A SKEIN APPROACH TO BENNEQUIN TYPE INEQUALITIES

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ABSTRACT. We give a simple unified proof for several disparate bounds on Thurston–Bennequin number for Legendrian knots and self-linking number for transverse knots in $\mathbb{R}^3$, and provide a template for possible future bounds. As an application, we give sufficient conditions for some of these bounds to be sharp.

1. INTRODUCTION

1.1. **Main results.** The problem of finding upper bounds for the Thurston–Bennequin and self-linking numbers of knots has garnered a fair bit of recent attention. Although this originated as a problem in contact geometry, it now lies more in the realm of knot theory and braid theory, with upper bounds given by the Seifert and slice genus, the Kauffman and HOMFLY-PT polynomials, and, more recently, Khovanov homology, Khovanov–Rozansky homology, and knot Floer homology. These bounds are sometimes collectively called "Bennequin type inequalities".

The original proofs of many Bennequin type inequalities were remarkably diverse and sometimes somewhat ad hoc. In this paper, we provide a template which simultaneously proves a number of the significant Bennequin type inequalities, thus providing a unified approach to many of these bounds. The proof of the template itself is a fairly easy induction argument based on the remarkable work of Rutherford [23]. Our template gives a means to prove future, yet to be discovered Bennequin type bounds, for example using Khovanov and Rozansky's proposed categorification of the Kauffman polynomial [12]. It also sheds some light on why particular bounds may be sharp for a certain Legendrian knot while others are not; see Section 3.

We briefly recall the relevant definitions; see also [5] or any number of other references. A *Legendrian* knot or link in $\mathbb{R}^3$ with the standard contact structure is a smooth, oriented knot or link along which $y = dz/dx$ everywhere. It is convenient to represent Legendrians by their *front projections* to the xz plane, which are (a collection of) oriented closed curves with no vertical tangencies, whose only singularities are transverse double points and semicubical cusps. Given a front, one obtains a link diagram by smoothing out cusps and resolving each double point to a crossing where the strand with larger slope lies below the strand with smaller slope; this link diagram,

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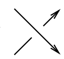
which we will call the *smoothed front*, is the topological type of the original Legendrian link. Any topological link has a Legendrian representative.

Given a front F, let $c(F)$ denote half of the number of cusps of F, and let $c_\downarrow(F)$ denote the number of cusps of F which are oriented downwards. Also let $w(F)$ denote the writhe of the corresponding smoothed front, the number of crossings counted with the standard signs ($+1$ for , -1 for ). Define the *Thurston–Bennequin number* and *self-linking number* of F, respectively, by

$$\mathrm{tb}(F) = w(F) - c(F)$$
$$\mathrm{sl}(F) = w(F) - c_\downarrow(F).$$

The Thurston–Bennequin and self-linking numbers are invariants of Legendrian links and comprise the "classical invariants" for Legendrians in $\mathbb{R}^3$. (In the literature, the role of the self-linking number for Legendrian links is usually played by the rotation number $\mathrm{r}(F) = \mathrm{tb}(F) - \mathrm{sl}(F)$, and the self-linking number is reserved for transverse links; our self-linking number for a Legendrian is the usual self-linking number of its positive transverse pushoff.)

Within a topological type, tb and sl for Legendrian representatives is always unbounded below; one can decrease tb and sl by adding zigzags to a front. However, in the early 1980's, Bennequin [2] proved the remarkable fact that tb and sl are bounded above for any link type L, by negative the minimal Euler characteristic for a Seifert surface bounding L; for knots K,

$$\mathrm{tb}(F), \mathrm{sl}(F) \leq 2g(K) - 1$$

for any front F representing K, where $g(K)$ is the Seifert genus.

Bennequin's inequality has subsequently been improved to a menagerie of different bounds on tb and sl. For a topological link L, define the *maximal Thurston–Bennequin number* $\overline{\mathrm{tb}}(L)$ (respectively *maximal self-linking number* $\overline{\mathrm{sl}}(L)$) to be the maximum tb (respectively sl) over all Legendrian realizations of L. Then we have the following Bennequin type inequalities for knots K:

$\overline{\mathrm{sl}}(K) \leq 2g(K) - 1$	(Bennequin bound [2])
$\overline{\mathrm{sl}}(K) \leq 2g_4(K) - 1$	(slice-Bennequin bound [22])
$\overline{\mathrm{sl}}(K) \leq 2\tau(K) - 1$	(τ bound [17])
$\overline{\mathrm{sl}}(K) \leq s(K) - 1$	(s bound [18, 24])
$\overline{\mathrm{sl}}(K) \leq -\operatorname{max-deg}_a P(K)(a,z) - 1$	(HOMFLY-PT bound [7, 13])
$\overline{\mathrm{sl}}(K) \leq -\operatorname{max-supp}_a \mathcal{P}^{a,q,t}(K) - 1$	(HOMFLY-PT homology bound [27])
$\overline{\mathrm{tb}}(K) \leq -\operatorname{max-deg}_a F(K)(a,z) - 1$	(Kauffman bound [21])
$\overline{\mathrm{tb}}(K) \leq -\operatorname{min-supp}_{q-t} HKh^{q,t}(K)$	(Khovanov bound [14])

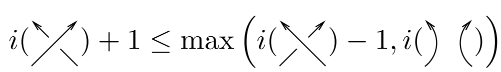
For notation and conventions, see Section 1.2. Several remarks are in order.

