Effectiveness of transverse knot invariants

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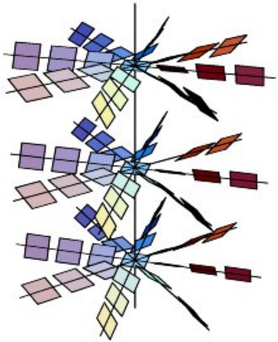
The standard contact structure

Standard contact $\mathbb{R}^3$: $\mathbb{R}^3$ equipped with the standard contact structure

$$\xi_{\mathrm{std}} = \ker \alpha_{\mathrm{std}}, \qquad \alpha_{\mathrm{std}} = dz - y\,dx.$$



The radially symmetric contact structure $\ker(dz - y\,dx + x\,dy)$.

Transverse knots

Definition

A knot K in $(\mathbb{R}^3, \xi_{\text{std}})$ is Legendrian if $\alpha = 0$ along K (i.e., K is everywhere tangent to ξ).

Definition

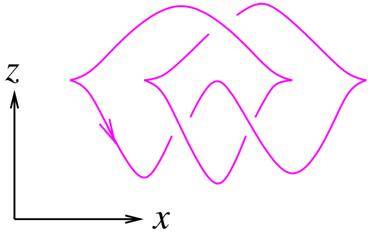
A knot K in $(\mathbb{R}^3, \xi_{\text{std}})$ is transverse if $\alpha > 0$ along K (in particular, $K \pitchfork \xi$). Two transverse knots are transversely isotopic if they are isotopic through transverse knots.

Transverse classification problem

Classify transverse knots of some particular topological type.

Legendrian knots

One way to study transverse knots: through Legendrian knots. A Legendrian knot is uniquely determined by its front (xz) projection (then $y = dz/dx$).

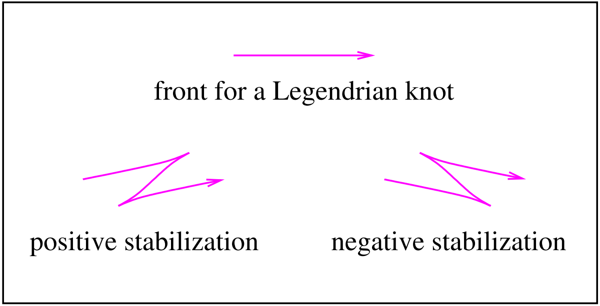


Legendrian classification problem

Classify Legendrian knots of some particular topological type.

Legendrian stabilization

There are two operations on Legendrian knots that produce new Legendrian knots of the same topological type: $+/-$ Legendrian stabilization.



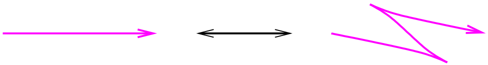
Legendrian and transverse knots

Any Legendrian knot has a “transverse pushoff”, and conversely any transverse knot has a “Legendrian approximation” (not unique).

Theorem (Epstein–Fuchs–Meyer, Etnyre–Honda 2001)

There is a one-to-one correspondence

$$\{\textit{transverse knots}\} \longleftrightarrow \{\textit{Legendrian knots}\}/$$
$$(\textit{negative Legendrian stabilization/destab}).$$



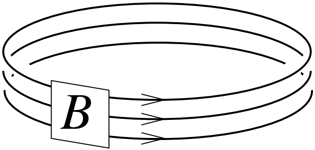
Braids and transverse knots

Another way to study transverse knots: braids.

Theorem (Bennequin 1983)

Any braid (conjugacy class) can be closed in a natural way to produce a transverse knot in $(\mathbb{R}^3, \xi_{std})$, and every transverse knot is transversely isotopic to a closed braid.

This is a transverse version of Alexander's Theorem.

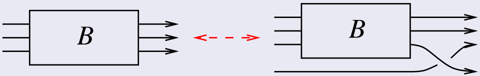


Transverse Markov Theorem

Transverse Markov Theorem (Orevkov–Shevchishin 2001, Wrinkle 2002)

Two braids represent the same transverse knot iff related by:

- *conjugation in the braid groups*
- *positive braid stabilization* $B \longleftrightarrow B\sigma_n$*:*



Cf. usual Markov Theorem: topological knots/links are equivalent to braids mod conjugation and positive/negative braid stabilization.

