},\ i \neq j;\ c_{ij}) \\ 2 & (e_{ij}) \end{cases}$
- hol. disks with ∂ on $\Lambda_K \longleftrightarrow$
  gradient flow trees on S^2 (Ekholm, based on Floer, Fukaya-Oh)

"Reduces" calculating differential to Morse theory.

Thm (Ekholm-Etnyre-N.-Sullivan 2011)
$\exists$ combinatorial formula for $(\mathcal{A}_K, \partial_K)$. (note: formula from N. ~'03-'05)

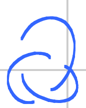
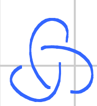
Note: $\mathcal{A}_K$ supported in deg ≥ 0.

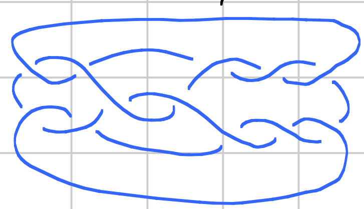
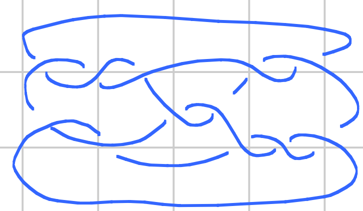
$\partial(a_{ij}) = 0$
$\partial(b_{ij}), \partial(c_{ij}) = \text{poly in } a_{ij}, Q^{\pm}, \lambda^{\pm}, \mu^{\pm}$
$\partial(e_{ij}) = \text{messy}.$

Ex unknot: $\mathcal{A}_U = R\langle \underset{1}{c}, \underset{2}{e} \rangle$ $\qquad \partial c = Q - \lambda - \mu + \lambda\mu$
$\partial e = 0$

Thm $(\mathcal{A}_K, \partial_K)$:

1. (N. '05) detects mirrors, mutants

 $\neq$ 

 $\neq$ 

KT $\qquad$ Conway

2. (N. '05) encodes the Alexander poly $\Delta_K(t)$
3. (N. '05, Gordon-Lidman '15) detects the unknot, torus knots, cabledness / compositeness, ...

Idea of pf of 1: count graded algebra maps $\epsilon : HC_*(K) \to \mathbb{Z}/3$ (!).

Note: $$HC_0(K) = R\langle a_{ij}\rangle \big/ (\partial b_{ij} = \partial c_{ij} = 0)$$

So ϵ is determined by $\epsilon(a_{ij}) \in \mathbb{Z}_3$ such that $\epsilon(\partial b_{ij}) = \epsilon(\partial c_{ij}) = 0$.

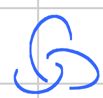
<u>Running examples</u>: for simplicity of computation, set $Q=1$.

DGA is over $R_0 = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$.

<u>Unknot</u>: $HC_0(\bigcirc) = R_0 \big/ (1-\lambda-\mu+\lambda\mu)$.

<u>trefoil</u>: $HC_0($  $) = R_0[\overset{\text{“}a_{12}\text{”}}{x}] \big/ (\overset{\text{“}\partial c_{12}\text{”}}{x^2 - \mu x + \lambda\mu^3 - \lambda\mu^4}, \overset{\text{“}\partial c_{11}\text{”}}{x^2 + \lambda\mu^2 x - \lambda\mu^2 + \lambda\mu^3})$

If $K_1 \cong K_2$ then $\exists$ iso $HC_0(K_1) \xrightarrow{\cong} HC_0(K_2)$ restricting to id on R_0.

Exc By counting ring homom $HC_0(K) \to \mathbb{Z}_3$, use this to prove that $\bigcirc$   are distinct.

(Note: $HC_0(m(K)) \cong HC_0(K)\big|_{\substack{\lambda \mapsto \lambda \\ \mu \mapsto \mu^{-1}}}$).

Augmentations

Def $(\mathcal{A},\partial)$ DGA over R, S = ring. An augmentation is a graded algebra map

$$\epsilon : (\mathcal{A},\partial) \to (S,0)$$

i.e. $\epsilon\circ\partial = 0$ and $\epsilon(a)=0$ unless $|a|=0$; $\epsilon(1)=1$.

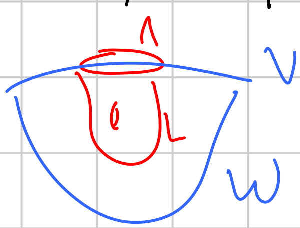
(Note induces $\epsilon : H_*(\mathcal{A},\partial)\to S$; also induces $\epsilon|_R : R\to S$.)