- Most of these inequalities have obvious generalizations to links; in particular, the last four translate unchanged to bounds for links.
- $\overline{\mathrm{tb}}(L) \le \overline{\mathrm{sl}}(L)$ always: rotating any front F $180°$ produces another front F' of the same topological type with $2\,\mathrm{tb}(F) = 2\,\mathrm{tb}(F') = \mathrm{sl}(F) + \mathrm{sl}(F')$. It follows that any upper bound for $\overline{\mathrm{sl}}$ is also an upper bound for $\overline{\mathrm{tb}}$. However, the Kauffman and Khovanov $\overline{\mathrm{tb}}$ bounds above do not extend to bounds on $\overline{\mathrm{sl}}$.
- Some Bennequin type inequalities imply others. The τ and s bounds (and presumably the HOMFLY-PT homology bound) imply slice-Bennequin, which in turn implies Bennequin; the HOMFLY-PT homology bound also implies the HOMFLY-PT (polynomial) bound. On the other hand, many pairs of the inequalities are incommensurable, notably the Kauffman and Khovanov bounds [14] (see also [6]).
- The above inequalities (in particular, the Kauffman, Khovanov, and HOMFLY-PT bounds) suffice to calculate $\overline{\mathrm{tb}}$ and $\overline{\mathrm{sl}}$ for all but a handful of knots with 11 or fewer crossings [15].
- The Kauffman and HOMFLY-PT bounds have been given many proofs in the literature, involving state models, plane curves, the Jaeger formula, etc. See [3, 6, 8, 25, 26] for additional proofs of the Kauffman bound, and [4, 8, 25] for HOMFLY-PT.
- This is not a complete list of known Bennequin type inequalities. In particular, Wu [27, 29] has derived bounds on $\overline{\mathrm{sl}}$ from Khovanov–Rozansky $\mathfrak{sl}_n$ homology.

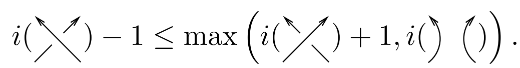
Our main results give general criteria for a link invariant to provide an upper bound for $\overline{\mathrm{tb}}$ or $\overline{\mathrm{sl}}$. These criteria are satisfied for many of the known bounds.

Theorem 1. *Suppose that $i(L)$ is a $\mathbb{Z}$-valued invariant of oriented links such that:*

(a) $i(\overbrace{\bigcirc\,\bigcirc\cdots\bigcirc}^{n}) \le n$;
(b) *we have*



and



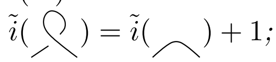
Then

$$\overline{\mathrm{sl}}(L) \le -i(L).$$

Corollary 1. *The HOMFLY-PT and HOMFLY-PT homology bounds on $\overline{\mathrm{sl}}$ hold for oriented links.*

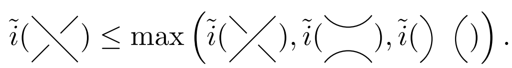
Theorem 2. *Suppose that* $\tilde{i}(D)$ *is a* $\mathbb{Z}$*-valued invariant of unoriented link diagrams such that:*

(a) $\tilde{i}(D)$ *is invariant under Reidemeister moves II and III;*

(b) 

(c) $\tilde{i}(\overbrace{\bigcirc\,\bigcirc\cdots\bigcirc}^{n}) \leq n$*;*

(d) *we have*



Then

$$\overline{\mathrm{tb}}(L) \leq -i(L),$$

where $i(L)$ *is the link invariant defined by* $i(D) = \tilde{i}(D) - w(D)$*.*

Corollary 2. *The Kauffman and Khovanov bounds on* $\overline{\mathrm{tb}}$ *hold for oriented links.*

We speculate that when τ and s are extended to oriented links, $i = 1 - 2\tau$ and $i = 1 - s$ should each satisfy the conditions of Theorem 1. This would demonstrate that the τ and s bounds can be proven using our template as well.

It also seems likely that Khovanov–Rozansky's proposed categorification of the Kauffman polynomial [12] would satisfy a skein relation which would allow one to apply Theorem 2. This would give an upper bound on $\overline{\mathrm{tb}}$ from Kauffman homology strengthening the Kauffman (polynomial) bound, just as the HOMFLY-PT homology bound on $\overline{\mathrm{sl}}$ strengthens the HOMFLY-PT (polynomial) bound.

