Infinitely Many Fillings Via Augmentations

Lenny Ng
Duke University (USA)

Legendrians, Cluster Algebras, and Mirror Symmetry
— 11 January 2021 —

Joint work with R. Casals

Some version of these notes are at: math.duke.edu/~ng/fillings.pdf

$\boxed{\text{Setting}}$ $(D^4, \omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ standard symplectic 4-ball

boundary: $(S^3, \alpha_{std} = x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2)$ standard contact 3-sphere

$\boxed{\text{Motivating problem}}$

Given a Legendrian link $\Lambda \subset S^3$,
study the Lagrangian fillings $L \subset D^4$ of Λ:
Lagrangians $L \subset D^4$ with $\partial L = \Lambda$.
$\rightarrow$ How many fillings does a given Λ have?

- Many Legendrian links have no fillings:
  slice-Bennequin inequality (Rudolph 1993) + Chantraine 2010
  $\Rightarrow$ if Λ has a filling and is a knot then $tb(\Lambda) = 2g_4(\Lambda) - 1$
  ($\Rightarrow$ topology of L is determined by Λ)
- the standard Legendrian unknot  has a unique filling (Eliashberg-Polterovich 1996)
- the standard Legendrian trefoil  has ≥ 5 fillings (Ekholm-Honda-Kálmán 2016)
- the standard Legendrian $(2,n)$ torus knot has $\geq C_n = \frac{1}{n+1}\binom{2n}{n}$ fillings (EHK + Pan 2017)

Thm (Casals-Gao, Jan 2020; Casals-Zaslow, July 2020; Gao-Shen-Weng, Sep 2020; Casals-N in progress)

There are Legendrian links in (S^3, α_{std}) with infinitely many fillings.

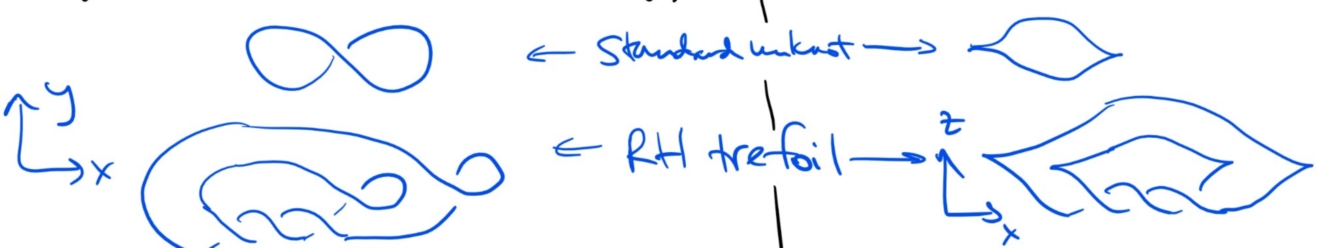
- Casals-Gao: microlocal sheaves, cluster algebras, loops of Legendrians
- Casals-Zaslow: microlocal sheaves, cluster algebras, Legendrian weaves
- Gao-Shen-Weng: Donaldson-Thomas transformations of cluster varieties
- Casals-N: Floer theory (Legendrian contact homology and augmentations)

Background

On $\mathbb{R}^3$: standard contact structure $\xi_{std} = \ker(\overbrace{dz - y\,dx}^{\alpha_{std}})$.

Def A link $\Lambda \subset \mathbb{R}^3$ is Legendrian if Λ is everywhere tangent to ξ_{std}.

Lagrangian projection to $\mathbb{R}^2_{xy}$: $z = \int y\,dx$	front projection to $\mathbb{R}^2_{xz}$: $y = \frac{dz}{dx}$



Reeb chords of Λ are integral curves for the Reeb vector field $\frac{\partial}{\partial z}$ on $\mathbb{R}^3$ with endpoints on Λ:

in Lagrangian projection, Reeb chords of Λ = crossings of $\Pi_{xy}(\Lambda)$.

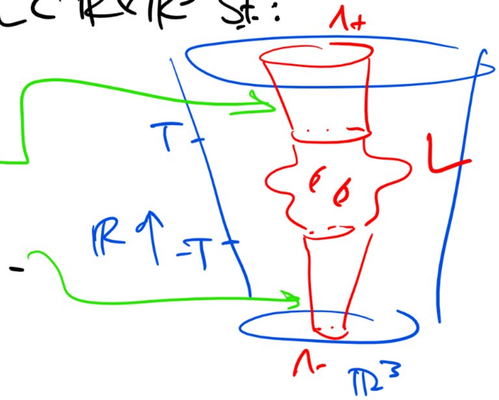
Def The symplectization of $\mathbb{R}^3$ is $(\mathbb{R}_t \times \mathbb{R}^3, \omega = d(e^t \alpha_{std}))$.
A submanifold $L \subset \mathbb{R} \times \mathbb{R}^3$ is Lagrangian if $\dim L = 2$ and $\omega|_L \equiv 0$.
ex: $\Lambda \subset \mathbb{R}^3$ Legendrian $\Rightarrow$ $\mathbb{R} \times \Lambda \subset \mathbb{R} \times \mathbb{R}^3$ is a Lagrangian cylinder.

Def $\Lambda_\pm \subset \mathbb{R}^3$ Legendrian. An exact Lagrangian cobordism to Λ_+ from Λ_- is a Lagrangian $L \subset \mathbb{R} \times \mathbb{R}^3$ s.t.:

⓪ L is embedded and oriented

① for some T, $L \cap \{t > T\} = (T, \infty) \times \Lambda_+$
$L \cap \{t < -T\} = (-\infty, -T) \times \Lambda_-$

② $\exists\ f : L \to \mathbb{R}$ s.t. $df = e^t \alpha_{std}|_L$
and f is constant on each end.



