Knots and the Symplectic Field Theory of cotangent bundles

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[1] featuring joint work in progress with Kai Cieliebak, Tobias Ekholm, and Janko Latschev

Outline

Cotangents and conormals

- Let M be a smooth n-manifold.
  - T^*M is naturally a *symplectic* $2n$-manifold;
  - ST^*M, the cosphere bundle of M, is naturally a *contact* $(2n-1)$-manifold.
- Let $K \subset M$ be any embedded submanifold. Define $LK \subset T^*M$ to be the *conormal bundle* to K:

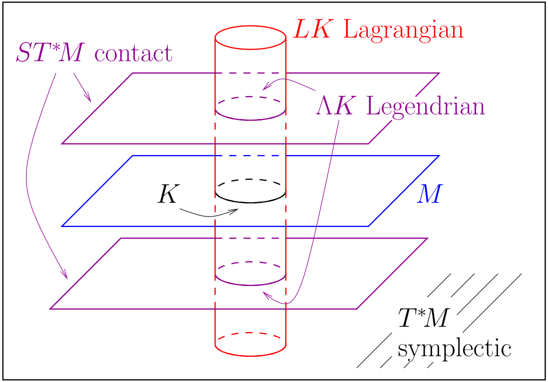
$$LK = \{(q,p) \in T^*M : q \in K, \langle p, v\rangle = 0 \, \forall \, v \in T_qK\}.$$

  Also define $\Lambda K \subset ST^*M$ to be the *unit conormal bundle* to K:

$$\Lambda K = LK \cap ST^*M.$$

  - $LK \subset T^*M$ is a *Lagrangian* submanifold ($\omega|_{LK} \equiv 0$);
  - $\Lambda K \subset ST^*M$ is a *Legendrian* submanifold (ΛK tangent to ξ) .

Schematic picture



($K \subset M$ submanifold; ST^*M cosphere bundle; LK conormal bundle to K; ΛK unit conormal bundle to K.)

Symplectic and topological invariants

Symplectic/contact invariants of T^*M, ST^*M yield smooth invariants of M.

- Symplectic homology of T^*M and loop space cohomology: Viterbo, Abbondandolo–Schwarz, Salamon–Weber
- Cylindrical contact homology of ST^*M and string topology: Cieliebak–Latschev
- related work of Abouzaid, Seidel, ...

Relative case: invariants of LK, ΛK under Lagrangian/Legendrian isotopy yield smooth-isotopy invariants of $K \subset M$.
Apply Legendrian contact homology ($\subset$ Symplectic Field Theory).

- For Legendrian Λ in contact V, counts holomorphic disks in $\mathbb{R} \times V$ with boundary on Lagrangian $\mathbb{R} \times \Lambda$ with one boundary puncture at $+\infty$ and some number of boundary punctures at $-\infty$.
- Eliashberg–Hofer; Ekholm–Etnyre–Sullivan for case $V = J^1(Q)$.

Knot contact homology

First reasonably nontrivial case:

- $M = \mathbb{R}^3$, $K \subset M$ knot or link
- $ST^*M = ST^*\mathbb{R}^3 = J^1(S^2)$
- Think of $\Lambda K \subset ST^*\mathbb{R}^3$ as the boundary of a tubular neighborhood of $K \subset \mathbb{R}^3$; topologically T^2
- ΛK is unknotted as a smooth torus but generally knotted as a Legendrian torus.

Definition

Let $K \subset \mathbb{R}^3$ be a knot. The Legendrian contact homology of $ST^*\mathbb{R}^3$ relative to ΛK is the knot contact homology of K,

$$HC_*(K) := HC_*(ST^*\mathbb{R}^3, \Lambda K).$$

This is a smooth knot invariant.

Knot contact homology, continued

Knot contact homology $HC_*(K)$ is the homology of a dg-algebra $(\mathcal{A}, d)$, where $\mathcal{A}$ is the graded tensor algebra over $\mathbb{Z}$ generated by:

- finitely many generators in degrees 0, 1, 2 (Reeb chords for ΛK)
- $\lambda^{\pm 1}, \mu^{\pm 1}$ in degree 0 (to keep track of the relative homology classes of boundaries of holomorphic disks).

The geometric content of $(\mathcal{A}, d)$ is encoded in the differential d, which counts the holomorphic disks.