As mentioned earlier, one benefit of our results is a better understanding of when particular Bennequin type inequalities are sharp. Rutherford [23] has demonstrated a necessary and sufficient condition for the Kauffman bound to be sharp, in terms of certain decompositions of fronts known as rulings. It would be nice to have similar characterizations for sharpness for, say, the HOMFLY-PT bound and the Khovanov bound. It seems that such characterizations should now be within reach, but for now we present some sufficient conditions for these bounds to be sharp; see Section 3.

Here is a rundown of the rest of the paper. In Section 1.2, we summarize the notation used in our presentation of the Bennequin type inequalities. The proofs of Theorems 1 and 2 and their rather easy consequences, Corollaries 1 and 2, are given in Section 2. In Section 3, we use the inductive proofs of our main results to construct trees which decompose any Legendrian link into simpler links, and use these trees to study sharpness of some Bennequin type inequalities.

1.2. **Notation.** Here we collect the definitions used in the Bennequin type inequalities mentioned above, including the particular conventions we use.

(These conventions coincide with those from `KnotTheory` [1] wherever applicable.)

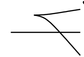
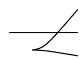
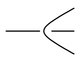
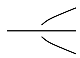
- max-$\deg_a$ is the maximum degree in a; max-supp_a is the maximum a degree in which the homology is supported; min-supp_{q-t} is the minimum value for $q-t$ over all bidegrees (q,t) in which the homology is supported.
- $g_4(K)$ is the slice genus of K.
- $\tau(K)$ is the concordance invariant from knot Floer homology [16], normalized so that $\tau = 1$ for right-handed trefoil.
- $s(K)$ is Rasmussen's concordance invariant from Khovanov homology [19].
- $P(K)(a,z)$ is the HOMFLY-PT polynomial of K, normalized so that $P = 1$ for the unknot and
$$aP(\nwarrow\!\!\!\!\nearrow) - a^{-1}P(\nwarrow\!\!\!\!\nearrow) = zP(\uparrow\,\uparrow).$$
- $\mathcal{P}^{a,q,t}(K)$ is the (reduced) Khovanov–Rozansky HOMFLY-PT homology [11] categorifying $P(K)$, normalized so that
$$\sum_{i,j,k}(-1)^{(k-i)/2}a^iq^j \dim \mathcal{P}^{i,j,k}(K) = P(K)(a, q-q^{-1}).$$
Note that our $\mathcal{P}^{a,q,t}(K)$ is Rasmussen's $\overline{H}^{q,-a,t}(K)$ from [20]. (For links, we need to use the "totally reduced" version $\overline{\overline{H}}$ of HOMFLY-PT homology in place of reduced HOMFLY-PT homology $\overline{H}$; see [20].)
- $F(K)(a,z)$ is the Kauffman polynomial of K, normalized so that for a diagram D representing K, $F(K)(a,z) = a^{-w(D)}\tilde{F}(D)(a,z)$, where $\tilde{F}$ is the framed Kauffman polynomial, the regular-isotopy invariant of unoriented link diagrams defined by $\tilde{F}(\bigcirc) = 1$, $\tilde{F}(\text{kink}) = a\tilde{F}(\frown)$, and
$$\tilde{F}(\times) + \tilde{F}(\times) = z\left(\tilde{F}(\asymp) + \tilde{F}()\,()\right).$$
- $HKh^{q,t}(K)$ is $(\mathfrak{sl}_2)$ Khovanov homology, normalized so that
$$\sum_{i,j}(-1)^jq^i \dim HKh^{i,j}(K) = (q+q^{-1})V_K(q^2),$$
where V_K is the Jones polynomial.

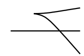
2. Proofs

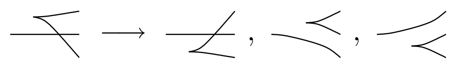
Theorems 1 and 2 have essentially the same proof. We establish Theorem 2 first, and then prove Theorem 1 and Corollaries 1 and 2.

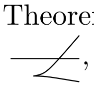
Proof of Theorem 2. View $\tilde{i}$ as a map on fronts by applying $\tilde{i}$ to the smoothed version of any front. Let L be an oriented link and let F be a Legendrian front of type L. We wish to show that $\mathrm{tb}(F) \leq -i(L)$, or equivalently, that

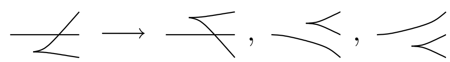
$c(F) - \tilde{i}(F) \geq 0$. Note that $c(F) - \tilde{i}(F) = -\operatorname{tb}(F) - i(F)$ is invariant under Legendrian isotopy.

The idea, which is essentially due to Rutherford [23], is to use skein moves to replace F by simpler fronts in such a way that $c - \tilde{i}$ does not decrease, and then to induct. The four fronts , , , and  are topologically , , , and , respectively, and are thus related by the four-term unoriented skein relation.

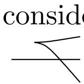
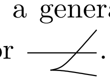
Suppose that F contains a Legendrian tangle . Successively replace

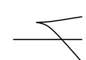
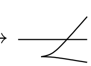
(1) 

in F, to obtain three new fronts, and suppose that $c - \tilde{i} \geq 0$ for each of those fronts. Since c is the same for all four fronts, assumption (d) in the statement of Theorem 2 then implies that $c - \tilde{i} \geq 0$ for F as well. Similarly, if F contains , and the three fronts obtained from F by

(2) 

all satisfy $c - \tilde{i} \geq 0$, then $c - \tilde{i} \geq 0$ for F as well.