Ex. For $(\mathcal{A}_u,\partial_u)$, there are maps $\epsilon|_R : \mathbb{Z}[Q^{\pm},\lambda^{\pm},\mu^{\pm}]\to S$ such that

$$\epsilon(Q)-\epsilon(\lambda)-\epsilon(\mu)+\epsilon(\lambda)\epsilon(\mu)=0.$$

Where do augmentations often come from? Exact Lagrangian fillings.

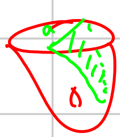
$\Lambda\subset V$, $V=\partial W$, W symplectic with convex boundary V , $\omega = d\theta$



$L\subset W$ is exact Lagrangian if $\omega|_L\equiv 0$, $\theta|_L = df$ for some $f: L\to\mathbb{R}$.

"SFT functoriality": think of L as a cobordism from Λ to $\emptyset$:

$\exists$ DGA map $(\mathcal{A}_\Lambda,\partial_\Lambda)\to(\mathcal{A}_\emptyset,\partial_\emptyset) = (R,0)$.

$\epsilon(a) = \#$ 

So: fillings $\Rightarrow$ augmentations.

Converse isn't true, but for $\Lambda\subset\mathbb{R}^3$, Ng-Rutherford-Shende-Sivek-Zaslow:

Constructible sheaves $\Longleftrightarrow$ augmentations.

Def The augmentation variety of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is

$$\mathrm{Aug}(\Lambda, S) = \{ \epsilon|_R \in \mathrm{Hom}(R, S) \; : \; \epsilon : \mathcal{A} \to S \text{ augmentation} \}.$$

Our case: $\Lambda = \Lambda_K$, $R = \mathbb{Z}[Q^{\pm}, \lambda^{\pm}, \mu^{\pm}]$, $S = \mathbb{C}$.

$$\epsilon|_R \longleftrightarrow (\epsilon(Q), \epsilon(\lambda), \epsilon(\mu)) \in (\mathbb{C}^\times)^3.$$

Def $K \subset \mathbb{R}^3$ knot. The augmentation variety of K, V_K, is the closure of the maximal-dimensional components of

$$\mathrm{Aug}(\Lambda_K, \mathbb{C}) \subset (\mathbb{C}^\times)^3.$$

(Complex algebraic variety)

Ex unknot: $\quad V_U = \{ Q - \lambda - \mu + \lambda\mu = 0 \} \subset (\mathbb{C}^\times)^3_{Q, \lambda, \mu}$.

More generally: for $K \subset \mathbb{R}^3$ an m-component link,

$R = \mathbb{Z}[Q^{\pm}, \lambda_1^{\pm}, \mu_1^{\pm}, \ldots, \lambda_m^{\pm}, \mu_m^{\pm}]$ and $V_K \subset (\mathbb{C}^\times)^{2m+1}$.

Empirical observation/conjecture

For fixed $Q \in \mathbb{C}^\times$, $\quad V_K$ is a complex Lagrangian in $(\mathbb{C}^\times)^{2m}$

w.r.t. the symplectic form $\sum_i d \log \lambda_i \wedge d \log \mu_i$.

in particular: if K is a knot, $V_K \subset (\mathbb{C}^\times)^3$ is codimension 1.

Def The augmentation polynomial of K is

$$\mathrm{Aug}_K(Q, \lambda, \mu) \in \mathbb{C}[Q, \lambda, \mu] \quad (\text{actually } \mathbb{Z} \text{ coeffs})$$

with $V_K = \{ \mathrm{Aug}_K = 0 \}$.

well-defined up to units in $\mathbb{Z}[Q^{\pm}, \lambda^{\pm}, \mu^{\pm}]$.

Ex $\mathrm{Aug}_U = Q - \lambda - \mu + \lambda\mu$.

in general: to calculate: write $\partial a_{ij}, \partial c_{ij}$ as

$$\begin{matrix} p_1(Q,\lambda,\mu,a_{ij}) \\ \vdots \\ p_N(Q,\lambda,\mu,a_{ij}) \end{matrix}$$

and find (Q,λ,μ) s.t. these have a common root in a_{ij}.