There is a purely algebraic/combinatorial dg-algebra $(\mathcal{A}^{\mathrm{comb}}, d^{\mathrm{comb}})$ associated to a braid or knot diagram for K; $\mathcal{A}^{\mathrm{comb}}$ is as above, but d^{comb} can be defined without PDEs.

Combinatorial knot contact homology

Here it is, for $B \in B_n$ a braid whose closure is K:

ϕ_B algebra automorphism of $\mathcal{A}_n$ defined by

$$\phi_{\sigma_k}: \begin{cases} a_{ki} & \mapsto & -a_{k+1,i} - a_{k+1,k}\mu^{-1}a_{ki} & i \neq k, k+1 \\ a_{ik} & \mapsto & -a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \\ a_{k+1,i} & \mapsto & a_{ki} & i \neq k, k+1 \\ a_{i,k+1} & \mapsto & a_{ik} & i \neq k, k+1 \\ a_{k,k+1} & \mapsto & a_{k+1,k}\mu^{-1} & \\ a_{k+1,k} & \mapsto & \mu a_{k,k+1} & \\ a_{ij} & \mapsto & a_{ij} & i, j \neq k, k+1; \end{cases}$$

$n \times n$ matrices Φ_B^L, Φ_B^R defined by

$$\phi_B(a_{i\cdot}) = \sum_{j=1}^{n}(\Phi_B^L)_{ij}a_{j\cdot} \qquad \text{and} \qquad \phi_B(a_{\cdot j}) = \sum_{i=1}^{n} a_{\cdot i}(\Phi_B^R)_{ij};$$

$n \times n$ matrix $\Lambda = \mathrm{diag}(\lambda, 1, \cdots, 1)$; generators a_{ij} $(i \neq j)$ of degree 0, b_{ij}, c_{ij} of degree 1, d_{ij}, e_i of degree 2 with $1 \leq i, j \leq n$, assembled into $n \times n$ matrices $A = (a_{ij})$ (with $a_{ii} = \mu - 1$), $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$;

$$d(A) = 0$$
$$d(B) = (1 - \Lambda \cdot \Phi_B^L) \cdot A$$
$$d(C) = A \cdot (1 - \Phi_B^R \cdot \Lambda^{-1})$$
$$d(D) = B \cdot (1 - \Phi_B^R \cdot \Lambda^{-1}) - (1 - \Lambda \cdot \Phi_B^L) \cdot C$$
$$d(e_i) = (B + \Lambda \cdot \Phi_B^L \cdot C)_{ii}.$$

Invariance

Theorem (—, 2002–2004)

The chain homotopy type of $(\mathcal{A}^{comb}, d^{comb})$ is diagram-independent and yields a knot invariant, combinatorial knot contact homology

$$HC_*^{comb}(K) := H_*(\mathcal{A}^{comb}, d^{comb}),$$

supported in degrees $* \geq 0$.

Theorem (Ekholm, Etnyre, Sullivan, —, in progress)

$(\mathcal{A}^{comb}, d^{comb})$ is homotopy equivalent (in fact, "stable tame isomorphic") to the complex $(\mathcal{A}, d)$ for Legendrian contact homology; in particular,

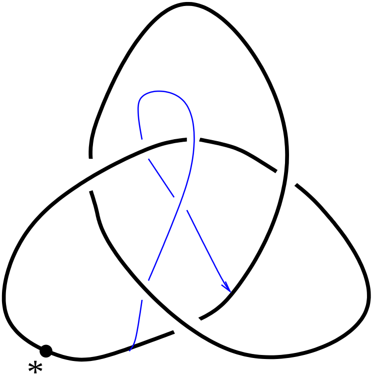
$$HC_*(K) \cong HC_*^{comb}(K).$$

Properties of knot contact homology $HC_*^{\mathrm{comb}}(K)$

- Encodes Alexander polynomial (via linearized HC_1^{comb}).
- HC_0^{comb} is a finitely generated, finitely presented noncommutative ring containing $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ (=group ring of $H_1(\Lambda K)$).
- HC_0^{comb} is closely related to A-polynomial; distinguishes the unknot (Kronheimer–Mrowka, Dunfield–Garoufalidis).
- HC_0^{comb} distinguishes mirrors, mutants (and noninvertible knots?); can be defined solely in terms of the knot quandle (?).
- HC_0^{comb} extends to tangles; spatial graphs; virtual knots; arbitrary codimension-2 submanifolds.