To prove that $c(F) - \tilde{i}(F) \geq 0$ for all F, we induct on the *singularity number* $s(F)$ of F, defined as the total number of singularities (crossings and cusps) of F. If $s(F) = 2$, then F is the standard Legendrian unknot and $c(F) = 1$, $\tilde{i}(F) \leq 1$. Now consider a general front F. Suppose that F contains a tangle of the form  or . If we replace this tangle successively by three tangles according to (1) or (2), then the last two of the resulting fronts have lower s than F and are covered by the induction assumption, while the first has the same s as F.

The strategy is now to apply "skein crossing changes"  $\leftrightarrow$  to obtain a simpler front. To do this, we perform a second induction, this time on a modified singularity number $s'(F)$, defined as the number of singularities to the right of the rightmost left cusp of F. Since Legendrian isotopy, Legendrian destabilization (the removal of a zigzag), and the removal of trivial unknots do not increase $c - \tilde{i}$, the Theorem follows by induction from the following result.

Lemma 1 (Rutherford [23], Lemma 3.3)**.** *Via skein crossing changes, Legendrian isotopy, Legendrian destabilization, and the removal of trivial unknots, we can turn F into a front which either has lower s, or the same s and lower s'.*

For completeness, we sketch here the proof of the lemma. Consider the portion of F immediately to the right of the rightmost left cusp of F. By using Legendrian Reidemeister moves II and III if necessary, we can assume that this portion of F has one of the forms shown on the left hand side of

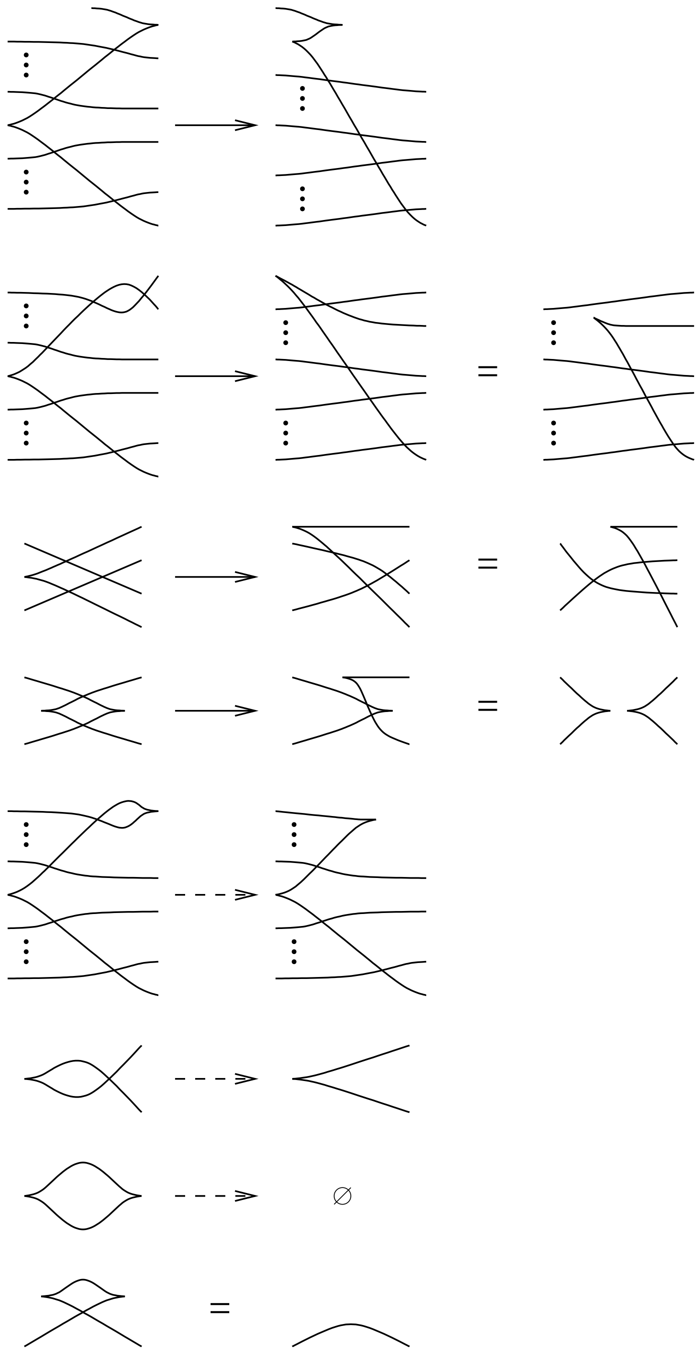


FIGURE 1. Inductively simplifying fronts. Solid arrows denote skein crossing changes; dashed arrows denote destabilizations or deletions of trivial unknots; equalities denote Legendrian isotopies. For each of the initial fronts (left column), the rightmost left cusp in the front is the unique left cusp depicted.

