Filtered knot contact homology and transverse knots

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Geometric Topology Seminar
Columbia
March 4, 2011

References:
T. Ekholm, J. Etnyre, L. Ng, and M. Sullivan, "Filtrations on the knot contact homology of transverse knots", arXiv:1010.0450.
L. Ng, "Combinatorial knot contact homology and transverse knots", arXiv:1010.0451.
T. Ekholm, J. Etnyre, L. Ng, and M. Sullivan, "Knot contact homology", in preparation.
L. Ng, "Framed knot contact homology", *Duke Math. J.* **141**, 365–406.

Outline

Cotangents and conormals

- Let M be a smooth n-manifold.
  - T^*M is naturally a *symplectic* $2n$-manifold;
  - ST^*M, the cosphere bundle of M, is naturally a *contact* $(2n-1)$-manifold.

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- Let $K \subset M$ be any embedded submanifold. Define $L_K \subset T^*M$ to be the *conormal bundle* to K:

$$L_K = \{(q,p) \in T^*M : \ q \in K, \ \langle p, v \rangle = 0 \, \forall \, v \in T_q K\}.$$

Also define $\Lambda_K \subset ST^*M$ to be the *unit conormal bundle* to K:

$$\Lambda_K = L_K \cap ST^*M.$$

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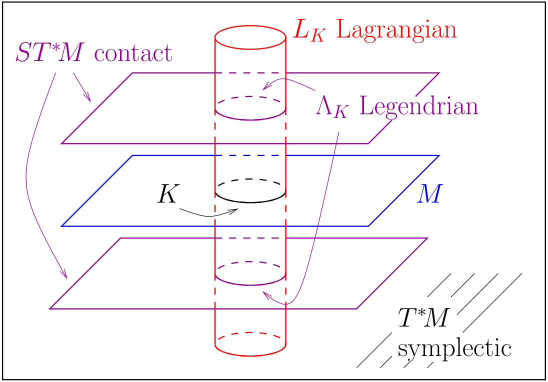
  - $L_K \subset T^*M$ is a *Lagrangian* submanifold ($\omega|_{L_K} \equiv 0$);
  - $\Lambda_K \subset ST^*M$ is a *Legendrian* submanifold (Λ_K tangent to ξ) .

Schematic picture



($K \subset M$ submanifold; ST^*M cosphere bundle; L_K conormal bundle to K; Λ_K unit conormal bundle to K.)

Symplectic and topological invariants

Symplectic/contact invariants of T^*M, ST^*M yield smooth invariants of M.

Question

Is T^*M up to symplectomorphism equivalent to M up to diffeomorphism? That is, does the symplectic topology of T^*M completely encode the smooth topology of M?

- Symplectic homology of T^*M and loop space cohomology: Viterbo, Abbondandolo–Schwarz, Salamon–Weber
- Cylindrical contact homology of ST^*M and string topology: Cieliebak–Latschev
- related work of Abouzaid, Seidel, . . .

Symplectic and topological invariants: the relative case

Relative case: invariants of L_K, Λ_K under Lagrangian/Legendrian isotopy yield smooth-isotopy invariants of $K \subset M$.

Question

Does the symplectic topology of the conormal bundle L_K completely encode the smooth topology of K? If Λ_{K_1} and Λ_{K_2} are Legendrian isotopic, does that imply that K_1 and K_2 are smoothly isotopic?

Symplectic and topological invariants: the relative case

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Does the symplectic topology of the conormal bundle L_K completely encode the smooth topology of K? If Λ_{K_1} and Λ_{K_2} are Legendrian isotopic, does that imply that K_1 and K_2 are smoothly isotopic?

Apply Legendrian contact homology ($\subset$ Symplectic Field Theory) due to Eliashberg–Hofer (for case $V = J^1(Q)$, work of Ekholm–Etnyre–Sullivan).