$\epsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \to (\mathbb{C}, 0)$ gives a point $(\epsilon(Q), \epsilon(\lambda), \epsilon(\mu), \epsilon(a_{ij}))$ in:

$(\mathbb{C}^\times)^3_{Q\lambda\mu} \times \mathbb{C}^{\text{large}}_{a_{ij}}$ $\qquad$ $(\mathbb{C}^\times)^3_{Q\lambda\mu}$

Rmk For 2-bridge knots, conjecturally can reduce to 1 variable: $p_1(Q,\lambda,\mu,x)$, $p_2(Q,\lambda,\mu,x)$. Then $\mathrm{Aug}_K(Q,\lambda,\mu) = \mathrm{resultant}(p_1, p_2; x)$.

Exc $HC_0(\;$$\;) \cong R[x] \big/ (Qx^2 - Q\mu x + \lambda\mu^3 - \lambda\mu^4,\ Qx^2 + \lambda\mu^2 x - Q\lambda\mu^2 + \lambda\mu^3)$.

Calculate Aug. answer: $(1-\lambda)Q^3 + (-\mu + \lambda\mu + 2\lambda\mu^2)Q^2$
$+ (-2\lambda\mu^2 + \lambda\mu^3 - \lambda\mu^4)Q + (-\lambda^2\mu^3 + \lambda^2\mu^4)$.

Easier: set $Q = 1$ and calculate "2-var aug poly"

$\mathrm{Aug}(\lambda,\mu)$. answer: $(\lambda - 1)(\mu - 1)(\lambda\mu^3 + 1)$.

cf. $\mathrm{Aug}_{\bigcirc}(\lambda,\mu) = (\lambda - 1)(\mu - 1)$.

What is Aug_K? Conjectural interpretation.

Colored HOMFLY polynomials (1 row):

$$K \rightsquigarrow P_K^n(a,q) = P_K^{\overbrace{\boxed{}\boxed{}\cdots\boxed{}}}(a,q) \qquad n=1: \text{HOMFLY}.$$

Garoufalidis-Lauda-Le: $\{P_K^n\}$ are q-holonomic: fix K, define operations $\hat{x}, \hat{y}$ by $\hat{x}P^n = P^{n+1}$, $\hat{y}P^n = q^n P^n$. $(\hat{y}\hat{x} = q\hat{x}\hat{y})$

Then $\{P_K^n\}$ satisfy a linear recurrence relation: $\exists$ poly $\hat{A}_K(a,q,\hat{x},\hat{y})$ s.t.

$$\hat{A}_K P_K^n = 0.$$

Conj $\hat{A}_K|_{q=1} = \mathrm{Aug}_K$ after suitable change of variables.

$a \longleftrightarrow Q$

$\hat{y} \longleftrightarrow \mu$

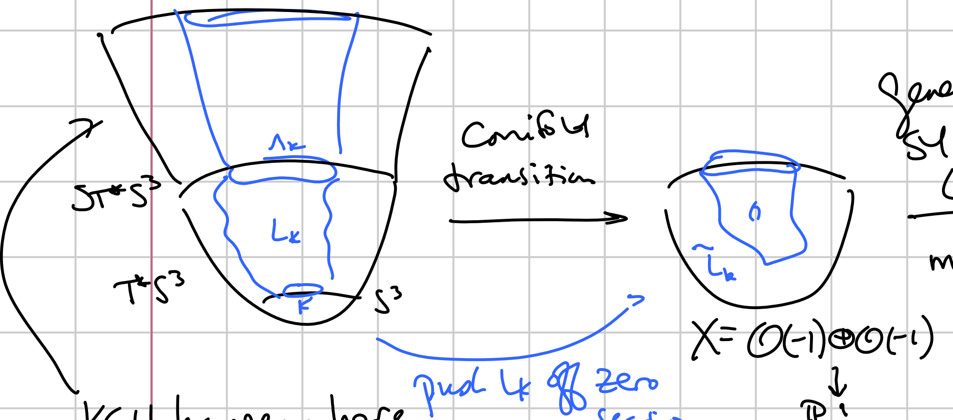
$\hat{x} \longleftrightarrow \lambda \frac{\mu-1}{\mu Q}$

Checked for unknot, some 2-bridge knots and some torus knots

This says: Aug_K = Classical limit of recurrence relation for colored HOMFLY

cf. "AJ Conjecture" (Garoufalidis): A-polynomial = ... Colored Jones poly.

Intermediary between aug poly + HOMFLY: topological string theory, in particular "Q-deformed A-polynomial" (Aganagic-Vafa).



KCH happens here ("at infinity")

Generalized SYZ conj (AV) $\xrightarrow{\text{mirror}}$ X_K = disk instanton corrected moduli space of sLag $T^3 \subset X$.