Figure 1. In each case, the use of skein crossing changes, Legendrian isotopy, Legendrian destabilization, and/or the removal of trivial unknots yields a simpler front (one with lower s, or the same s and lower s'). The lemma, and Theorem 2, follows. □

Proof of Theorem 1. This is a minor modification of the proof of Theorem 2. Define an invariant $\tilde{i}$ of oriented link diagrams by $\tilde{i}(D) = i(D) + w(D)$; then

$$\tilde{i}(\cdot) \le \max\left(\tilde{i}(\cdot), \tilde{i}(\cdot\ \cdot)\right)$$

and

$$\tilde{i}(\cdot) \le \max\left(\tilde{i}(\cdot), \tilde{i}(\cdot\ \cdot)\right).$$

For (oriented) Legendrian fronts F, we wish to show that $\mathrm{sl}(F) \le -i(F)$, or equivalently, that $c_{\downarrow}(F) - \tilde{i}(F) \ge 0$.

As before, we use skein moves to induct on the singularity number of F. Note that $c_{\downarrow}(F) - \tilde{i}(F)$ is invariant under Legendrian isotopy and non-increasing under Legendrian destabilization. If F contains a tangle (respectively), then we can successively replace it by (respectively) and whichever of and inherits an orientation from F, to obtain two new fronts. If $c_{\downarrow} - \tilde{i} \ge 0$ for these two fronts, then $c_{\downarrow} - \tilde{i} \ge 0$ for F as well. We now apply Lemma 1 as before. □

Proof of Corollary 1. Define $i(L) = \text{max-deg}_a P(L)(a, z) + 1$. By the skein relation and normalization for the HOMFLY-PT polynomial, the conditions in Theorem 1 hold, and Theorem 1 then gives the HOMFLY-PT bound.

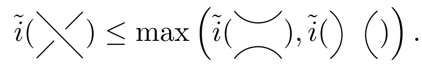
For the HOMFLY-PT homology bound, define $i(L) = \text{max-supp}_a \mathcal{P}^{a,q,t}(L) + 1$. The skein relation for the HOMFLY-PT polynomial (see [20]) categorifies to an exact triangle relating $\mathcal{P}(\cdot)$, $\mathcal{P}(\cdot)$, and $\mathcal{P}(\cdot\ \cdot)$, and this exact triangle yields condition (b) in the statement of Theorem 1. The normalization condition (a) is easy to check, and thus Theorem 1 yields the HOMFLY-PT homology bound. □

Proof of Corollary 2. For the Kauffman bound, define $\tilde{i}(D) = \text{max-deg}_a \tilde{F}(D) + 1$ for unoriented link diagrams D; by the skein relation for $\tilde{F}$, the conditions for Theorem 2 are satisfied, and the Kauffman bound follows.

For the Khovanov bound, collapse the q, t-bigrading on Khovanov homology to a single grading $* = q - t$. One can define framed Khovanov homology $\widetilde{HKh}^*(D)$ for unoriented link diagrams D such that $HKh^*(L) = \widetilde{HKh}^{*-w(D)}(D)$ if D is of link type L; indeed, the complex for Khovanov homology is first defined this way. The exact triangle in Khovanov homology (in this context, see, e.g., [14]) is given by

$$\cdots \longrightarrow \widetilde{HKh}^*(\cdot\ \cdot) \longrightarrow \widetilde{HKh}^*(\cdot) \longrightarrow \widetilde{HKh}^*(\cdot) \longrightarrow \widetilde{HKh}^{*-1}(\cdot\ \cdot) \longrightarrow \cdots.$$

If we now define $\tilde{i}(D) = -\operatorname{min-supp}_* \widetilde{HKh}^*(D)$, then the exact triangle implies that



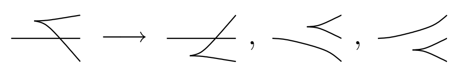
In particular, condition (d) in Theorem 2 holds, and the Khovanov bound follows. ☐

3. Skein trees

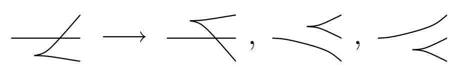
The nature of the proofs of Theorems 1 and 2 allows us to give necessary conditions and sufficient conditions for various Bennequin type inequalities to be sharp, and to compare these inequalities with each other. We can decompose any Legendrian knot via a skein tree, much as one would do to calculate knot polynomials using skein relations, and the skein tree can often tell us whether one bound or another is sharp.

Starting with an unoriented Legendrian front, construct the *unoriented skein tree* by following Rutherford's strategy described in the proof of Theorem 2:

- at each step, do a tangle replacement

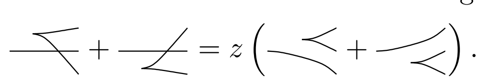
  or

  to obtain three new fronts;
- simplify the results by Legendrian isotopy, and repeat;
- stop when the result is either a stabilization (isotopic to a front with a zigzag) or a standard Legendrian unlink (the disjoint union of tb $=$ -1 Legendrian unknots).