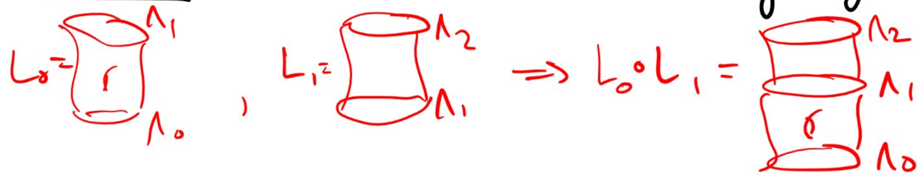
Def $\Lambda \subset \mathbb{R}^3$ Legendrian. A(n exact Lagrangian) filling of Λ is an exact Lagrangian cobordism to Λ from ϕ.

Usually consider cobordisms/fillings up to Hamiltonian isotopy: in this setting, isotopy through exact Lagrangian cobordisms with fixed ends.

Restated main question: Given a Legendrian link $\Lambda \subset \mathbb{R}^3$, how many fillings does Λ have, up to Hamiltonian isotopy?

Constructing exact Lagrangian cobordisms

Observation: Can concatenate exact Lagrangian cobordisms to get another:



Def The Lagrangian cobordism category $\mathcal{LCob}$ is defined by:

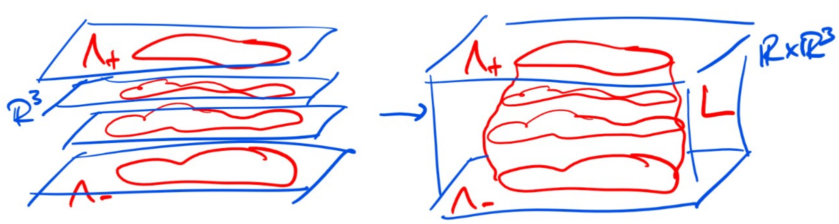
$\mathrm{Ob}\,\mathcal{LCob} =$ oriented Legendrian links in $\mathbb{R}^3$

$\mathrm{Hom}(\Lambda_-, \Lambda_+) = \{$exact Lagrangian cobordisms to Λ_+ from $\Lambda_-\}$

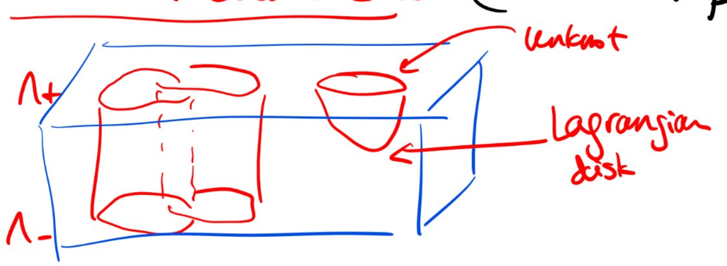
Can construct exact Lagrangian cobordisms by concatenating elementary cobordisms (Ekholm-Honda-Kálmán):

- isotopy cylinders:

  Given any Legendrian isotopy between Λ_- and Λ_+, the trace of the isotopy can be deformed to give an exact Lagrangian cobordism: each component is topologically a cylinder.

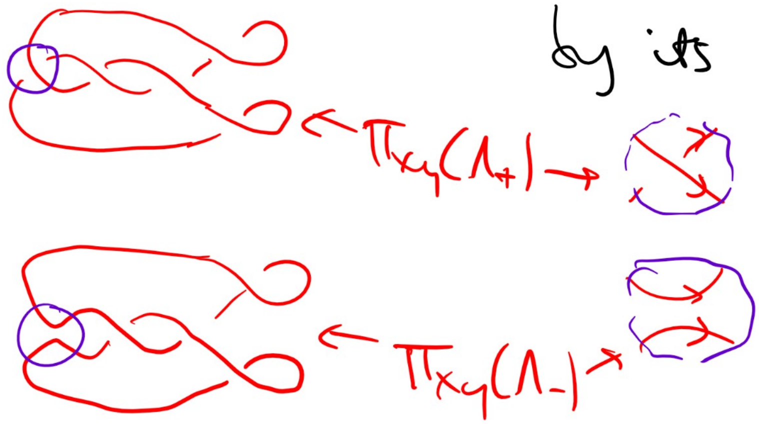


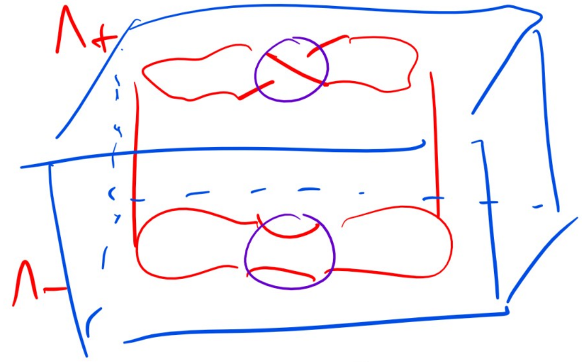
- minimum cobordisms (disk fillings):



$\exists$ standard Lagrangian disk giving a cobordism to standard Legendrian unknot  from ϕ.