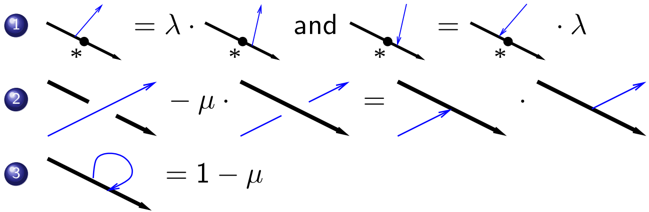
Cords

$HC_0^{\mathrm{comb}}(K)$ can be defined topologically: the cord algebra. A cord of K is a continuous oriented path in $\mathbb{R}^3$ with endpoints on K (not equal to a fixed point $* \in K$) and no other points on K.



The cord algebra

The cord algebra of K is the tensor algebra over $\mathbb{Z}$ generated by $\lambda^{\pm 1}, \mu^{\pm 1}$ and homotopy classes of cords, modulo "skein relations"[2]:



The cord algebra is finitely generated and finitely presented for any knot, and is evidently an invariant of (framed) smooth knots.

[2] Skein relation 2 is slightly incorrect; it would be correct if μ commuted with cords. One fixes this using "framed cords".

More on the cord algebra

Theorem (—, 2003–2004)

$HC_0^{comb}(K)$ is isomorphic to the cord algebra of K as rings containing $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$.

Possibly illustrative examples:

$$
\begin{aligned}
HC_0^{\mathrm{comb}}(\mathrm{unknot}) &= \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]/((\lambda - 1)(\mu - 1)) \\
HC_0^{\mathrm{comb}}(\mathrm{RH\ trefoil}) &= \mathbb{Z}\langle \lambda^{\pm 1}, \mu^{\pm 1}, x\rangle \,/\, \langle \lambda\mu - \mu\lambda, x\lambda - \lambda x, \\
&\quad x\mu x + \mu x\mu - \lambda\mu^{-2} - \lambda\mu^{-1}, \\
&\quad x\mu^2 x + \mu^{-1} x\lambda\mu^{-1} - \lambda\mu^{-2} - \lambda\mu^{-1}\rangle.
\end{aligned}
$$

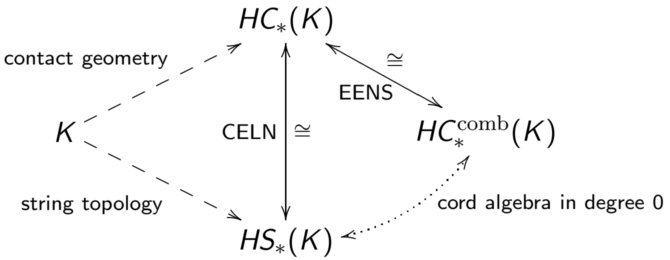
Would like a topological interpretation (à la the cord algebra) for the entire knot contact homology $HC_*(K)$.

Topological reinterpretation of $HC_*(K)$

Joint with Cieliebak, Ekholm, and Latschev: associate to a knot K a new knot invariant, the string homology $HS_*(K)$, which is evidently an isotopy invariant.

Theorem (Cieliebak, Ekholm, Latschev, —)

$HS_*(K) \cong HC_*(K)$.



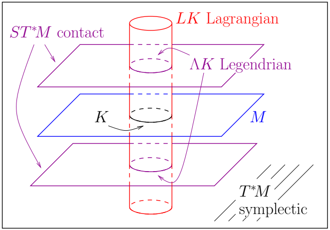
$M \cup_K N$

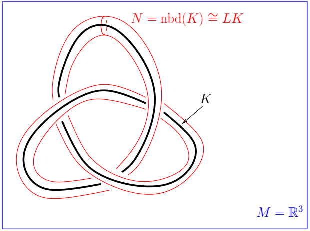
Recall: $M = \mathbb{R}^3$, $K \subset M$. There are two natural Lagrangians in T^*M, intersecting in K:

- the zero section M
- the conormal bundle LK.

Think of LK as a tubular neighborhood $N = \text{nbd}(K) \subset M$; then K sits naturally in both M and N.

Let $M \cup_K N$ be the space constructed as the *disjoint union* of M and N glued along K.