An example is given in Figure 2.

One can easily use an unoriented skein tree for F to calculate the coefficient of $a^{-\operatorname{tb}(F)-1}$ in the Kauffman polynomial for F with any orientation (this coefficient is nonzero if and only the Kauffman bound is sharp): for terminal leaves in the tree, the coefficient is 1 at a standard Legendrian unlink and 0 at a stabilized front; use the skein relation for the framed Kauffman polynomial to backwards-construct the coefficient along the tree:



This is simply a restatement of a result of Rutherford [23].

We now see a heuristic reason for why the Kauffman bound sometimes fails even for Legendrian knots which maximize tb. Consider for example the Legendrian $(3,-4)$ torus knot F shown in Figure 2. Two of the terminal leaves of the unoriented skein tree are standard Legendrian unlinks. Either leaf by itself gives a contribution to the Kauffman polynomial of $T(3,-4)$ which would imply that the Kauffman bound on F is sharp; but the two

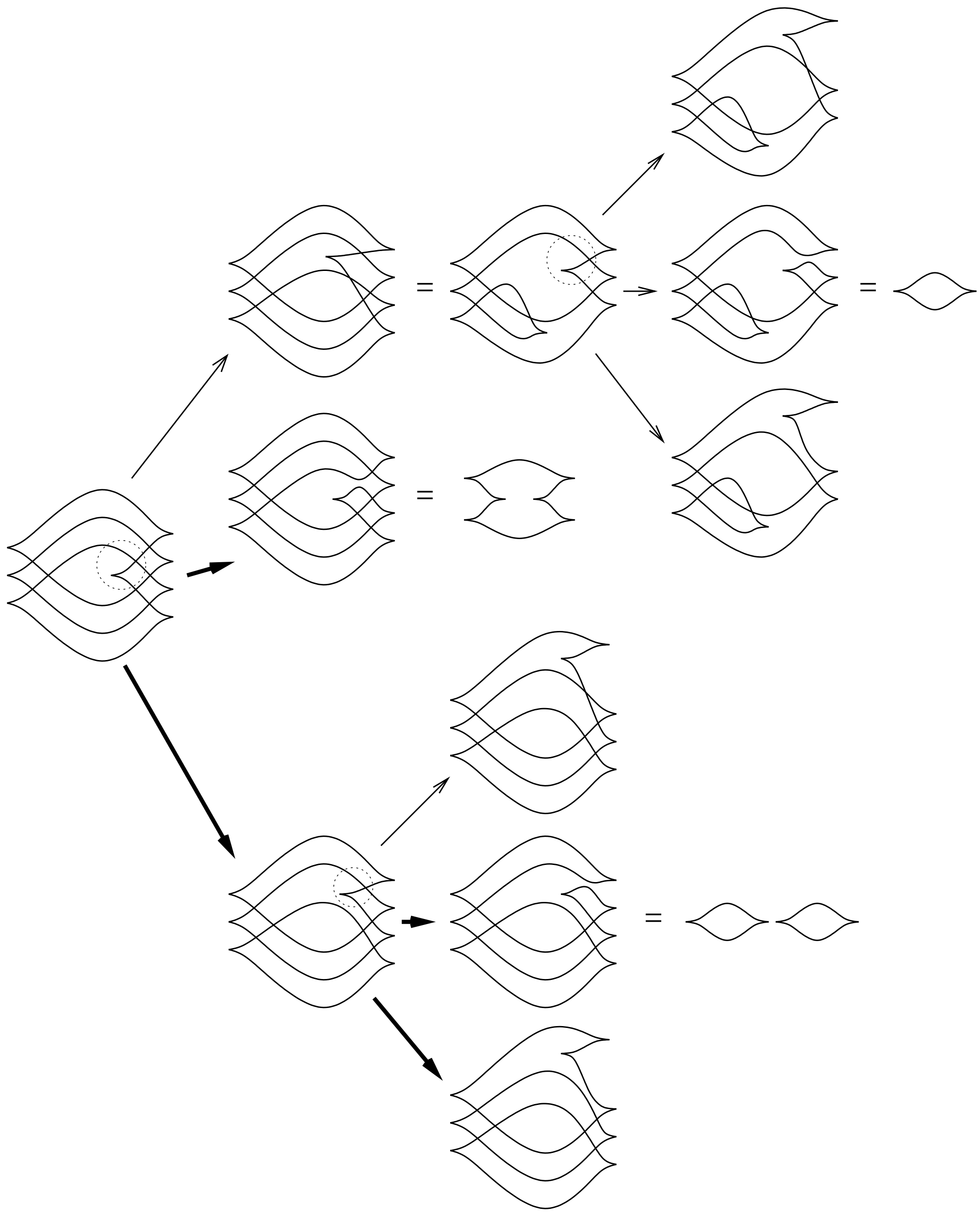


FIGURE 2. Unoriented skein tree for a Legendrian $(3,-4)$ torus knot. The Jones skein tree is the subtree given by the bold arrows. The circled tangles are where the tangle replacements are made.

