Grid Diagrams, Braids, and Contact Geometry

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ABSTRACT. We use grid diagrams to present a unified picture of braids, Legendrian knots, and transverse knots.

1. Introduction

Grid diagrams, also known in the literature as arc presentations, are a convenient combinatorial tool for studying knots and links in $\mathbb{R}^3$. Although grid diagrams (or equivalent structures) have been studied for over a century ([Bru, Cro, Dyn]), they have recently regained prominence due to their role in the combinatorial formulation of knot Floer homology ([MOS, MOST]).

It has been known for some time that grid diagrams are closely related to contact geometry as well as to braid theory. Our purpose here is to indicate the extent to which the relationships are similar. Indeed, braids, like the Legendrian and transverse knots in contact geometry, can be viewed as certain equivalence classes of grid diagrams, and we will see that the various equivalences fit into one single description. Furthermore, this description is compatible with the various maps between these objects, like the transverse knot constructed from a braid. Much of the picture we will present has previously appeared, but we believe that the full picture (especially the part concerning braids) is new.

Definition 1. A *grid diagram* with *grid number* n is an $n \times n$ square grid with n X's and n O's placed in distinct squares, such that each row and each column contains exactly one X and one O.

We will employ the word "knot" throughout as shorthand for "oriented knot or oriented link". Then any grid diagram yields a diagram of a knot in a standard way: connect O to X in each row, connect X to O in each column, and have the vertical line segments pass over the horizontal ones (Figure 1). In addition, one can associate to any grid diagram not only a topological knot but also a braid, a Legendrian knot, and a transverse knot.
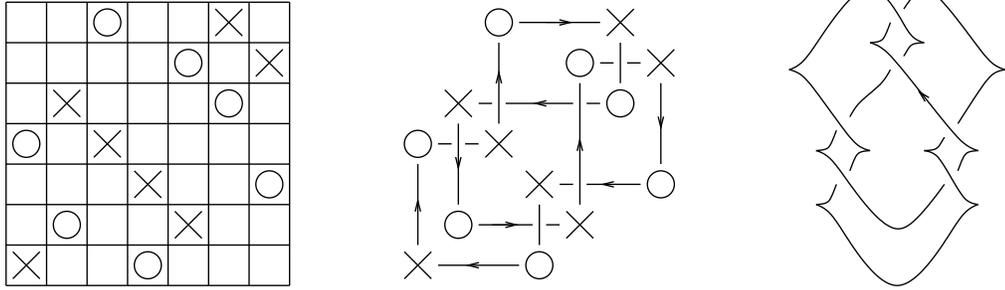
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FIGURE 1. A grid diagram and corresponding knot diagram and Legendrian front.

We will use the following notation:

$$\mathcal{G} = \{\text{grid diagrams}\}$$
$$\mathcal{K} = \{\text{isotopy classes of topological knots}\}$$
$$\mathcal{B} = \{\text{isotopy classes of braids modulo conjugation and exchange}\}$$
$$\mathcal{L} = \{\text{Legendrian isotopy classes of Legendrian knots}\}$$
$$\mathcal{T} = \{\text{transverse isotopy classes of transverse knots}\}.$$

(For definitions, see Section 2.)

In Section 2, we will review maps between these various sets that fit together into the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \mathcal{L} \\
\downarrow & \searrow & \downarrow \\
\mathcal{B} & \longrightarrow & \mathcal{T} \\
 & & \searrow \\
 & & \mathcal{K}.
\end{array}
\tag{1}
$$

Here the map from $\mathcal{G}$ to $\mathcal{K}$ is as described above. For the other maps, see also [Ben, Cro, Dyn, KN, MM, OST].