SU(N) CS for K $\xleftrightarrow{\text{Ooguri-Vafa}}$ open top. strings on X with $\tilde{L}_K$ brane (A-model) $\xleftrightarrow{\text{Aganagic-Vafa}}$ top. strings on X_K (B-model)

$$X_K = \{(u,v,x,y) \mid uv = A_K(Q,e^x,e^y)\} \subset \mathbb{C}^4 \qquad Q = e^t = \text{Kähler parameter for } \mathbb{P}^1$$

A_K ← "Q-def A-poly"

Conj (AENV) $A_K = \mathrm{Aug}_K$. (goes in both directions: colored HOMFLY; computability)

Idea for approach: a Lagrangian filling for Λ_K, like $\tilde{L}_K$, parametrizes part of the augmentation variety.

filling $\tilde{L}_K$ - - - → Gromov-Witten potential $W(Q,x)$ counting hol disks with ∂ on $\tilde{L}_K$

and if $p = \frac{\partial W}{\partial x}$ then $(Q, e^x, e^p) \in V_K$. (from symplectic topology)

But in fact the same GW potential appears in AV, parametrizing $\{A_K = 0\}$ = locus of singular fibers of mirror.

▷ Not known how to promote Aug to 3-var poly à la super-A-polynomial.

▷ In progress (?): using symplectic methods to promote to $\hat{\mathrm{Aug}}(Q,q,\lambda,\mu)$.
$(\mu\lambda = q\lambda\mu)$.

String topology + the knot group

write $\pi = \pi_1(\mathbb{R}^3 - K)$.

From now on, set $Q=1$: coefficients in DGA in $\mathbb{Z}[H_1(\Lambda_K)] \cong \mathbb{Z}[\lambda^{\pm}, \mu^{\pm}]$ rather than $\mathbb{Z}[H_2(ST^*\mathbb{R}^3, \Lambda_K)]$. (then can use "fully noncommutative" DGA)

As before: $\epsilon : (\mathcal{A}_K, \partial_K) \to \mathbb{C}$ gives $(\epsilon(\lambda), \epsilon(\mu)) \in (\mathbb{C}^\times)^2$
$\rightsquigarrow$ 2-variable aug. poly $\mathrm{Aug}_K(\lambda, \mu)$.

Thm (N.-Cornwell) $\mathrm{Aug}_K(\lambda,\mu) =$ stable A-polynomial of K.

$\rho : \pi \to GL(n, \mathbb{C})$ satisfying $\rho(m) = \begin{bmatrix} \mu & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ "KCH representation"

satisfies $\rho(\ell) = \begin{bmatrix} \lambda & 0 \\ 0 & \boxed{*} \end{bmatrix}$ for some λ $\rightsquigarrow (\lambda, \mu) \in (\mathbb{C}^\times)^2$.

The set of all such (λ, μ) is a variety $= \{ \tilde{A}_K(\lambda, \mu) = 0 \}$
where $\tilde{A}_K =$ stable A-poly.
($n=2$ gives reps to $SL(2,\mathbb{C})$ after renormalization; $\tilde{A}_K =$ multiple of A-polynomial A_K).

Cornwell: uses this to get bounds on meridional rank: min'l # of meridians generating π. $(\mathrm{ar}(K) \le \mathrm{mr}(K) \le b(K))$

Where does this come from? Any augmentation $\epsilon : (\mathcal{A}_K, \partial_K) \to (\mathbb{C}, 0)$ corresponds to a map $\epsilon : HC_0(K)|_{Q=1} \to \mathbb{C}$
λ, μ^k

Thm (Cieliebak, Ekholm, Latschev, N. '15)
Define $R_K \subset \mathbb{Z}\pi$ by R_K = subring gen'd by

$\cdot\ \lambda^{\pm 1}, \mu^{\pm 1}$

$\cdot\ (\mu - 1)x, \quad x \in \pi.$

Then if $K \neq$ unknot,

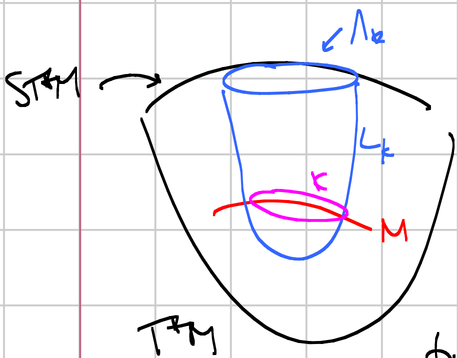
$$HC_0(K)|_{Q=1} \cong R_K \qquad \text{(ring isom.)}$$