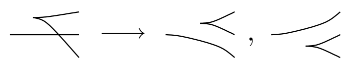
contributions cancel, and the maximum framing degree of the Kauffman polynomial of $T(3,-4)$ is one less than necessary for sharpness.

Next we examine the sharpness of the Khovanov bound on tb. For a front F, define

$$KhB(F) = -\operatorname{tb}(F) + \text{min-supp}_{q-t}\, HKh^{q,t}(F)$$

(this is $c - \tilde{i}$ from the proof of Theorem 2); then $KhB(F) \geq 0$ with equality if and only if the Khovanov bound is sharp for F.

Define the *Jones skein tree* to be the subtree of the unoriented skein tree consisting only of tangle replacements



and

(In each case, the two replacements are the 0- and 1-resolution, respectively.) Since this only counts 0- and 1-resolutions and not crossing changes, this is the same tree used to calculate the Jones polynomial for a knot.

At each stage in the Jones skein tree, a front F is connected to its 0-resolution F_0 and its 1-resolution F_1. The skein exact sequence for Khovanov homology implies the following.

Lemma 2. *If $KhB(F_0) = 0$, or $KhB(F_1) = 0$ and $KhB(F_0) \geq 2$, then $KhB(F) = 0$.*

We now have the following sufficient condition for the Khovanov bound to be sharp.

Theorem 3. *Let F be a front. Circle particular fronts in the Jones skein tree for F as follows: circle all terminal leaves which are standard Legendrian unknots; then work backwards, circling a front F' if either*

- *$(F')_0$ is circled or*
- *$(F')_1$ is circled and $(F')_0$ is isotopic to a front stabilized at least twice (i.e., with two zigzags).*

If F is circled by this process, then the Khovanov bound is sharp for F.

Proof. Since $KhB \geq 2$ for a front stabilized at least twice, Lemma 2 implies that all circled fronts satisfy $KhB = 0$. □

As an illustration of Theorem 3, the Khovanov bound is sharp for the Legendrian $(3,-4)$ torus knot shown in Figure 2; see Figure 3. Comparing with Figure 2 gives some indication of why the Khovanov bound is sharp here but the Kauffman bound is not: two terminal leaves of the unoriented skein tree are standard unlinks and their contributions to the Kauffman polynomial cancel, while only one of these terminal leaves is counted in the Jones skein tree and it makes a nonvanishing contribution to Khovanov homology.

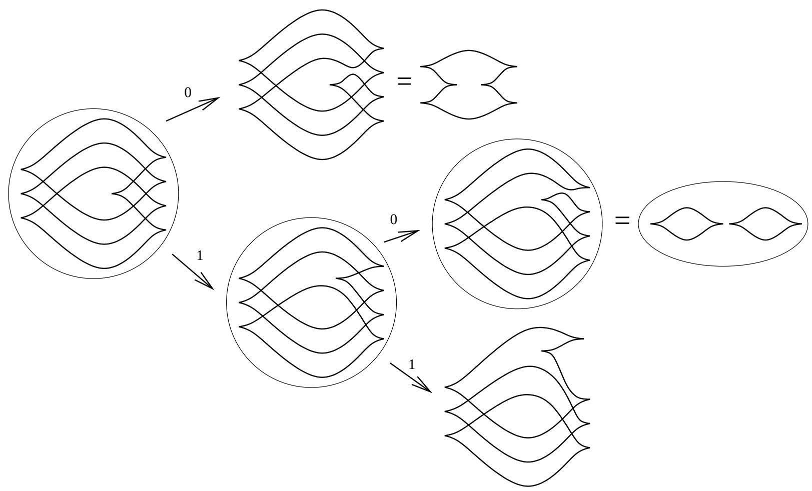


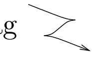
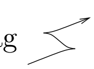
FIGURE 3. The Jones skein tree for the $(3, -4)$ torus knot. The numbers refer to 0- or 1-resolutions; the circled fronts are Khovanov sharp.

We remark that Theorem 3 implies, but is generally much stronger than, the sufficient condition for Khovanov sharpness given in [14]. Recall from [14] that the 0-resolution of a front, obtained by replacing each double point by its 0-resolution, is *admissible* if each component of the 0-resolution is a standard Legendrian unknot, and no component contains both pieces of any resolved double point.

Corollary 3 ([14, Proposition 7])**.** *The Khovanov bound is sharp for any front with admissible* 0*-resolution.*

Proof. Suppose that F is a front with admissible 0-resolution; apply the procedure from Theorem 3. It is easy to check that in the Jones skein tree for F, F and all of its iterated 0-resolutions are circled. □

We do not know how the condition of Theorem 3 compares to the sufficient condition for Khovanov sharpness given by Wu [28].