In [Cro] (see also [Dyn]), Cromwell provides a list of alterations of grid diagrams that do not change topological knot type, the grid-diagram equivalent of Reidemeister moves for knot diagrams. These are collectively known as *Cromwell moves* and consist of translations, commutations, and stabilizations/destabilizations. The last we distinguish into four types, *X:NW*, *X:NE*, *X:SW*, and *X:SE*, following [OST].
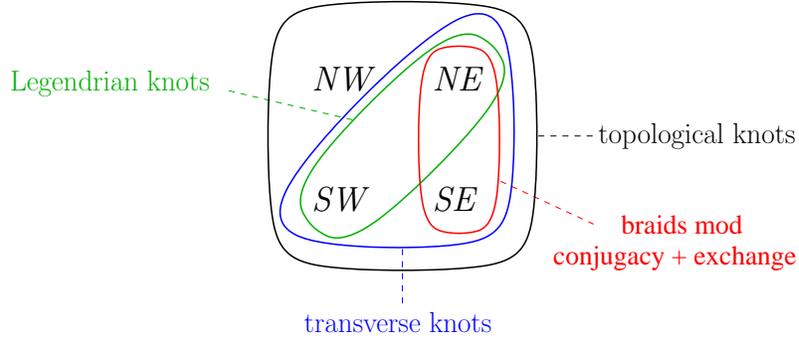
FIGURE 2. Quotienting $\tilde{\mathcal{G}}$, the set of grid-diagram orbits under translation and commutation, by various combinations of X (de)stabilizations yields equivalence classes of braids and various types of knots.

Proposition 1 (Cromwell [Cro])**.** *The map $\mathcal{G} \to \mathcal{K}$ sending grid diagrams to topological knots induces a bijection*

$$\mathcal{K} \longleftrightarrow \mathcal{G}/(\text{translation, commutation, (de)stabilization}).$$

We will see that the maps from $\mathcal{G}$ to $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{T}$ can be similarly understood. More precisely, we have the following result.

Proposition 2. *Let $\tilde{\mathcal{G}}$ denote the quotient set $\mathcal{G}/(\text{translation, commutation})$. The maps $\mathcal{G} \to \mathcal{B}$, $\mathcal{G} \to \mathcal{L}$, and $\mathcal{G} \to \mathcal{T}$ induce bijections*

$$\mathcal{B} \longleftrightarrow \tilde{\mathcal{G}}/(\text{X:NE,X:SE (de)stabilization})$$

$$\mathcal{L} \longleftrightarrow \tilde{\mathcal{G}}/(\text{X:NE,X:SW (de)stabilization})$$

$$\mathcal{T} \longleftrightarrow \tilde{\mathcal{G}}/(\text{X:NE,X:SW,X:SE (de)stabilization}).$$

It follows from this result that the maps between $\mathcal{B}, \mathcal{L}, \mathcal{T}, \mathcal{K}$ can also be interpreted in terms of grid diagrams. For instance, the map $\mathcal{B} \to \mathcal{T}$ is the quotient

$$\tilde{\mathcal{G}}/(\text{X:NE,X:SE (de)stabilization}) \longrightarrow \tilde{\mathcal{G}}/(\text{X:NE,X:SW,X:SE (de)stabilization}).$$

Similarly, the maps $\mathcal{B} \to \mathcal{K}$, $\mathcal{L} \to \mathcal{T}$, $\mathcal{L} \to \mathcal{K}$, $\mathcal{T} \to \mathcal{K}$, in terms of grid diagrams, are quotients by various (de)stabilizations.

Proposition 2 is summarized diagrammatically in Figure 2. The bijections in Proposition 2 involving $\mathcal{L}$ and $\mathcal{T}$ have already been established in [OST]; the new content in this note is the bijection involving $\mathcal{B}$.

We note that stabilization operations on braids and Legendrian and transverse knots can be expressed in terms of Cromwell moves. More precisely, we have the following.

Proposition 3. *Under the identifications of Proposition 2, we have*

$$positive\ braid\ stabilization \longleftrightarrow X{:}SW\ stabilization$$
$$negative\ braid\ stabilization \longleftrightarrow X{:}NW\ stabilization$$
$$positive\ Legendrian\ stabilization \longleftrightarrow X{:}NW\ stabilization$$
$$negative\ Legendrian\ stabilization \longleftrightarrow X{:}SE\ stabilization$$
$$transverse\ stabilization \longleftrightarrow X{:}NW\ stabilization.$$

Proposition 3 follows from an inspection of the effect of the various X stabilizations on the corresponding braid or Legendrian or transverse knot. See also the table at the end of Section 2.4.