Recap

$K \subset M$ smooth

↓

Λ_K Legendrian $\subset ST^*M$ contact

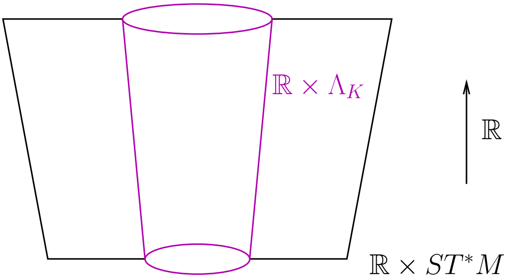
↓ Legendrian contact homology

$HC_*(ST^*M, \Lambda_K) =: HC_*(K)$, invariant of Legendrian Λ_K

When Legendrian contact homology is well-defined, this gives an isotopy invariant of K.

Legendrian contact homology

The LCH complex for $\Lambda_K \subset ST^*M$ is $(\mathcal{A}, \partial)$, where $\mathcal{A}$ is the tensor algebra freely generated by Reeb chords of Λ_K. The differential ∂ counts certain holomorphic disks with $\partial \subset \mathbb{R} \times \Lambda_K$.



The Lagrangian cylinder $\mathbb{R} \times \Lambda_K$ inside the symplectization $\mathbb{R} \times ST^*M$.

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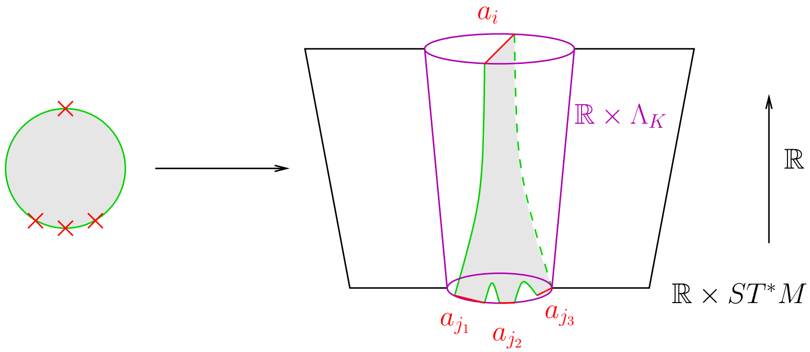

Holomorphic-disk contribution of $a_{j_1} a_{j_2} a_{j_3}$ to $\partial(a_i)$, where a_i, a_{j_1}, a_{j_2}, a_{j_3} are Reeb chords.

Knot contact homology

First reasonably nontrivial case:

- $M = \mathbb{R}^3$, $K \subset M$ knot (or link)
- $ST^*M = ST^*\mathbb{R}^3 = J^1(S^2)$
- Think of $\Lambda_K \subset ST^*\mathbb{R}^3$ as the boundary of a tubular neighborhood of $K \subset \mathbb{R}^3$; topologically T^2
- Λ_K is unknotted as a smooth torus but generally knotted as a Legendrian torus.

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Definition

Let $K \subset \mathbb{R}^3$ be a knot. The Legendrian contact homology of $\Lambda_K \subset ST^*\mathbb{R}^3$ is the knot contact homology of K,

$$HC_*(K) := HC_*(ST^*\mathbb{R}^3, \Lambda_K).$$

This is a smooth knot invariant.

Knot contact homology, continued

Knot contact homology $HC_*(K)$ is the homology of a differential graded algebra $(\mathcal{A}, \partial)$, where $\mathcal{A}$ is the graded tensor algebra over

$$R := \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$$

generated by finitely many generators in degrees 0, 1, 2 (Reeb chords for Λ_K). The coefficient ring keeps track of the relative homology classes of boundaries of holomorphic disks.

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There is a purely algebraic/combinatorial DGA $(\mathcal{A}^{\mathrm{comb}}, \partial^{\mathrm{comb}})$ associated to a braid or knot diagram for K; $\mathcal{A}^{\mathrm{comb}}$ is as above, but ∂^{comb} can be defined without PDEs.