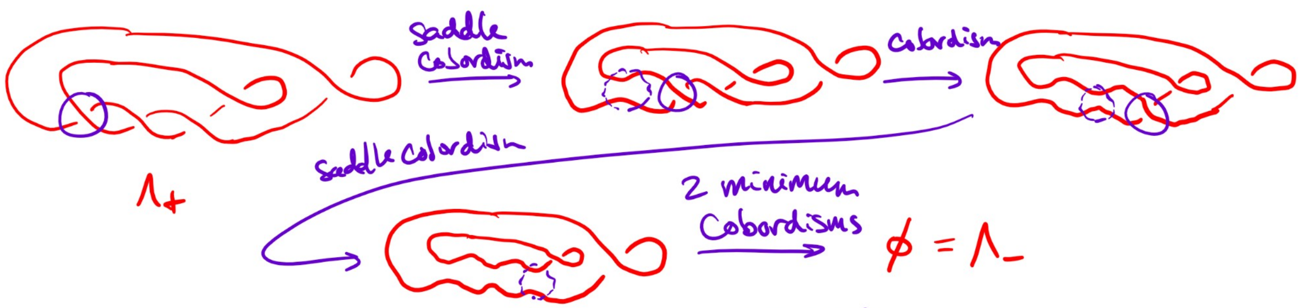
• **saddle cobordisms**: replacing a crossing (contractible Reeb chord) in xy projection of Λ_+ by its resolution.



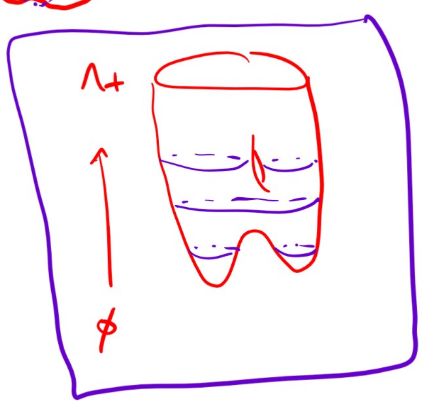


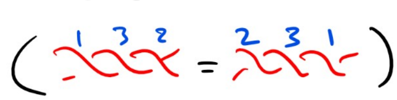
There is an exact Lagrangian cobordism to Λ_+ from Λ_-: <u>saddle cobordism</u>.

<u>Ex</u> Filling of the Legendrian trefoil.



$\Rightarrow$ genus 1 filling of the trefoil



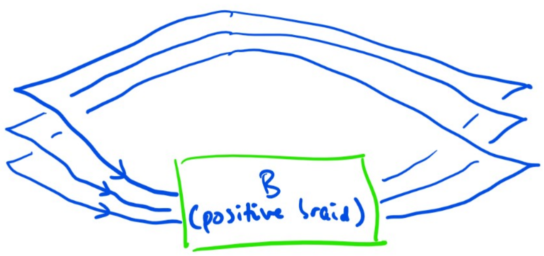
Any of the 3! orderings of doing saddle cobordisms at the 3 crossings  gives a filling. 

Ekholm-Honda-Kálmán: exactly 5 of these are distinct.

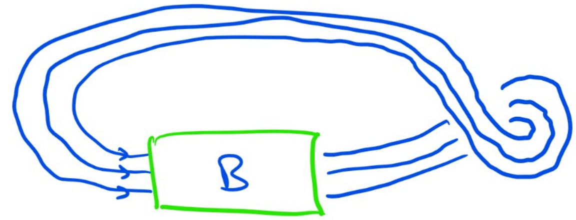
Infinite families of fillings: a template

We'll consider Legendrian links that are (-1)-closures of positive braids:

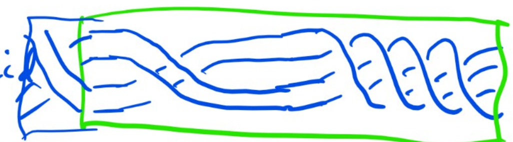
xz projection



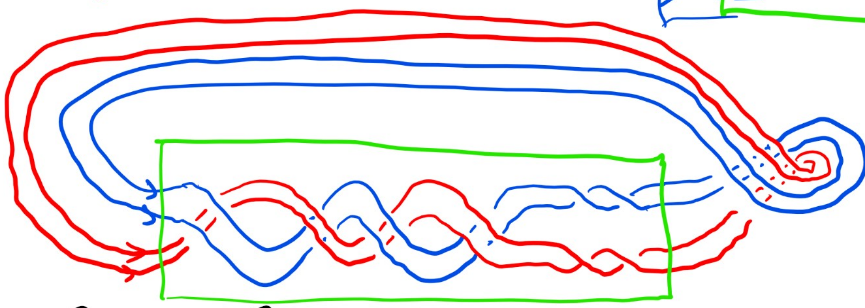
xy projection



Affine D_4 Legendrian $\Lambda(\tilde{D}_4)$: braid

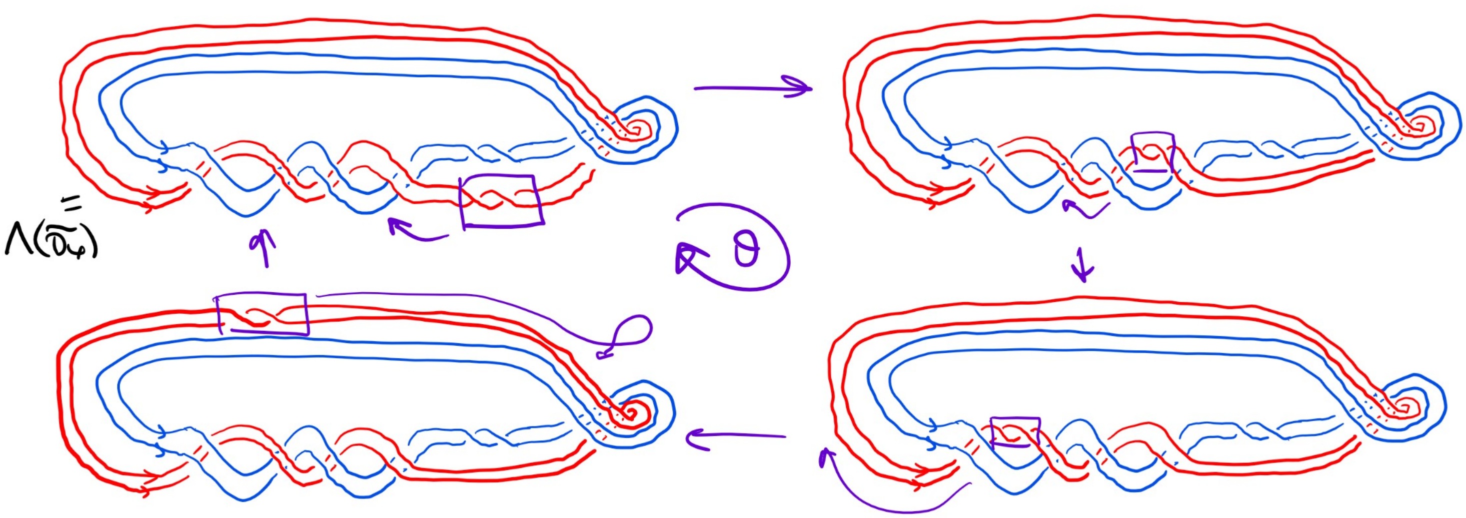


$\Lambda(\tilde{D}_4) =$



Write $\mathcal{L}(\Lambda) = \{$ Legendrian links in $\mathbb{R}^3$ that are Legendrian isotopic to $\Lambda \}$

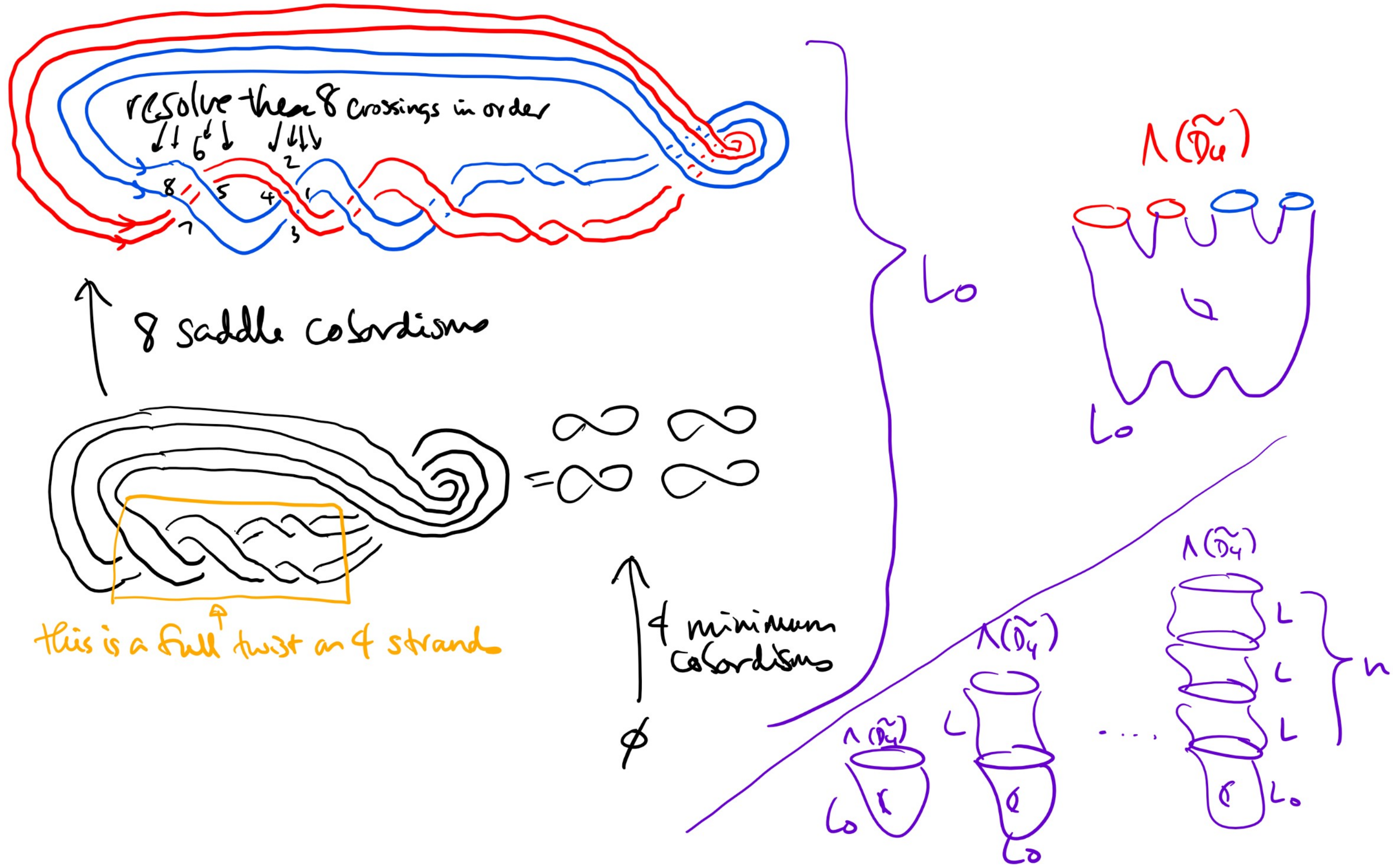
Define a loop $\Theta \subset \mathcal{L}(\Lambda(\tilde{D}_4))$ based at $\Lambda(\tilde{D}_4)$:



For $n \in \mathbb{Z}$, Θ^n is a Legendrian isotopy from $\Lambda(\tilde{D}_4)$ to itself

$\rightsquigarrow$ exact Lagrangian isotopy cylinder $L^n \in \mathrm{Hom}(\Lambda(\tilde{D}^4), \Lambda(\tilde{D}^4))$.