Propositions 2 and 3 give an alternate proof via grid diagrams of the following result.

Proposition 4 (Transverse Markov Theorem [OSh, Wr])**.** *Two braids represent isotopic transverse knots if and only if they are related by a sequence of conjugations and positive braid stabilizations and destabilizations.*

In the usual formulation of Proposition 4, the map from braids to transverse knots uses a contact-geometric construction of Bennequin [Ben] (cf. Section 2.4), rather than the map we use here; see [KN] for a proof that the two maps coincide.

In Section 2, we recall the various relevant constructions and discuss the effects of grid-diagram symmetries on the maps in Formula (1). We prove our main result, Proposition 2, in Section 3.

2. Definitions and Maps

2.1. Grid diagrams

The Cromwell moves on grid diagrams, translation, commutation, and stabilization/destabilization, are illustrated in Figure 3 and defined below. From that figure it is clear that each Cromwell move preserves the topological type of the corresponding knot.

Translation views a grid diagram as lying on a torus by identifying opposite ends of the grid, and changes the diagram by translation in the torus. Any translation is a composition of some number of *vertical translations*, which move the top row of the diagram to the bottom or vice versa, and *horizontal translations*, which move the leftmost column of the diagram to the rightmost or vice versa.

Commutation interchanges two adjacent rows (*vertical commutation*) or two adjacent columns (*horizontal commutation*). These adjacent rows or columns are required to be disjoint or nested in the following sense. For rows, the four X's and O's in the adjacent rows must lie in distinct columns, and the horizontal line segments connecting O and X in each row must be either disjoint or nested (one contained in the other) when projected to a single horizontal line; there is an obvious analogous condition for columns.

An X (resp. O) *destabilization* replaces a 2×2 subgrid containing two X's and one O (resp. two O's and one X) with a single square containing an X (resp. O), eliminating
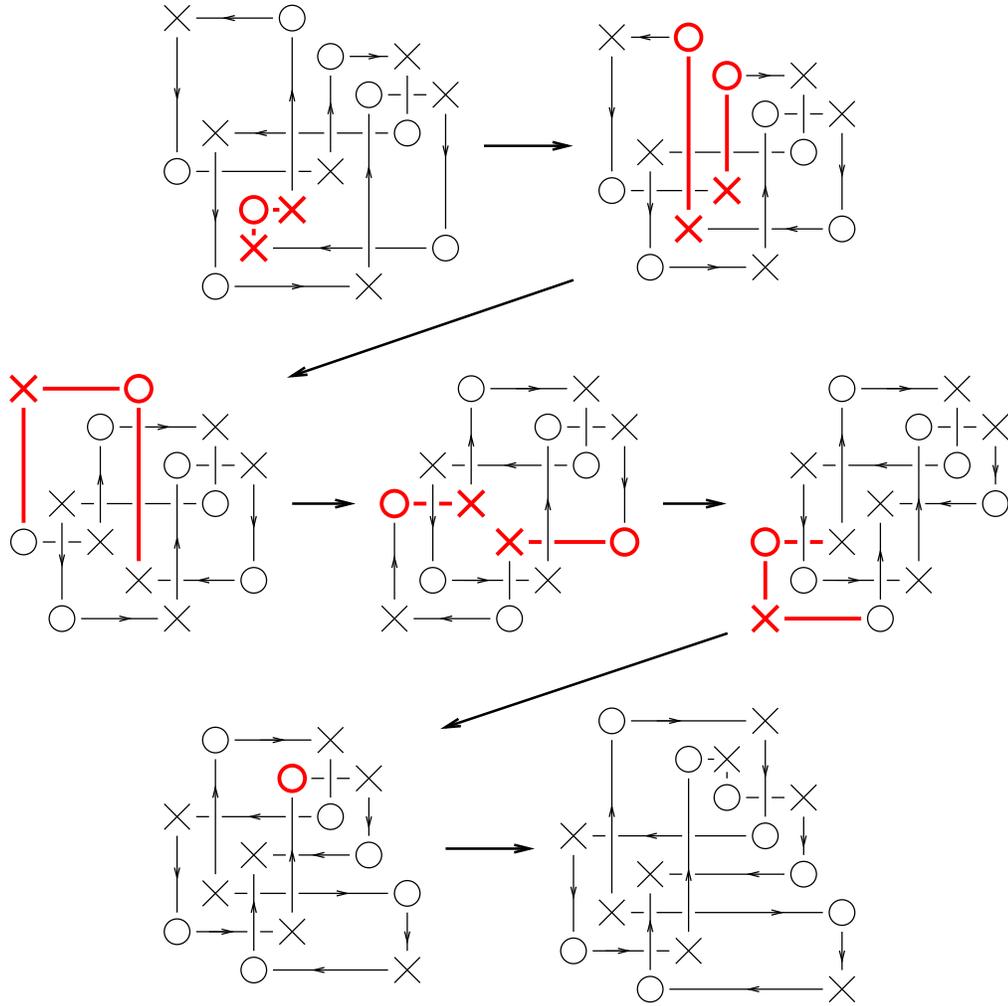
FIGURE 3. Illustration of a sequence of Cromwell moves. In succession: *X:SE* destabilization; horizontal commutation; vertical torus translation; vertical commutation; horizontal torus translation; *O:SW* stabilization. The highlighted sections of each diagram indicate the portion that changes under the following move.