Transverse classification

If a transverse knot T is the closure of a braid B, the self-linking number of T is

$$sl(T) = w(B) - n(B)$$

where $w(B)$ = algebraic crossing number of B and $n(B)$ = braid index of B.

Definition

A topological knot is transversely simple if its transverse representatives are completely determined by self-linking number; otherwise transversely nonsimple.

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Examples of transversely simple knots:

- unknot (Eliashberg 1993)
- torus knots (Etnyre 1999) and the figure 8 knot (Etnyre–Honda 2000)
- some twist knots (Etnyre–N.–Vértesi 2010)

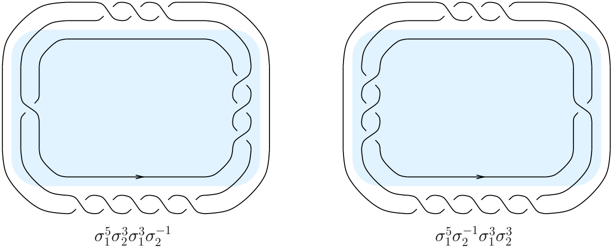
Transverse nonsimplicity

Three general approaches to proving that a knot type is transversely nonsimple:

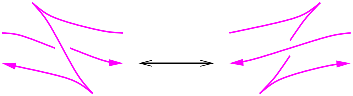
- dividing-curve techniques for classifying Legendrian knots: e.g. $(2,3)$-cable of $(2,3)$ torus knot (Etnyre–Honda 2003) and other torus knot cables (Etnyre–LaFountain–Tosun 2011)
- braid-foliation techniques: Birman–Menasco "Markov Theorem without stabilization"
- invariants of transverse knots.

Negative flypes

One way to produce candidates for possibly different transverse knots of the same topological type and self-linking number: negative flype (Birman–Menasco 1993).



This corresponds to the "SZ move" for Legendrian knots (Lipshitz–N.–Sarkar 2013).

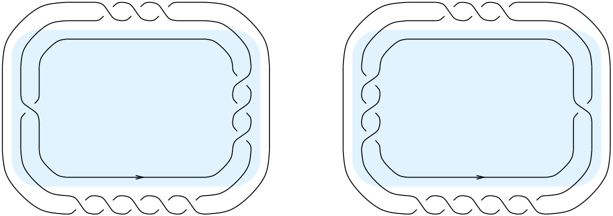


Transversely nonsimple knots: Birman–Menasco examples

Birman–Menasco 2008: family of knots with braid index 3 that are transversely nonsimple. The transverse knots given by the closures of the 3-braids

$$\sigma_1^a \sigma_2^b \sigma_1^c \sigma_2^{-1}, \quad \sigma_1^a \sigma_2^{-1} \sigma_1^c \sigma_2^b,$$

related by a negative flype, are transversely nonisotopic for particular choices of (a, b, c).



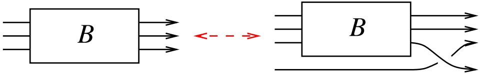
$\sigma_1^5 \sigma_2^3 \sigma_1^3 \sigma_2^{-1}$ $\sigma_1^5 \sigma_2^{-1} \sigma_1^3 \sigma_2^3$

Example: $11a_{240}$ is transversely nonsimple.

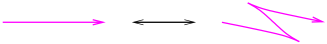
Transverse invariants

Invariants of transverse knots T are typically defined via a braid representative B or a Legendrian approximation L:

- an invariant of braids, invariant(B), that is also invariant under braid conjugation and positive braid stabilization (transverse Markov theorem);



- an invariant of Legendrian knots, invariant(L), that is also invariant under negative Legendrian stabilization.



Lee generators

Let K be a knot diagram corresponding to the closure of a braid B. Lee, 2002: the filtered Khovanov complex $C_{Kh}(K)$ has two cycles

$$\widetilde{\psi}^{\pm}(B) \in C_{Kh}(K)$$

supported in the oriented resolution of K and generating Lee homology $Lee(K) = H(C_{Kh}(K))$.

Plamenevskaya, 2004: let

$$\psi(B) \in Gr\, C_{Kh}(K)$$

be the lowest filtered piece of $\widetilde{\psi}^{\pm}(B)$: this labels each circle in the oriented resolution by x where the Frobenius algebra for Khovanov homology is $\mathbb{Z}[x]/(x^2)$.