Also let L_0 be the filling of $\Lambda(\tilde{D}_4)$ constructed out of elementary cobordisms (saddle + minimum) as follows:



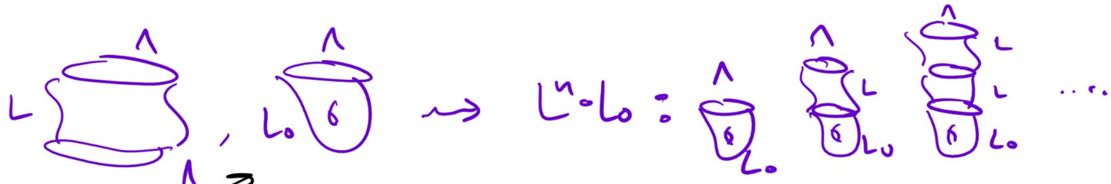
Thm (Casals–N) Let $\Lambda = \Lambda(\tilde{D}_4)$. The fillings $L^n \circ L_0$ of Λ for $n \in \mathbb{Z}$ are all distinct under Hamiltonian isotopy. In particular, Λ has infinitely many fillings.

Rmk: All of these fillings are topologically isotopic.

Cor ① The loop θ in $\mathcal{L}(\Lambda)$ induces an infinite-order class $[\theta] \in \pi_1(\mathcal{L}(\Lambda))$.

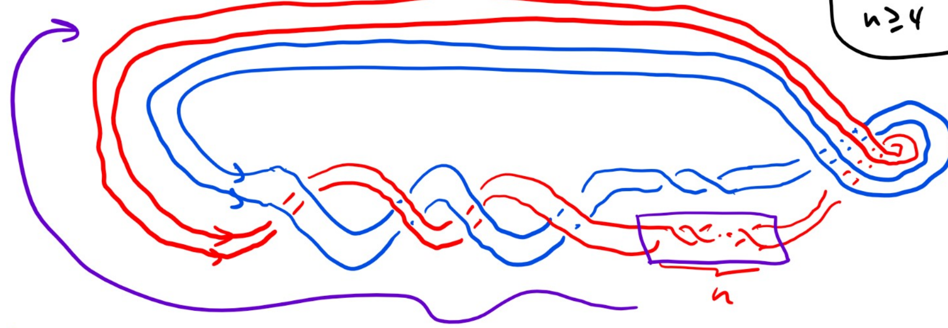
② θ induces a Lagrangian self-concordance L of Λ that has infinite order in the monoid

$$\{\text{Lagrangian self-concordances of } \Lambda\} / (\text{Hamiltonian isotopy}).$$

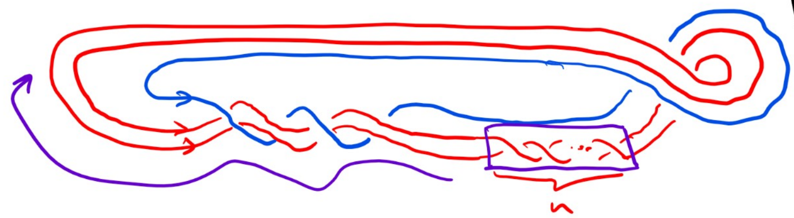


This template works to prove that other Legendrian links Λ have infinitely many fillings too.

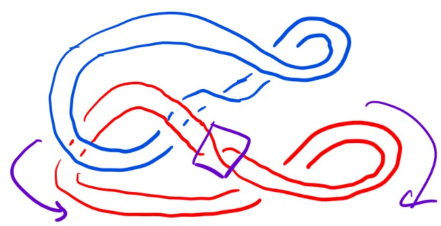
In each of these, L is induced by the "purple-box isotopy" sending ▭ around the parallel red components.



$\Lambda(\tilde{D}_n)$
$n \geq 4$



Λ_n
$n \geq 1$



$\Lambda(\tilde{A}_{1,1})$

<u>Thm</u> (Casals-N)

Each of these Legendrian links

$\Lambda(\tilde{D}_n)$, Λ_n , $\Lambda(\tilde{A}_{1,1})$

has infinitely many fillings $L^n \circ L_0$, $n \in \mathbb{Z}$

where
- L comes from the purple-box isotopy
- L_0 is given by a certain sequence of saddle + minimum cobordisms in each case.

Cobordisms and Legendrian Contact Homology

Floer-theoretic input: Legendrian contact homology

$\Lambda \subset \mathbb{R}^3$ Legendrian $\rightsquigarrow$ Chekanov-Eliashberg differential graded algebra $(\mathcal{A}_\Lambda, \partial)$

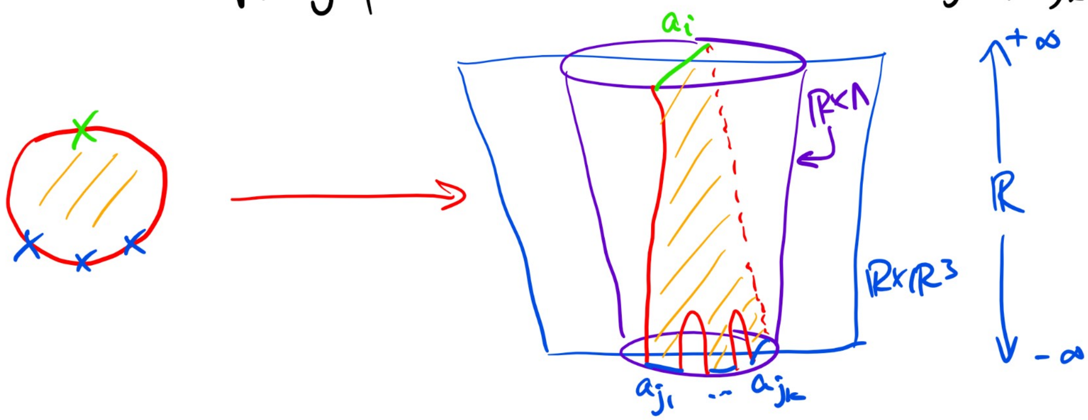
- $\mathcal{A}_\Lambda$ = tensor algebra over $\mathbb{Z}[H_1(\Lambda)]$ generated by Reeb chords of Λ, graded by Conley-Zehnder indices
- $\partial: \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ counts holomorphic disks:

  $a_i, a_{j_1}, \dots, a_{j_k}$ Reeb chords of Λ $(k \geq 0)$

  $\Rightarrow \mathcal{M}(a_i; a_{j_1}, \dots, a_{j_k})$ = moduli space of holomorphic disks

  $(D^2$ with boundary punctures, $\partial D^2) \longrightarrow (\mathbb{R}\times\mathbb{R}^3, \mathbb{R}\times\Lambda)$

  mapping punctures to a_i at $+\infty$ and $a_{j_1}, \dots, a_{j_k}$ at $-\infty$.



$$\boxed{\partial(a_i) = \sum_{\substack{a_{j_1},\dots,a_{j_k}\ (k\geq 0)\\ \dim \mathcal{M}(a_i; a_{j_1},\dots,a_{j_k})/\mathbb{R} = 0}} \ \sum_{\Delta \in \mathcal{M}(a_i; a_{j_1},\dots,a_{j_k})} (\operatorname{sgn}\Delta)\, e^{[\partial\Delta]} a_{j_1}\cdots a_{j_k}}$$

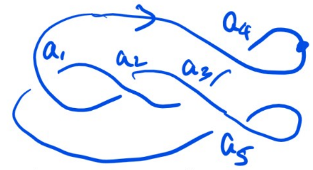
$[\partial\Delta] \in H_1(\Lambda)$

Chekanov, Etnyre-N-Sabloff, Ekholm-Etnyre-Sullivan:

$\partial^2 = 0$, $\deg \partial = -1$, and $(\mathcal{A}_\Lambda, \partial)$ is invariant (up to chain homotopy equivalence) under Legendrian isotopy of Λ.

$H_*(\mathcal{A}_\Lambda, \partial)$ is the Legendrian contact homology of Λ.

Ex: trefoil



$\mathcal{A}_\Lambda = (\mathbb{Z}[t^{\pm 1}])\langle a_1, a_2, a_3, a_4, a_5 \rangle$

$|a_1| = |a_2| = |a_3| = 0$, $|a_4| = |a_5| = 1$

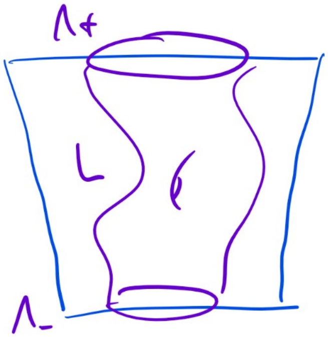
$\partial(a_1) = \partial(a_2) = \partial(a_3) = 0$

$\partial(a_4) = t + a_1 + a_3 + a_1 a_2 a_3$

$\partial(a_5) = 1 - a_1 - a_3 - a_3 a_2 a_1$

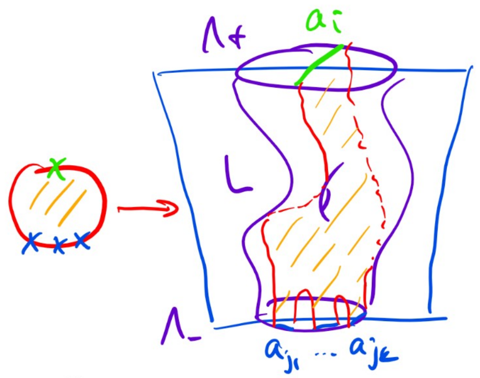
Ekholm-Honda-Kálmán, Karlsson:

Exact Lagrangian cobordisms induce chain maps between DGAs.



$\Longrightarrow$

$$\begin{array}{c} (\mathcal{A}_{\Lambda_+}, \partial) \\ \big\downarrow \Phi_L \\ (\mathcal{A}_{\Lambda_-}, \partial) \end{array}$$



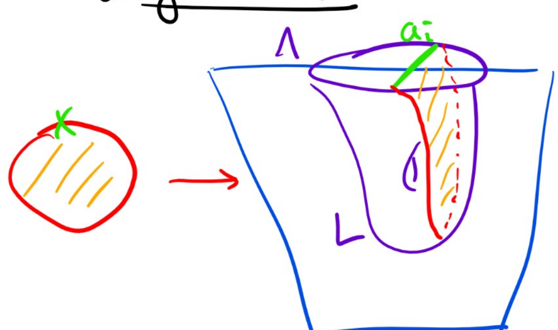
$$\Phi_L(a_i) = \sum_{\dim \mathcal{M}(a_i; a_{j_1}, \dots, a_{j_k}) = 0} (\text{coefficient})\, a_{j_1} \cdots a_{j_k}$$

Special case: when L is a filling of Λ then we get a DGA map

$$\Phi_L : (\mathcal{A}_\Lambda, \partial) \longrightarrow (\mathbb{Z}[H_1(L)], 0) :$$

"$\mathcal{A}_\emptyset$"

an augmentation.