one row and one column in the process. *Stabilization* is the inverse of destabilization. Each (de)stabilization is identified by its type, X or O, along with the corner in the 2×2 subgrid not occupied by a symbol. This yields eight possibilities: *X:NW*, *X:NE*, *X:SW*,

X:SE, O:NW, O:NE, O:SW, O:SE. It is easy to check that any *O:NW* (resp. *O:NE, O:SW, O:SE*) (de)stabilization can be expressed as a composition of translations, commutations, and one *X:SE* (resp. XSW, *X:NE, X:NW*) (de)stabilization. Thus we restrict our set of Cromwell moves to include only X (de)stabilizations.

Remark 5. By the argument of [OST, Lemma 4.3], we can instead drop torus translations and keep matching O (de)stabilizations to yield alternate definitions for topological, Legendrian, and transverse knots in terms of grid diagrams. In particular, *X:NE, X:SW, O:SW*, and *O:NE* (de)stabilizations, combined with commutations, generate all torus translations. The same argument can also be adapted for braids: that is, $\mathcal{B}$ is also $\mathcal{G}$ modulo commutation and *X:NE, X:SE, O:NW*, and *O:SW* (de)stabilization, as follows. Sequences of moves similar to those from [OST, Lemma 4.3] show that any horizontal torus translation can be achieved by these moves, as can any vertical torus translation where the O appears to the left of the X. But any vertical torus translation can be put into the correct position by horizontal torus translations.

2.2. Braids

As usual, a *braid* of braid index n is an element of the group $\mathcal{B}_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$. Note the natural inclusion $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ sending σ_i to itself for $i \leq n-1$. The relevant moves to consider on braids are:

- braid conjugation: $B \mapsto B'B(B')^{-1}$ for $B, B' \in \mathcal{B}_n$;
- exchange move [BM]: $B_1 \sigma_{n-1} B_2 \sigma_{n-1}^{-1} \mapsto B_1 \sigma_{n-1}^{-1} B_2 \sigma_{n-1}$ on $\mathcal{B}_n$, where $B_1, B_2 \in \mathcal{B}_{n-1} \subset \mathcal{B}_n$;
- braid stabilization: either positive braid stabilization $(B \in \mathcal{B}_n) \mapsto (B\sigma_n \in \mathcal{B}_{n+1})$ or negative braid stabilization $(B \in \mathcal{B}_n) \mapsto (B\sigma_n^{-1} \in \mathcal{B}_{n+1})$; and
- braid destabilization: the inverse of braid stabilization.