The Plamenevskaya invariant

Theorem (Plamenevskaya 2004)

$$[\psi(B)] \in H(Gr\, C_{Kh}(K)) = Kh(K)$$

is invariant under braid isotopy, conjugation, and positive stabilization. This yields a transverse invariant of a transverse knot T of topological type K, the Plamenevskaya invariant

$$\psi(T) \in Kh^{0,sl(T)}(K).$$

In fact the element $\psi^{\pm}(B)$ in the filtered complex $C_{Kh}(K)$ is also a transverse invariant (Lipshitz–N.–Sarkar 2013).

Effective transverse invariants

Definition

A transverse invariant is effective if it can distinguish different transverse knots with the same self-linking number and topological type (i.e., prove that some topological knot is transversely nonsimple).

Question

Is the Plamenevskaya invariant (or the filtered Plamenevskaya invariant) effective?

This is still open!

Theorem (Lipshitz–N.–Sarkar 2013)

The Plamenevskaya invariant and the filtered Plamenevskaya invariant are invariant under negative flypes.

Generalizations of the Plamenevskaya invariant

Plamenevskaya's invariant is a distinguished element in Khovanov homology.

Wu, 2005: distinguished elements in Khovanov–Rozansky $\mathfrak{sl}_n$ homology. These are also invariant under conjugation and positive braid stabilization: transverse invariants.

Unknown whether the Wu $\mathfrak{sl}_n$ invariants are effective.

What transverse invariants *are* effective?

Known to be effective:

- Ozsváth–Szabó–Thurston 2006: HFK grid invariant: distinguished element in knot Floer homology via grid diagrams
  - Lisca–Ozsváth–Stipsicz–Szabó 2008: LOSS invariant: distinguished element in knot Floer homology via open book decompositions: same as HFK grid invariant (Baldwin–Vela-Vick–Vértesi 2011)
- Ekholm–Etnyre–N.–Sullivan 2010: transverse homology: filtered version of knot contact homology

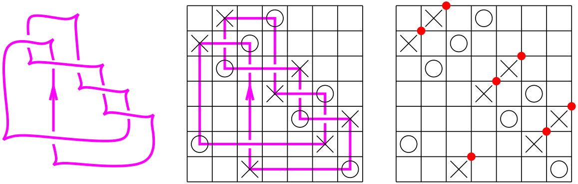
HFK grid invariant

Ozsváth–Szabó–Thurston 2006:

transverse knot T of topological type K

$$\downarrow$$

distinguished element $\theta^-(T) \in HFK^-(m(K))$.



In combinatorial model for *CFK* via grid diagrams (Manolescu–Ozsváth–Sarkar), $\theta^-(T)$ is the generator given by the upper-right corners of the X's for a Legendrian approximation of T.

HFK grid invariant, continued

Result (after mapping $HFK^- \to \widehat{HFK}$) for T transverse of type K:

$$\widehat{\theta}(T) \in \widehat{HFK}_{sl(T)+1}(m(K), \tfrac{sl(T)+1}{2}).$$

Theorem (Ozsváth–Szabó–Thurston 2006)

The HFK grid invariant $\widehat{\theta}$ is a transverse invariant.

Crude way to apply $\widehat{\theta}$: if T_1, T_2 are transverse knots with $\widehat{\theta}(T_1) = 0$ and $\widehat{\theta}(T_2) \neq 0$, then they're distinct.

Theorem (N.–Ozsváth–Thurston 2007)

The HFK grid invariant $\widehat{\theta}$ is an effective transverse invariant.

E.g., can be used to recover Etynre–Honda's result that the $(2,3)$-cable of the $(2,3)$ torus knot is transversely nonsimple.

Limitations of crude approach

$$\widehat{\theta}(T) \in \widehat{HFK}_{sl(T)+1}\left(m(K), \frac{sl(T)+1}{2}\right):$$

- If this group is 0, then $\widehat{\theta}(T) = 0$ carries no information.
- If $\widehat{\theta}(T_1), \widehat{\theta}(T_2) \neq 0$, how to tell them apart?