$[\partial\Delta] \in H_1(L)$

$$\boxed{\Phi_L(a_i) = \sum_{\substack{\dim \mathcal{M}(a_i) = 0 \\ \Delta \in \mathcal{M}(a_i)}} (\operatorname{sgn} \Delta)\, e^{[\partial\Delta]}}$$

Conclusion in the spirit of Symplectic Field Theory:

Legendrian contact homology gives a contravariant functor

$$\underline{LCob} \longrightarrow \underline{DGA}$$

Λ Legendrian $\longrightarrow (\mathcal{A}_\Lambda, \partial)$ Chekanov-Eliashberg DGA

$\begin{array}{c} \Lambda_+ \\ L \uparrow \\ \Lambda_- \end{array}$ exact Lagrangian cobordism $\longrightarrow$ $\begin{array}{c} (\mathcal{A}_{\Lambda_+}, \partial) \\ \downarrow \Phi_L \\ (\mathcal{A}_{\Lambda_-}, \partial) \end{array}$ Ekholm-Honda-Kálmán cobordism map of DGAs

Prop (Ekholm-Honda-Kálmán, Karlsson)

① This is functorial:

$\Phi_{L_1 \circ L_0} = \Phi_{L_0} \circ \Phi_{L_1}$

$$\begin{tikzcd} (\mathcal{A}_{\Lambda_2}, \partial) \arrow[r, "\Phi_{L_1}"] \arrow[d, "\Phi_{L_1 \circ L_0}"'] & (\mathcal{A}_{\Lambda_1}, \partial) \arrow[dl, "\Phi_{L_0}"] \\ (\mathcal{A}_{\Lambda_0}, \partial) & \end{tikzcd}$$

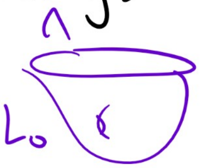
② If L, L' are cobordisms between Λ_- and Λ_+ that are <u>Hamiltonian isotopic</u> then $\Phi_L = \Phi_{L'}$ $\underbrace{\text{up to chain homotopy}}$.

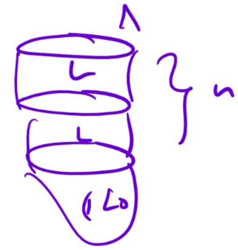
(for us, actually equal for degree reasons)

> Key takeaway:
> If L, L' are fillings of Λ that are Hamiltonian isotopic then
> $$\Phi_L = \Phi_{L'} : (\mathcal{A}_\Lambda, \partial) \to (\mathbb{Z}[H_1(L)], 0).$$

Applying this to our setting:

(from isotopy of Λ) ,

L_0 

$\rightsquigarrow L^n \circ L_0 =$ 

① Calculate the augmentation map

$$\Phi_{L_0} : (\mathcal{A}_\Lambda, \partial) \to (\mathbb{Z}[H_1(L_0)], 0) \cong (\mathbb{Z}[s_1^{\pm 1}, \dots, s_k^{\pm 1}], 0)$$

by using a signed version of the explicit saddle cobordism maps constructed by Ekholm-Honda-Kálmán.

② Calculate the monodromy map

$$\Phi_L : (\mathcal{A}_\Lambda, \partial) \xrightarrow{\cong} (\mathcal{A}_\Lambda, \partial)$$

by using the chain maps for Legendrian isotopies (Chekanov, Etnyre-N-Sullivan, Kálmán, Ekholm-Honda-Kálmán)

③ Show that the maps $\Phi_{L_0} \circ \Phi_L^n : (\mathcal{A}_\Lambda, \partial) \to (\mathbb{Z}[s_1^{\pm 1}, \dots, s_k^{\pm 1}], 0)$ are all distinct (by composing with the 2^k possible maps $\mathbb{Z}[s_1^{\pm 1}, \dots, s_k^{\pm 1}] \to \mathbb{Z}$).

④ The fillings $L^n \circ L_0$ of Λ are all distinct!

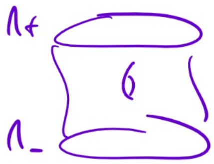
Creating more Legendrians with ∞ many fillings:

Def Legendrian $\Lambda \subset \mathbb{R}^3$ is *aug-infinite* if $\exists$ ∞ many augmentations $(\mathcal{A}_\Lambda, \partial) \to (\mathbb{Z}, 0)$ coming from fillings.

Observation (Λ aug-infinite) $\Rightarrow$ (Λ has infinitely many fillings)

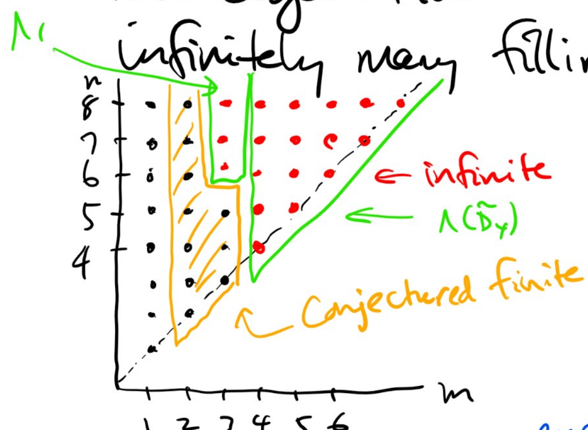
What we've proven $\Lambda(\tilde{D}_n)$, Λ_n, $\Lambda(\tilde{A}_{1,1})$ are aug-infinite.

Prop If $\exists$ exact Lagrangian cobordism to Λ_+ from Λ_- and Λ_+ has only nonnegatively graded Reeb chords then



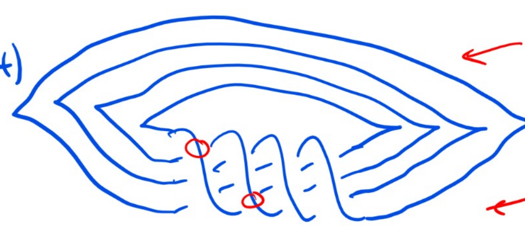
(Λ_- aug-infinite) $\Rightarrow$ (Λ_+ aug-infinite).