Combinatorial knot contact homology

Here it is, for $B \in B_n$ a braid whose closure is K:

ϕ_B automorphism of the algebra generated by a_{ij}, $1 \le i, j \le n$, $i \neq j$, defined by

$$\phi_{\sigma_k} : \begin{cases} a_{ki} & \mapsto & -a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} & \mapsto & -a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \\ a_{k+1,i} & \mapsto & a_{ki} & i \neq k, k+1 \\ a_{i,k+1} & \mapsto & a_{ik} & i \neq k, k+1 \\ a_{k,k+1} & \mapsto & a_{k+1,k} & \\ a_{k+1,k} & \mapsto & a_{k,k+1} & \\ a_{ij} & \mapsto & a_{ij} & i, j \neq k, k+1; \end{cases}$$

$n \times n$ matrices Φ_B^L, Φ_B^R defined by

$$\phi_B(a_{i\cdot}) = \sum_{j=1}^{n} (\Phi_B^L)_{ij} a_{j\cdot} \qquad \text{and} \qquad \phi_B(a_{\cdot j}) = \sum_{i=1}^{n} a_{\cdot i} (\Phi_B^R)_{ij};$$

$n \times n$ matrix $\Lambda = \mathrm{diag}(\lambda, 1, \cdots, 1)$; generators a_{ij} $(i \neq j)$ of degree 0, b_{ij} $(i \neq j)$, c_{ij}, d_{ij} of degree 1, e_{ij}, f_{ij} of degree 2 with $1 \le i, j \le n$, assembled into $n \times n$ matrices A, B, C, D, E, F, with $A_{ij} = a_{ij}$ if $i > j$, μa_{ij} if $i < j$, $-1-\mu$ if $i = j$; $B_{ij} = b_{ij}$ if $i > j$, μb_{ij} if $i < j$, 0 if $i = j$; $C_{ij} = c_{ij}$, $D_{ij} = d_{ij}$, $E_{ij} = e_{ij}$, $F_{ij} = f_{ij}$;

$$\begin{aligned} \partial(A) &= 0 \\ \partial(B) &= A - \Lambda \cdot \Phi_B^L \cdot A \cdot \Phi_B^R \cdot \Lambda^{-1} \\ \partial(C) &= A - \Lambda \cdot \Phi_B^L \cdot A \\ \partial(D) &= A - A \cdot \Phi_B^R \cdot \Lambda^{-1} \\ \partial(E) &= B - C - \Lambda \cdot \Phi_B^L \cdot D \\ \partial(F) &= B - D - C \cdot \Phi_B^R \cdot \Lambda^{-1}. \end{aligned}$$

Invariance

Theorem (N., 2003)

The chain homotopy type of $(\mathcal{A}^{comb}, \partial^{comb})$ *is diagram-independent and yields a knot invariant, combinatorial knot contact homology*

$$HC_*^{comb}(K) := H_*(\mathcal{A}^{comb}, \partial^{comb}),$$

supported in degrees $* \geq 0$.

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Theorem (Ekholm–Etnyre–N.–Sullivan, in progress)

$(\mathcal{A}^{comb}, \partial^{comb})$ *is homotopy equivalent (in fact, "stable tame isomorphic") to the complex* $(\mathcal{A}, \partial)$ *for Legendrian contact homology; in particular,*

$$HC_*(K) \cong HC_*^{comb}(K).$$

Properties of knot contact homology $HC_*^{\mathrm{comb}}(K)$

Theorem (N., 2005)

- *HC_0^{comb} is a finitely generated, finitely presented noncommutative algebra over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ (=group ring of $H_1(\Lambda_K)$).*
- *Encodes Alexander polynomial (via linearized HC_1^{comb}).*
- *HC_0^{comb} is closely related to A-polynomial; distinguishes the unknot (Kronheimer–Mrowka, Dunfield–Garoufalidis).*
- *HC_0^{comb} extends to arbitrary codimension-2 submanifolds.*

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Corollary (Ekholm–Etnyre–N.–Sullivan)

$K \subset \mathbb{R}^3$ knot. If Λ_K is Legendrian isotopic to Λ_{unknot}, then K is the unknot.

Transverse knots

Definition

A knot K in a contact 3-manifold (M, ξ) is transverse if it is everywhere transverse to ξ. Two transverse knots are transversely isotopic if they are isotopic through transverse knots.

Bennequin: (closure of) braids $\longleftrightarrow$ transverse knots/links.