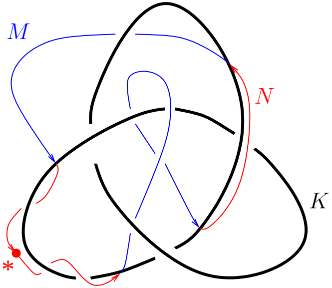


Broken closed strings

Definition

Fix K and a point $* \in N \setminus K$, and let ℓ be a nonnegative integer. A broken closed string of length ℓ is a C^1 map $\gamma : [0,1] \to M \cup_K N$ such that there exist $0 < t_1 < t_2 < \cdots < t_{2\ell} < 1$ for which:

- $\gamma(0) = \gamma(1) = *$; $\gamma(t_i) \in K$ for all i
- γ maps $[0, t_1] \to N$, $[t_1, t_2] \to M$, $[t_2, t_3] \to N$, $[t_3, t_4] \to M$, $\ldots$, $[t_{2\ell}, 1] \to N$
- $\dot\gamma(t_i^-) = \dot\gamma(t_i^+)$ for all i.



Chains of broken closed strings

Define:

$$\Sigma_\ell = \{\text{broken closed strings of length } \ell\}$$
$$C_n(\Sigma_\ell) = n\text{-chains on } \Sigma_\ell$$
$$C_n = \bigoplus_{\ell \geq 0} C_n(\Sigma_\ell).$$

There is the usual boundary operator

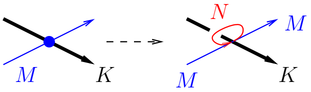
$$\partial : \ C_n(\Sigma_\ell) \to C_{n-1}(\Sigma_\ell)$$

giving a map $\partial : \ C_n \to C_{n-1}$ and satisfying $\partial^2 = 0$.

However, the homology $H_*(C_*, \partial)$ is independent of the knot K. To define string homology, we perturb ∂ to a new differential d.

The δ maps

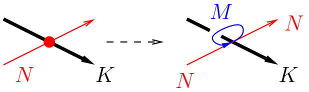
If a broken closed string of length ℓ has an M-string that passes through K in an extra point, then we can break that M-string in two to obtain a new broken closed string of length $\ell + 1$:



Since the extra-point condition has codimension 1, this operation turns a generic n-chain in Σ_ℓ into an $(n-1)$-chain in $\Sigma_{\ell+1}$:

$$\delta_M : C_n(\Sigma_\ell) \to C_{n-1}(\Sigma_{\ell+1}) \quad \rightsquigarrow \quad \boxed{\delta_M : C_n \to C_{n-1}}.$$

Similarly, one can break an N-string:



to obtain a map

$$\delta_N : C_n(\Sigma_\ell) \to C_{n-1}(\Sigma_{\ell+1}) \quad \rightsquigarrow \quad \boxed{\delta_N : C_n \to C_{n-1}}.$$

String homology

"Theorem"

$\boxed{d := \partial + \delta_M + \delta_N}$ is a differential on C_*: $d^2 = 0$.

Problem with well-definedness of δ_M, δ_N: transversality/genericity issues; definition of δ_M, δ_N on chain level rather than homology.

Solution: use slightly different complex $\tilde{C}_*$ defined using generic 0-, 1-, and 2-chains of broken closed strings, and define $\tilde{d}$ on $\tilde{C}_*$ analogously to d.

Theorem (Cieliebak, Ekholm, Latschev, —)

$(\tilde{C}_*, \tilde{d})$ is a complex: $\tilde{d}^2 = 0$. Furthermore, the string homology $HS_*(K) = H_*(\tilde{C}, \tilde{d})$ is a knot invariant.

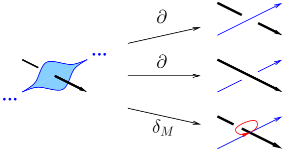
String homology and knot contact homology

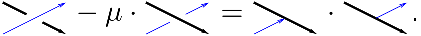
Theorem (Cieliebak, Ekholm, Latschev, —)