In fact, by an observation of Birman and Wrinkle [BW], an exchange move can be expressed as a combination of one positive stabilization, one positive destabilization, and a number of conjugations. (Here the positive stabilization and positive destabilization can equally well be replaced by a negative stabilization and negative destabilization.) For reference, we include the calculation here.

$$B_1 \sigma_{n-1} B_2 \sigma_{n-1}^{-1} \xmapsto{\text{conj}} \sigma_{n-1} B_1 \sigma_{n-1} B_2 \sigma_{n-1}^{-2} \xmapsto{+\text{ stab}} \sigma_{n-1} B_1 \sigma_{n-1} B_2 \sigma_{n-1}^{-2} \sigma_n$$

$$\xmapsto{\text{conj}} B_1 \sigma_{n-1} B_2 \sigma_{n-1}^{-2} \sigma_n \sigma_{n-1} = B_1 \sigma_{n-1} B_2 \sigma_n \sigma_{n-1} \sigma_n^{-2}$$

$$\xmapsto{\text{conj}} \sigma_n^{-2} B_1 \sigma_{n-1} \sigma_n B_2 \sigma_{n-1} = B_1 \sigma_{n-1} \sigma_n \sigma_{n-1}^{-2} B_2 \sigma_{n-1}$$

$$\xmapsto{\text{conj}} \sigma_{n-1}^{-2} B_2 \sigma_{n-1} B_1 \sigma_{n-1} \sigma_n \xmapsto{+\text{ destab}} \sigma_{n-1}^{-2} B_2 \sigma_{n-1} B_1 \sigma_{n-1}$$

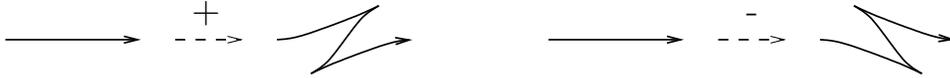$$\xmapsto{\text{conj}} B_1 \sigma_{n-1}^{-1} B_2 \sigma_{n-1}.$$

FIGURE 4. Positive and negative Legendrian stabilizations of the front projection of a Legendrian knot.

We will depict braids horizontally from left to right, with strands numbered from top to bottom; for instance, σ_1 interchanges the top two strands, with the top strand passing over the other as we move from left to right.

2.3. Legendrian and transverse knots

We give a quick description of Legendrian and transverse knots, which occur naturally in contact geometry; see, e.g., [Et] for more details. A *Legendrian knot* is a knot in $\mathbb{R}^3$ along which the standard contact form $dz - y\,dx$ vanishes everywhere; a *transverse knot* is a knot in $\mathbb{R}^3$ along which $dz - y\,dx > 0$ everywhere. (Note for the condition $dz - y\,dx > 0$ that the knot is oriented.) We consider Legendrian (resp. transverse) knots up to *Legendrian isotopy* (resp. *transverse isotopy*), which is simply isotopy through Legendrian (resp. transverse) knots.

One convenient way to depict a Legendrian knot is through its *front projection*, or projection in the xz plane. A generic front projection has three features: it has no vertical tangencies; it is immersed except at cusp singularities; and at all crossings, the strand of larger slope passes underneath the strand of smaller slope. Any front with these features corresponds to a Legendrian knot, with the y coordinate given by $y = dz/dx$.

The knot diagram corresponding to any grid diagram can be viewed as the front projection of a Legendrian knot by rotating it 45° counterclockwise and smoothing out the corners, creating cusps where necessary; see Figure 1 for an example. This yields a map $\mathcal{G} \to \mathcal{L}$ from grid diagrams to isotopy classes of Legendrian knots. Note that our convention differs from the convention of [OST]: the convention there is to reverse all crossings in the grid diagram and then rotate 45° clockwise. See also Section 2.5.