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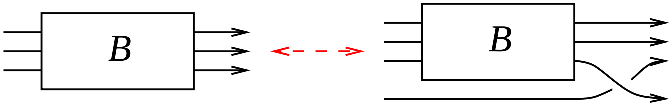
For $(M, \xi) = (\mathbb{R}^3, \xi_{\text{std}})$, the transverse Markov Theorem (Orevkov–Shevchishin, Wrinkle) states that transverse knots/links are equivalent to braids modulo:

- conjugation in the braid groups
- positive stabilization $B \longleftrightarrow B\sigma_n$:



Transverse classification

Question

Classify transverse knots of some particular topological type.

There is one “classical” invariant of transverse knots: self-linking number.

Definition

A topological knot is transversely simple if its transverse representatives are completely determined by self-linking number; otherwise transversely nonsimple.

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Transversely simple:

- unknot (Eliashberg)
- torus knots and the figure 8 knot (Etnyre–Honda)
- some twist knots (Etnyre–N.–Vértesi)
- ...

Transverse nonsimplicity

Transversely nonsimple:

- some torus knot cables (Etnyre–Honda, Etnyre–LaFountain–Tosun)
- some 3-braids (Birman–Menasco)
- a number of knots distinguished by Heegaard Floer homology.

Historically difficult problem: find effective invariants of transverse knots.

Definition

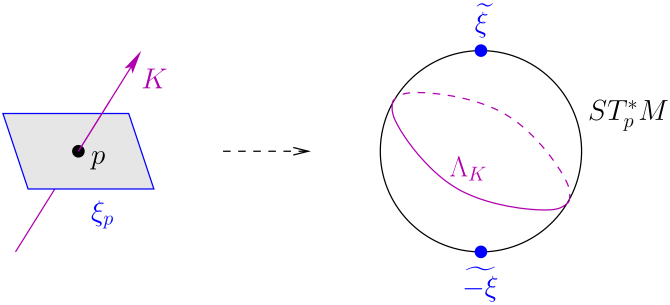
A transverse invariant is effective if it can distinguish different transverse knots with the same self-linking number and topological type (i.e., prove that some topological knot is transversely nonsimple).

Heegaard Floer homology provided the first.

Lifting a contact structure

Given a contact manifold (M, ξ), the contact structure ξ itself has a conormal lift to ST^*M:

$$\widetilde{\xi} \cup \widetilde{-\xi} = \{(q,p) \in ST^*M : \langle p, v\rangle = 0 \, \forall \, v \in \xi_q\}.$$



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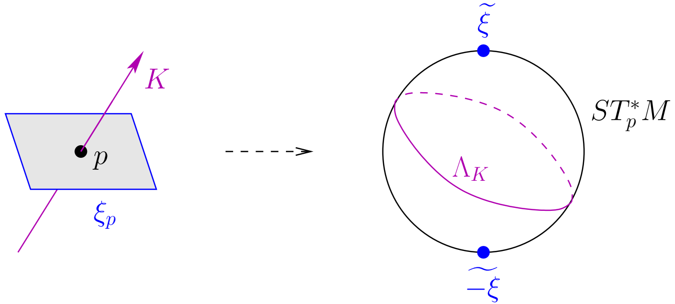
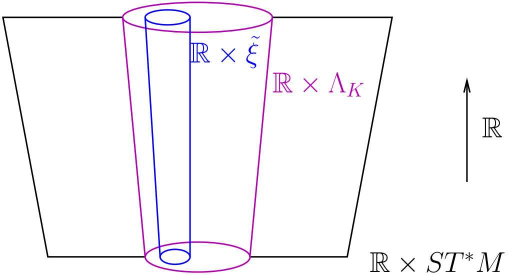