The complexes for string homology and knot contact homology are homotopy equivalent. In particular,

$$HS_*(K) \cong HC_*(K).$$

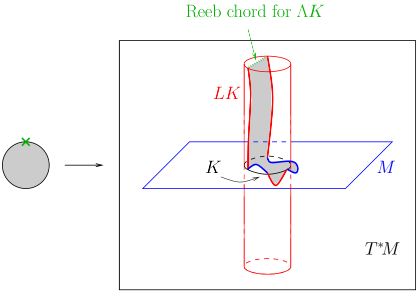
Note: when $* = 0$, the maps δ_M, δ_N precisely give the skein relations in the cord algebra, and so $HS_0(K)$ is the cord algebra.



cf. skein relation: 

Proof of equivalence

The proof of the result $HS_*(K) \cong HC_*(K)$ examines moduli spaces of holomorphic disks with one boundary puncture mapping to $T^*M = T^*\mathbb{R}^3$ with boundary on $M \cup N = R^3 \cup LK$, and a length-shortening argument.



The cord algebra via homotopy groups

We can reformulate the cord algebra $HC_0(K)$ purely in terms of the knot group and its peripheral subgroup.
Let K be a knot as before. Write

$$\begin{aligned} \pi &= \text{knot group} = \pi_1(\mathbb{R}^3 \setminus K) \\ \hat{\pi} &= \text{peripheral subgroup} = \pi_1(\partial(\text{tubular nbd of } K)) \cong \mathbb{Z}^2 \\ \mathbb{Z}\pi &= \text{group ring of } \pi \\ \mathbb{Z}\hat{\pi} &= \text{group ring of } \hat{\pi} \cong \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]. \end{aligned}$$

The map $\hat{\pi} \to \pi$ turns $\mathbb{Z}\pi$ into a $\mathbb{Z}\hat{\pi}$-bimodule.
One can construct a "noncommutative tensor product" $\mathbb{Z}\pi \otimes_{\mathbb{Z}\hat{\pi}} \mathbb{Z}\pi$, which is also a $\mathbb{Z}\hat{\pi}$-bimodule.

The cord algebra via homotopy groups, continued

Define $A_{\pi,\hat{\pi}}$ to be the "noncommutative tensor algebra of $\mathbb{Z}\pi$ over $\mathbb{Z}\hat{\pi}$", which is a ring and a $\mathbb{Z}\hat{\pi}$-bimodule:

$$A_{\pi,\hat{\pi}} = \mathbb{Z}\hat{\pi} \oplus \mathbb{Z}\pi \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}\hat{\pi}} \mathbb{Z}\pi) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}\hat{\pi}} \mathbb{Z}\pi \otimes_{\mathbb{Z}\hat{\pi}} \mathbb{Z}\pi) \oplus \cdots$$

Let $I_{\pi,\hat{\pi}} \subset A_{\pi,\hat{\pi}}$ be the two-sided ideal generated by:

- $x_1 x_2 - x_1 \mu x_2 - x_1 \otimes x_2$ for all $x_1, x_2 \in \pi$
- $1_{\hat{\pi}} - \mu - 1_{\pi}$, where $1_{\hat{\pi}}, \mu \in \mathbb{Z}\hat{\pi}$ and $1_{\pi} \in \mathbb{Z}\pi$.

Theorem

$A_{\pi,\hat{\pi}}/I_{\pi,\hat{\pi}}$ is isomorphic to $HC_0(K)$ as rings containing $\mathbb{Z}\hat{\pi} = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$.

The cord algebra is (almost) the group ring of the knot group

Theorem

The map

$$\mathbb{Z}[\lambda^{\pm 1}] \oplus \mathbb{Z}\pi \longrightarrow A_{\pi,\hat{\pi}}/I_{\pi,\hat{\pi}} \quad (\cong HC_0(K))$$

induces an isomorphism of $\mathbb{Z}\hat{\pi}$-bimodules.

Corollary

The cord algebra $HC_0(K)$ distinguishes the unknot.

The proof of the corollary uses nothing more complicated than the Loop Theorem.

Open questions—please solve!

- Can we generalize homotopy-theoretic definition to all degrees? I.e., use $\pi, \hat{\pi}$ to construct a complex homotopy equivalent to the complexes for contact homology, string homology.
- Full(er) Symplectic Field Theory invariant for $\Lambda K \subset ST^*\mathbb{R}^3$?
- Transverse knots in $(\mathbb{R}^3, \xi_{\text{std}})$ give a filtration on knot contact homology; what is it?
- Knots in other manifolds? Cord algebra detects knottedness of spun S^2's in $\mathbb{R}^4$.
- Cobordism/concordance of knots?