In [OST], it is verified that changing a grid diagram by translation, commutation, or (in our convention) *X:SW*, *X:NE* (de)stabilization does not change the isotopy class of the corresponding Legendrian knot. Changing by *X:NW* (resp. *X:SE*) stabilization does change the Legendrian knot type, by *positive Legendrian stabilization* (resp. *negative Legendrian stabilization*). Legendrian stabilizations can be described in the front projection as adding a zigzag, as shown in Figure 4.

Any Legendrian knot is isotopic to one obtained from some grid diagram. It is shown in [OST] that the set of equivalence classes of Legendrian knots under Legendrian isotopy corresponds precisely to grid diagrams modulo translation, commutation, and *X:NE*, *X:SW* (de)stabilization, as presented in Proposition 2.

A Legendrian knot can be C^0 perturbed to a transverse knot, its positive transverse pushoff. The resulting map $\mathcal{L} \to \mathcal{T}$ is not injective; negative Legendrian stabilization does

not change the transverse isotopy type of the positive transverse pushoff. It is a standard fact in contact geometry [EFM] that this gives a bijection

$$\mathcal{T} \longleftrightarrow \mathcal{L}/(\text{negative Legendrian stabilization}).$$

Since negative Legendrian stabilization corresponds to an *X:SE* Cromwell move, the characterization in Proposition 2 of $\mathcal{T}$ as a quotient of $\mathcal{G}$ holds. Note that positive Legendrian stabilization becomes the "transverse stabilization" operation on transverse knots.

2.4. Maps between $\mathcal{G}, \mathcal{B}, \mathcal{L}, \mathcal{T}, \mathcal{K}$

Here we collect the constructions of the maps in Formula (1). It suffices to define $\mathcal{G} \to \mathcal{L}$, $\mathcal{G} \to \mathcal{B}$, $\mathcal{L} \to \mathcal{T}$, $\mathcal{B} \to \mathcal{T}$, and $\mathcal{T} \to \mathcal{K}$, since the other maps follow by composition. We note that the commutativity of the square

$$
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
\mathcal{B} & \longrightarrow & \mathcal{T}
\end{array}
$$

was established in [KN], and in fact our description of the maps is essentially identical to the one given there. The maps $\mathcal{G} \to \mathcal{L}$ and $\mathcal{L} \to \mathcal{T}$ have already been discussed; since the map $\mathcal{T} \to \mathcal{K}$ is obvious, we are left with $\mathcal{G} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{T}$.

We begin with the map $\mathcal{G} \to \mathcal{B}$, as described in [Cro, Dyn]; this is also called a "flip" in [MM]. Any braid in B_n can be viewed as a braid diagram: a tangle diagram of n strands in the strip $[0,1] \times \mathbb{R}$, oriented so that the orientation points rightward at all points, with some collection of n distinct points $x_1, \ldots, x_n \in \mathbb{R}$ for which the braid intersects $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ in $\{(0, x_1), \ldots, (0, x_n)\}$ and $\{(1, x_1), \ldots, (1, x_n)\}$ respectively. Define a *rectilinear braid diagram* (cf. "braided rectangular diagram" [MM]) to be a tangle diagram in $[0,1] \times \mathbb{R}$ with the same boundary conditions as a braid diagram, but consisting exclusively of horizontal and vertical line segments, satisfying the following properties:

- vertical segments always pass over horizontal segments;
- each strand can be oriented so that every horizontal segment is oriented rightwards.

Any rectilinear braid diagram can be perturbed into a standard braid diagram by perturbing vertical segments slightly to point rightwards, as in Figure 5.

Now given a grid diagram, one obtains a knot diagram as usual by drawing horizontal and vertical lines. Turn this into a rectilinear braid diagram by replacing any horizontal line oriented leftwards from O to X by two horizontal lines, one pointing rightwards from the O, one pointing rightwards to the X, and have these new horizontal lines pass under all vertical line segments as usual. The rectilinear braid diagram corresponds to a braid as described above. This produces the desired map $\mathcal{G} \to \mathcal{B}$.