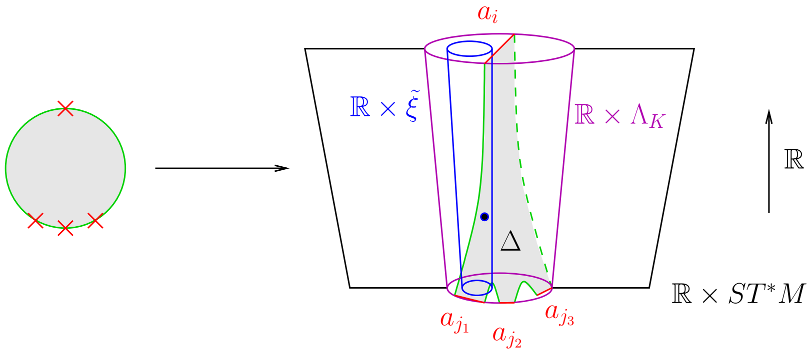
If K is transverse to ξ, then the conormal lifts of K and ξ are disjoint: $\Lambda_K \cap \widetilde{\pm\xi} = \emptyset$.

Filtering the LCH differential



- $(\mathbb{R} \times \Lambda_K) \cap (\mathbb{R} \times \widetilde{\pm\xi}) = \emptyset$
- $\dim(\mathbb{R} \times \widetilde{\pm\xi}) = 4$
- $\mathbb{R} \times \widetilde{\pm\xi}$ is holomorphic (given suitable choices).

Filtering the LCH differential



We can then filter the LCH differential for Λ_K by counting intersections with the holomorphic 4-manifolds $\mathbb{R} \times \widetilde{\pm\xi}$:

$$\partial^-(a_i) = U^{n_+(\Delta)} V^{n_-(\Delta)} a_{j_1} a_{j_2} a_{j_3} + \cdots ,$$

where $n_\pm(\Delta) \geq 0$ are the number of intersections of the holomorphic disk Δ with $\mathbb{R} \times \widetilde{\pm\xi}$.

Transverse homology

Definition

The (minus) transverse complex of a transverse knot K is the LCH algebra $(CT^-_*(K) = \mathcal{A}, \partial^-)$ over the base ring $R[U, V] = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U, V]$, with the differential ∂^- filtered by intersections with $\widetilde{\pm\xi}$. The transverse homology of K is $HT^-_*(K) = H_*(CT^-(K), \partial^-)$.

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Theorem

There is a combinatorial formula for $(CT_*^-(K), \partial^-)$ in terms of a braid representative of K.

This formula is a small tweak of the combinatorial formula for the complex for knot contact homology.

Combinatorial transverse homology

Here it is, for $B \in B_n$ a braid whose closure is K:

As before, algebra is generated by $a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}$, assembled into $n \times n$ matrices A, B, C, D, E, F; auxiliary $n \times n$ matrices $\hat{A}, \check{A}, \hat{B}, \check{B}$ defined by

$$\hat{A}_{ij} = \begin{cases} a_{ij} & i > j \\ \mu U a_{ij} & i < j \\ -1 - \mu U & i = j \end{cases} \qquad \check{A}_{ij} = \begin{cases} V a_{ij} & i > j \\ \mu a_{ij} & i < j \\ -V - \mu & i = j \end{cases}$$

$$\hat{B}_{ij} = \begin{cases} b_{ij} & i > j \\ \mu U b_{ij} & i < j \\ 0 & i = j \end{cases} \qquad \check{B}_{ij} = \begin{cases} V b_{ij} & i > j \\ \mu b_{ij} & i < j \\ 0 & i = j; \end{cases}$$

then the differential is given by

$$\begin{aligned}
\partial^-(A) &= 0 \\
\partial^-(B) &= A - \Lambda \cdot \Phi_B^L \cdot A \cdot \Phi_B^R \cdot \Lambda^{-1} \\
\partial^-(C) &= \hat{A} - \Lambda \cdot \Phi_B^L \cdot \check{A} \\
\partial^-(D) &= \check{A} - \hat{A} \cdot \Phi_B^R \cdot \Lambda^{-1} \\
\partial^-(E) &= \hat{B} - C - \Lambda \cdot \Phi_B^L \cdot D \\
\partial^-(F) &= \check{B} - D - C \cdot \Phi_B^R \cdot \Lambda^{-1}.
\end{aligned}$$

Main invariance results

Theorem

Up to stable tame isomorphism over $R[U, V]$, the transverse complex (CT_*^-, ∂^-) is invariant under transverse isotopy. In particular, transverse homology HT_*^- is an invariant of transverse knots.