Cor (Casals-Gao, reproven by Casals-N)
The Legendrian torus link $T(m,n)$ has infinitely many fillings for $n \geq m \geq 4$ and $m=3, n\geq 6$:



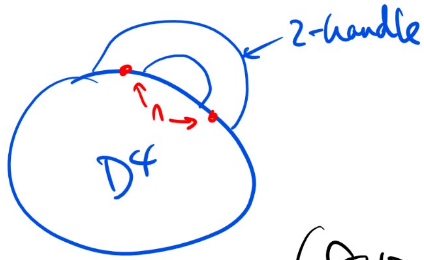
(use cobordisms from $\Lambda(\tilde{D}_4)$ and Λ_1)

ex: $T(4,4)$



saddle moves here give $\Lambda(\tilde{D}_4)$

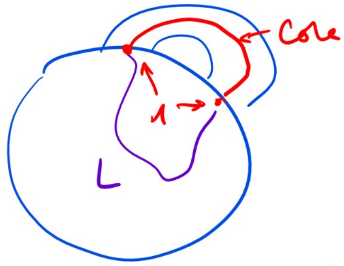
Rmk If Λ is aug-infinite then the Weinstein domain

$D^4(\Lambda) =$ 

has infinitely many closed Lagrangian surfaces:

(filling surface for Λ in D^4) $\cup$ (core of 2-handle)

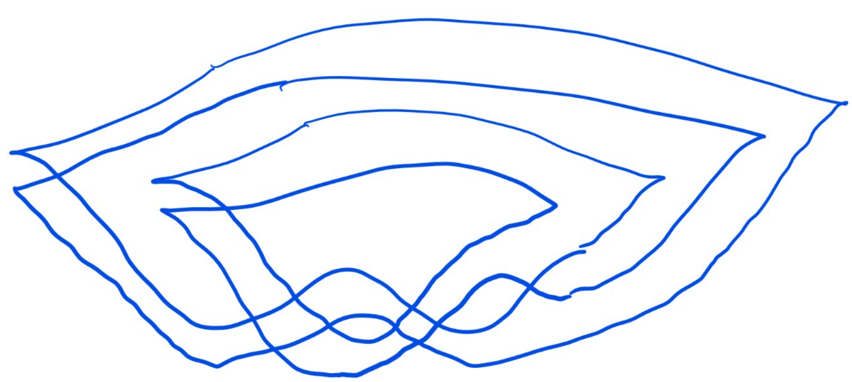
represent a different element of $\mathcal{W}\mathrm{Fuk}(D^4(\Lambda))$ for fillings that produce different augmentations.



Cor $g \geq 2$. There is a Weinstein 4-manifold $\simeq S^2$ containing infinitely many closed exact Lagrangian surfaces of genus g (and none of smaller genus).

($g \geq 6$: Casals-Gao; $g \geq 4$: Gao-Shen-Weng)

$g = 2$: use $D^4(\Lambda)$ where

$\Lambda =$ 

$m(10_{145})$ knot
$tb = 3 = 2g-1$

"Small" examples of Legendrian links with infinite fillings

One reasonable measure of "smallness": $tb(\Lambda) = 2g + m - 2$

genus of filling   # of components

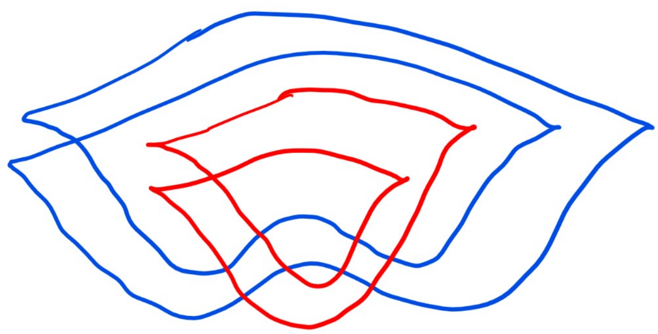
Since Λ_+

Λ_-

$\Rightarrow tb(\Lambda_+) \geq tb(\Lambda_-)$.

Smallest known link with ∞ many fillings:

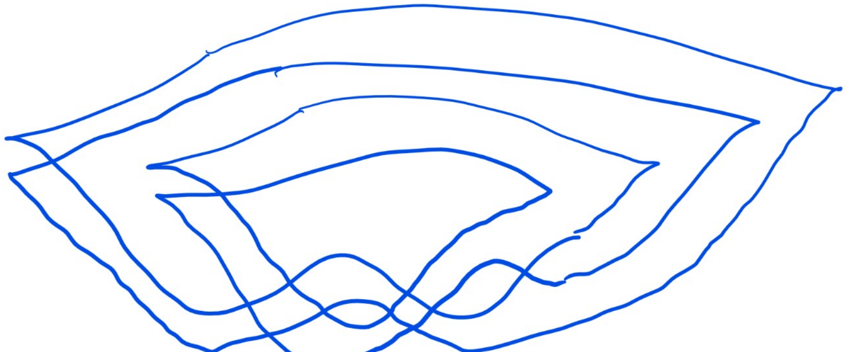
$\Lambda(\widetilde{A}_{1,1})$



$tb = 2$, $g = 1$, $m = 2$

Smallest known knot

$m(10_{145})$



$tb = 3$, $g = 2$, $m = 1$

Note neither of these is a rainbow closure of a positive braid (the setting where sheaf techniques work best).