Then, given $\rho: \pi \to GL(n, \mathbb{C})$ with $\rho(m) = \begin{bmatrix} \mu & 0 \\ 0 & \boxed{1} \end{bmatrix}$,
define $\epsilon: R_K \to \mathbb{C}$ by $\epsilon(\lambda) = \lambda$, $\epsilon(\mu) = \mu$,
$\epsilon((1-m)\gamma) = \langle e_1, \rho(\gamma) e_1 \rangle \qquad \gamma \in \pi.$

Exc this gives a ring homom. ($\Rightarrow$ augmentation).

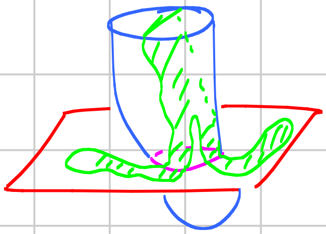
So $\tilde{A}_K \mid \mathrm{Aug}_K$. Converse: Cornwell.

Where does this relation to π come from? String topology / "filling by a singular Lagrangian".



Write $M=\mathbb{R}^3$. Then M, L_K are Lagrangian in T^*M, intersecting in K.

Consider a holomorphic disk in T^*M asymptotic to a Reeb chord for Λ_K with ∂ on $M \cup L_K$.



The boundary of this disk is a based loop in $L_K \cup M$: a "<u>broken string</u>".

Let $C_m = \mathbb{Z}\langle m\text{-chains of broken strings}\rangle$.

Then an index calculation $\Rightarrow$ moduli space of hol. disks asymptotic to a, $|a|=m$, is m-dim'l.

$\Rightarrow$ get a graded map $\mathcal{A}_K \longrightarrow C_*$.

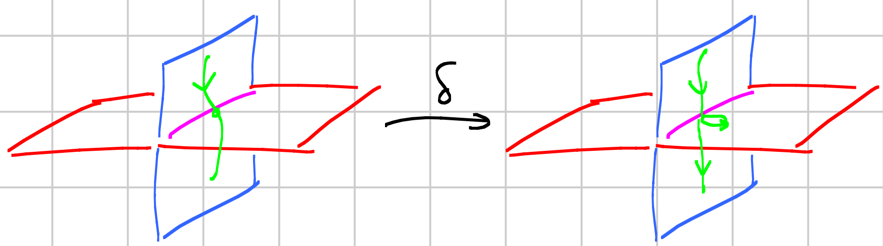
Thm (CELN) This gives a chain map

$$(\mathcal{A}_K, \partial_K)|_{Q=1} \longrightarrow (C_*, d)$$

inducing an isomorphism on homology. (at least in degree 0)

Here d isn't usual ∂, but $d = \partial + \delta$ ← <u>string coproduct</u>.

$\delta : C_m \to C_{m-1}$



$H_*(C_*, d)$ is the <u>string homology</u> of K: $H_0(C_*, d) \cong HC_0(K)|_{Q=1}$

Can be written in terms of π as given before.

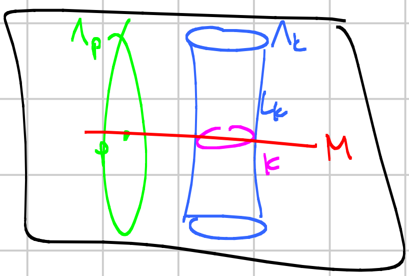
Complete Invariant

Using $HC_0(K)|_{Q=1}$, can prove:

- the conormal detects the unknot: N. 65
$$\Lambda_K = \Lambda_U \overset{\text{EENS}}{\Longrightarrow} HC_0(K)|_{Q=1} \cong HC_0(U)|_{Q=1} \Rightarrow K = U$$
- the conormal detects torus knots, compositeness/cabledness (Gordon–Lidman).

Open question: is KCH a complete knot invariant?

Observation: can "enhance" KCH by adding a cotangent fiber Λ_p and looking at $LCH_*(\Lambda_K \cup \Lambda_p)$ rather than just $LCH_*(\Lambda_K)$.



So: choose $p \in \mathbb{R}^3 - K$ and let $\Lambda_p = S^2$ fiber over p in $ST^*\mathbb{R}^3$. (conormal to $\{p\}$).
Then Λ_p is a Legendrian disjoint from Λ_K.