Two proofs:

- geometric (Ekholm–Etnyre–N.–Sullivan), by explicit computation of the holomorphic disks in LCH
- combinatorial (N.), via the transverse Markov Theorem.

Flavors of transverse homology

From $(CT^-(K), \partial^-)$ chain complex over $R[U, V]$ (with $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$), obtain:

-
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- $(CT^{\infty}_*(K), \partial^{\infty})$ chain complex over $R[U^{\pm 1}, V^{\pm 1}]$, by tensoring with $R[U^{\pm 1}, V^{\pm 1}]$ $\longrightarrow$ topological invariant
- $(CC_*(K), \partial)$ chain complex over R, by setting $(U, V) = (1, 1)$ $\longrightarrow$ topological invariant; original formulation of knot contact homology

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The homologies of these chain complexes are various flavors of transverse homology.

Effectiveness

Theorem (N., 2010)

Transverse homology (more precisely, $\widehat{HT}_0$) is an effective invariant of transverse knots in $(\mathbb{R}^3, \xi_{std})$.

Previous transverse invariants:

- Plamenevskaya, Wu: distinguished elements of Khovanov and Khovanov–Rozansky homology; not known to be effective (and guessed not to be?)

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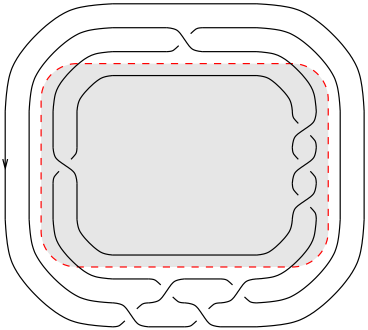
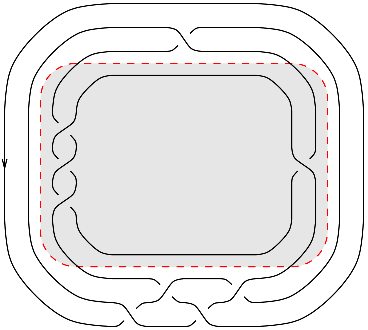
- Ozsváth–Szabó–Thurston: distinguished element of knot Floer homology via grid diagrams; known to be effective (work of Baldwin, Chongchitmate, Khandhawit, N., Ozsváth, Thurston, Vértesi, . . .)
- Lisca–Ozsváth–Stipsicz–Szabó: distinguished element of knot Floer homology via open book decompositions; known to be effective.

Example: $m(7_6)$ knot



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_3^3\sigma_2\sigma_3^{-1})$



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_3^{-1}\sigma_2\sigma_3^3)$

These two transverse representatives of the $m(7_6)$ knot, which are related by a "negative flype", can be distinguished by $\widehat{HT}_0$: one has no ring homomorphisms to $\mathbb{Z}/3$, the other has 5.

They can't be distinguished by the (hat) HFK invariant, which is an element of $\widehat{HFK}_{0,0}(m(7_6)) = 0$.

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK					
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK					
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK					
HT					

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK					
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK		✓		✓	
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK					
HT					

2007: N.–Ozsváth–Thurston, using grid diagrams

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓				
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK		✓		✓	
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK					
HT					

2008: Ozsváth–Stipsicz, using naturality of LOSS invariant

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓				
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK		✓		✓	
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓		✓	✓	
HT					

2010: Chongchitmate–N., using grid diagrams

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	×	×
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	×	✓	×	✓	
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT					

HFK invariants can't distinguish these.

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	×	×
HT	✓	✓	✓		✓
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	×	✓	×	✓	
HT		✓	✓	✓	
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓		✓	✓	

2010: N., using transverse homology

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	×	×
HT	✓	✓	✓	×?	✓

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	×	✓	×	✓	
HT	×?	✓	✓	✓	

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓	×?	✓	✓	

These are “transverse mirrors”, as are the Birman–Menasco knots.

Transverse nonsimplicity computations

Legendrian knot atlas (Chongchitmate–N.): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	×	×
HT	✓	✓	✓	×?	✓
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	×	✓	×	✓	
HT	×?	✓	✓	✓	
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓	×?	✓	✓	