Slightly more precise statement of invariance:

Theorem (Ozsváth–Szabó–Thurston 2006)

If T_1, T_2 are isotopic transverse knots and G_1, G_2 are grid diagrams of corresponding Legendrian approximations, then the transverse isotopy gives a sequence γ of grid moves from G_1 to G_2 inducing a combinatorially-defined isomorphism

$$\gamma_* : \widehat{HFK}(G_1) \to \widehat{HFK}(G_2)$$

and $\gamma_*(\widehat{\theta}(G_1)) = \widehat{\theta}(G_2)$.

Enter naturality

Naturality statement (conjectural)

Let G_1, G_2 be grid diagrams for the same topological knot, and let γ be a sequence of grid moves from G_1 to G_2. Then the isomorphism

$$\gamma_* : HFK^-(G_1) \to HFK^-(G_2)$$

depends only on the homotopy class of the path $\gamma \subset \{smooth\ knots\}$.

Cf. related work of Juhász–Thurston (2012) on naturality in HF.

Note: a similar statement for $\widehat{HFK}$ fails to hold, at least for unpointed knots (Sarkar 2011).

Naturality and the HFK grid invariant

Can use naturality in conjunction with $\widehat{\theta}$.

Definition

Let K be an oriented topological knot. The mapping class group of K is

$$MCG(K) = \pi_1(\{\text{smooth knots isotopic to } K\}).$$

Corollary of naturality

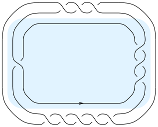
Let T_1, T_2 be transverse of type K with $MCG(K) = 1$, and let G_1, G_2 be grid diagrams for T_1, T_2. If T_1, T_2 are transversely isotopic, then for any sequence γ of grid diagrams from G_1 to G_2,

$$\gamma_*(\widehat{\theta}(G_1)) = \widehat{\theta}(G_2).$$

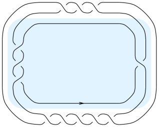
Birman–Menasco transverse knots

Corollary of naturality (N.–Thurston)

The Birman–Menasco pair $\sigma_1^5\sigma_2^3\sigma_1^3\sigma_2^{-1}$ and $\sigma_1^5\sigma_2^{-1}\sigma_1^3\sigma_2^3$ can be distinguished by $\widehat{\theta}$.



$\sigma_1^5\sigma_2^3\sigma_1^3\sigma_2^{-1}$



$\sigma_1^5\sigma_2^{-1}\sigma_1^3\sigma_2^3$

Here $MCG(11a_{240}) = 1$, and the $\widehat{\theta}$ invariants constitute distinct nonzero elements of

$$\widehat{HFK}_8(11a_{240}, 4) \cong (\mathbb{Z}/2)^2.$$

This argument can be extended to other Birman–Menasco pairs (possibly $\sigma_1^a\sigma_2^b\sigma_1^c\sigma_2^{-1}, \sigma_1^a\sigma_2^{-1}\sigma_1^c\sigma_2^b$ for $a, b, c \geq 3$ with $a \neq c$), but not all of them.

Transverse mapping class group

Definition

Let T be a transverse knot. The transverse mapping class group of T is

$$TMCG(T) = \pi_1(\{\text{transverse knots transversely isotopic to } T\}).$$

For a transverse knot K, there is an obvious map

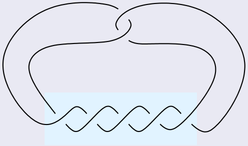
$$TMCG(K) \to MCG(K).$$

Naturality and $\widehat{\theta}$ can be used to show that this map is not an isomorphism for some transverse knots K.

Transverse mapping class group, continued

Corollary of naturality (N.–Thurston, preliminary)

Consider any twist knot K where the number of crossings in the shaded region is odd and ≥ 3.



There is a transverse knot T of type K such that the map

$$TMCG(T) \to MCG(K) \ (\cong \mathbb{Z}/2)$$

is not surjective.

Cf. Kálmán 2004: there are Legendrian knots L for which the map $LMCG(L) \to MCG(K)$ is not *injective*.