One can similarly construct an *oriented skein tree* for any oriented front, at each step replacing a front by the two fronts related to it by the oriented skein relation. Rather than stopping at all stabilized fronts, as for the unoriented skein tree, we stop only at fronts which are positive stabilizations (i.e., isotopic to a front with a downward zigzag ). If we encounter a negative stabilization (i.e., a front isotopic to one with an upward zigzag ), we eliminate the upward zigzag (this does not change sl) and proceed. All terminal leaves of the oriented skein tree are either positive stabilizations or standard Legendrian unlinks. See Figure 4.

As for unoriented skein trees and the Kauffman polynomial, we can use an oriented skein tree for a front F to calculate the coefficient of $a^{-\operatorname{sl}(F)-1}$

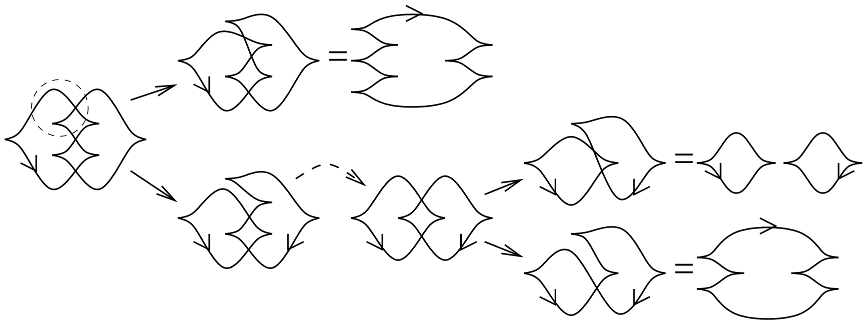


FIGURE 4. Oriented skein tree for a Legendrian trefoil. Solid arrows represent the skein relation; the dashed arrow is a negative destabilization.

in the HOMFLY-PT polynomial for F (this coefficient is nonzero if and only if the HOMFLY-PT bound is sharp). Standard Legendrian unlinks have coefficient 1; positive stabilizations have coefficient 0; the coefficient is preserved under negative stabilizations; we can backwards construct the coefficient along the tree using the skein relation, e.g.,

This again is very similar to a result of Rutherford [23].

It is sometimes easy to tell by inspection of an oriented skein tree whether the HOMFLY-PT bound is sharp. For example:

Theorem 4. *If an odd number of terminal leaves of the oriented skein tree of F are standard Legendrian unlinks, then the HOMFLY-PT polynomial bound on* sl *is sharp for F.*

Obviously this sufficient condition is rather weak, but it does for instance imply that the Legendrian trefoil in Figure 4 maximizes sl.

Skein trees can also show the limitations of Theorems 1 and 2. Consider the unoriented skein tree for the $m(10_{132})$ knot F shown in Figure 5. Each of the terminal leaves of the tree is a stabilization. Now suppose that $\tilde{i}(D)$ is any invariant satisfying the conditions of Theorem 2. Then by Theorem 2, $c - \tilde{i} \geq 0$ for all fronts; since each of the fronts on the right hand side of Figure 5 is a stabilization, $c - \tilde{i} \geq 1$ for these. Condition (d) from Theorem 2 implies that $c(F) - \tilde{i}(F) \geq 1$ as well, and so $i(m(10_{132})) \leq -\operatorname{tb}(F) - 1 = 0$.

It follows that the best possible bound given by Theorem 2 is $\overline{\mathrm{tb}}(m(10_{132})) \leq 0$. We however know from [15] that $\overline{\mathrm{tb}}(m(10_{132})) = -1$; hence the template of Theorem 2 can never give a sharp bound for $\overline{\mathrm{tb}}(m(10_{132}))$. A similar argument shows that the template of Theorem 1 can never give a sharp bound for $\overline{\mathrm{sl}}(m(10_{132}))$ $(= -1$ by [15]).

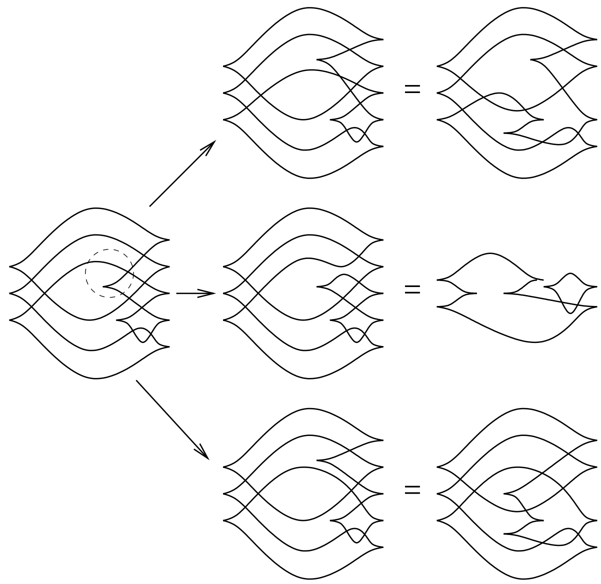


FIGURE 5. Unoriented skein tree for a Legendrian $m(10_{132})$ knot.

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