Key point: if K, K' are knots with $\Lambda_K \cong \Lambda_{K'}$ (Leg. isotopic) then $\Lambda_K \cup \Lambda_p \cong \Lambda_{K'} \cup \Lambda_p$. (just move p away from the isotopy).

Thus $\underset{H_*(\mathcal{A}, \partial)}{\overset{\shortparallel}{LCH_*(\Lambda_K \cup \Lambda_p)}}$ is an invariant of Λ_K.

$\mathcal{A}$ gen'd by Reeb chords from $\begin{Bmatrix}\Lambda_K \\ \Lambda_p\end{Bmatrix}$ to $\begin{Bmatrix}\Lambda_K \\ \Lambda_p\end{Bmatrix}$:
a quotient of this is $(\mathcal{A}_K, \partial_K)$.

Thm (Ekholm–Ng–Shende) If $LCH_*(\Lambda_K \cup \Lambda_p) \cong LCH_*(\Lambda_{K'} \cup \Lambda_p)$ then $K = K'$.

Cor Λ_K is a complete invariant of K.
(first proved by Shende using microlocal sheaves)

Idea of pf.
Write

$$LCH_*(\Lambda_K \cup \Lambda_p) = (LCH_*)_{\Lambda_K,\Lambda_K} \oplus \underbrace{(LCH_*)_{\Lambda_K,\Lambda_p}}_{R_{Kp}} \oplus \underbrace{(LCH_*)_{\Lambda_p,\Lambda_K}}_{R_{pK}} \oplus (LCH_*)_{\Lambda_p,\Lambda_p}$$

in minimal degree: R_{KK}

$R_{KK} = HC_0(K)$. $\longleftrightarrow$ based loops in $L_K \cup M \cup L_p$ based at L_K

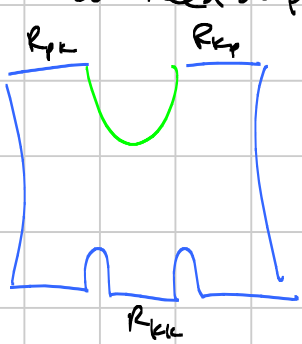
R_{Kp} $\longleftrightarrow$ paths in $L_K \cup M \cup L_p$ from L_K to L_p.

R_{pK} $\longleftrightarrow$ L_p to L_K.

think of this as concatenating paths $L_p \to L_K \to L_p$

Prop $\mathbb{Z} \oplus (R_{pK} \otimes_{R_{KK}} R_{Kp}) \cong \mathbb{Z}\pi$.

This is originally a $\mathbb{Z}$-module isom, but in fact it's a ring isom.
For this we need a product on LHS. This comes from



$$R_{Kp} \otimes R_{pK} \longrightarrow R_{KK}$$

2-input disks.
(Legendrian SFT: Ekholm '08)

Thm: if $\Lambda_K \cong \Lambda_{K'}$ then $LCH_*(\Lambda_K \cup \Lambda_p) \cong LCH_*(\Lambda_{K'} \cup \Lambda_p)$

$$\Rightarrow \quad \mathbb{Z}[\pi_1(\mathbb{R}^3 - K)] \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 - K')]$$

$$\Rightarrow \quad \pi_1(\mathbb{R}^3 - K) \cong \pi_1(\mathbb{R}^3 - K').$$

Exc Justify this as follows: G is <u>left orderable</u> if it has a total ordering $<$ invt under left multiplication.

Fact: knot groups are left orderable.

Prove: if G, G' are left orderable and $\mathbb{Z}[G] \cong \mathbb{Z}[G']$ then $G \cong G'$.

With a bit more work: this isom. sends longitude, meridian to longitude, meridian

(Waldhausen '68) $\Rightarrow \quad K \cong K'$.

[Rmk about parametrized Legendrian isotopy.]

$$\begin{tikzcd}
K \arrow[r] & \Lambda_K \subset ST^*\mathbb{R}^3 \arrow[r] \arrow[d] & \text{COMPLETE INVARIANT} \\
& LCH_*(\Lambda_K \cup \Lambda_p) \arrow[ur] \arrow[d] \arrow[ddl] & \\
& LCH_*(\Lambda_K) \text{ knot contact homology} \arrow[uur, dashed, "?"'] \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Alexander module of } K & \text{detects unknot etc.} & \text{augmentation variety.} \arrow[d] \\
& & \text{top. strings, colored HOMFLY, ...}
\end{tikzcd}$$