Knot contact homology

Definition

K knot. The knot contact homology of K is the Legendrian contact homology of the unit conormal bundle N^*K in the contact 5-manifold $ST^*\mathbb{R}^3$:

$$HC_*(K) := LCH_*(ST^*\mathbb{R}^3, N^*K).$$

- invariant of smooth knots
- combinatorial description in terms of a braid B whose closure is K (N. 2003, Ekholm–Etnyre–N.–Sullivan 2011)
- $HC_*(K) = H_*(\mathcal{A}, \partial)$, where $(\mathcal{A}, \partial)$ is a differential graded algebra over the ring

$$\mathbb{Z}[H_1(N^*K)] \cong \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}].$$

Transverse homology

When T is a transverse knot, the coefficient ring for the DGA $(\mathcal{A}, \partial)$ can be improved:

$$\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}] \rightsquigarrow \mathbb{Z}[U, \lambda^{\pm 1}, \mu^{\pm 1}].$$

This comes from positivity of intersection with a holomorphic 4-manifold, the conormal lift of the contact structure ξ_{std}.

Theorem (N., Ekholm–Etnyre–N.–Sullivan 2010)

The homology of the DGA $(\mathcal{A}, \partial)$ defined over $\mathbb{Z}[U, \lambda^{\pm 1}, \mu^{\pm 1}]$ is a transverse invariant, the (minus) transverse homology $HT_*^-(T)$.

Transverse homology and knot Floer homology

The complex for transverse homology $HT_*^-(T)$ (over $\mathbb{Z}[U, \lambda^{\pm 1}, \mu^{\pm 1}]$) can be thought of as an additional filtration on the complex for knot contact homology $HC_*(K)$ (more precisely, set $U = 1$ on the chain level). In this sense it's similar to knot Floer homology:

$$HT_*^-(T) : HC_*(K) :: HFK_*^-(K) : \widehat{HF}_*(Y).$$

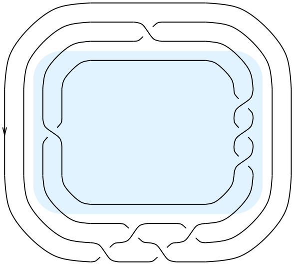
As in Heegaard Floer, one can define other flavors of transverse homology:

- $\widehat{HT}_*(T)$ (set $U = 0$)
- $HT_*^\infty(T)$ ($\otimes \mathbb{Z}[U, U^{-1}]$): in fact, this is an invariant of the underlying smooth knot.

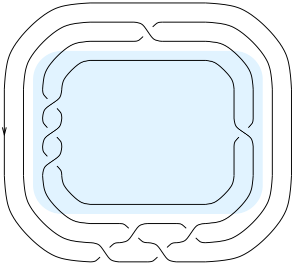
Effectiveness of transverse homology

Theorem (N. 2010)

$\widehat{HT}_*(T)$ *is an effective transverse invariant.*



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_3^3\sigma_2\sigma_3^{-1})$



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_3^{-1}\sigma_2\sigma_3^3)$

These two transverse $m(7_6)$ knots can be distinguished by $\widehat{HT}_0$: count number of augmentations (ring homomorphisms)

$$\widehat{HT}_0 \to \mathbb{Z}/3.$$

Comparison of transverse invariants

Legendrian knot atlas (Chongchitmate–N. 2010): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK					
HT					
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK					
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK					
HT					

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HFK					
HT					
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK		✓		✓	
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK					
HT					

N.–Ozsváth–Thurston 2007, using HFK grid invariant

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HFK		✓		✓	
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓		✓	✓	
HT					

Chongchitmate–N. 2010, using HFK grid invariant

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HFK	✓				
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK		✓		✓	
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓		✓	✓	
HT					

Ozsváth–Stipsicz 2008, using LOSS invariant

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓			✓(?)	
HT					
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓(?)	✓		✓	
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓		✓	✓	
HT					

N.–Thurston, using HFK grid invariant and conjectural naturality

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓(?)	×
HT					
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓(?)	✓	×	✓	
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT					

HFK invariants don't work: $\widehat{HFK} = 0$ in relevant bidegree.

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓(?)	×
HT	✓	✓	✓		✓
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓(?)	✓	×	✓	
HT		✓	✓	✓	
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HFK	✓	×	✓	✓	
HT	✓		✓	✓	

N. 2010, using transverse homology

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓(?)	×
HT	✓	✓	✓	?	✓

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓(?)	✓	×	✓	
HT	?	✓	✓	✓	

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓	?	✓	✓	

These are "transverse mirrors", as are the Birman–Menasco knots.

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓(?)	×
HT	✓	✓	✓	?	✓

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓(?)	✓	×	✓	
HT	?	✓	✓	✓	

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓	?	✓	✓	