It remains to define the map $\mathcal{B} \to \mathcal{T}$. The original contact-geometric definition from [Ben] is as follows. Identify ends of B to obtain a knot or link in the solid torus $S^1 \times D^2$.
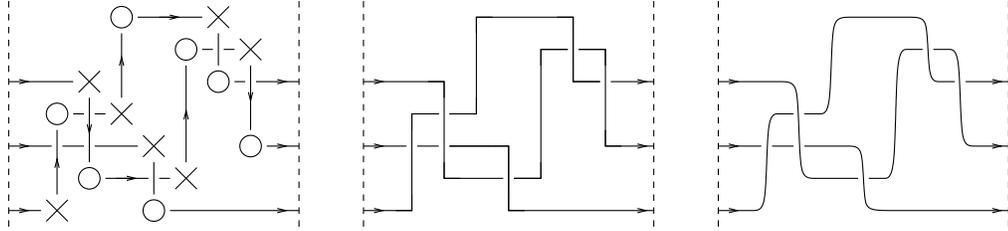
FIGURE 5. Braid version (left) of the grid diagram in Figure 1. Omitting the X's and O's produces a rectilinear braid diagram, which can be perturbed to become a braid, in this case $\sigma_2^{-1}\sigma_1\sigma_2^2\sigma_1^2 \in \mathcal{B}_3$.
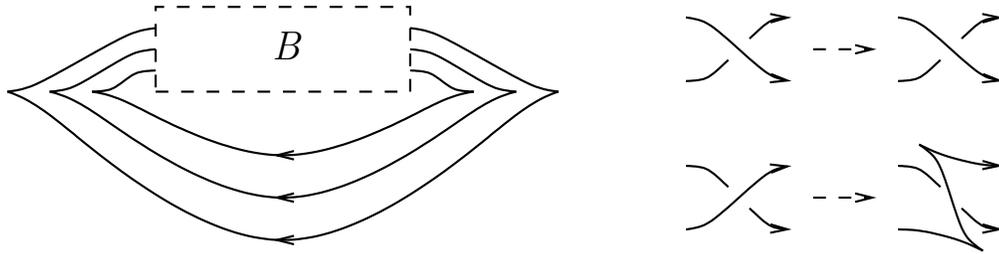

FIGURE 6. A Legendrian front for a braid B.

View the solid torus as a small (framed) tubular neighborhood of the standard transverse unknot in $\mathbb{R}^3$ with self-linking number -1. Then B becomes a transverse knot in a neighborhood of the transverse unknot.

There is also a combinatorial description for the map $\mathcal{B} \to \mathcal{T}$, which we now describe. (This description is proven to coincide with the contact-geometric description in [KN]; see also [MM, OSh]). Create a front by replacing each braid crossing as shown in Figure 6 and joining corresponding braid ends. (Joining ends introduces $2n$ cusps for a braid with n strands; see Figure 6.) This construction produces a Legendrian knot from any braid.

It is an easy exercise in Legendrian Reidemeister moves to show that changing the braid by isotopy changes the Legendrian knot by isotopy and negative Legendrian (de)stabilization; the stabilization is needed when one introduces cancelling terms $\sigma_i\sigma_i^{-1}$ or $\sigma_i^{-1}\sigma_i$ in the braid. Similarly, a conjugation or exchange move on a braid produces a Legendrian isotopy of the Legendrian knot. See Figure 7 for the exchange move.

The map $\mathcal{B} \to \mathcal{T}$ is now given as follows: given a braid, the corresponding Legendrian front is well-defined up to isotopy and negative Legendrian stabilization, and hence its positive transverse pushoff is well-defined. This transverse knot (equivalently, the class of the Legendrian knot modulo negative Legendrian (de)stabilization) is unchanged by braid conjugation and exchange.
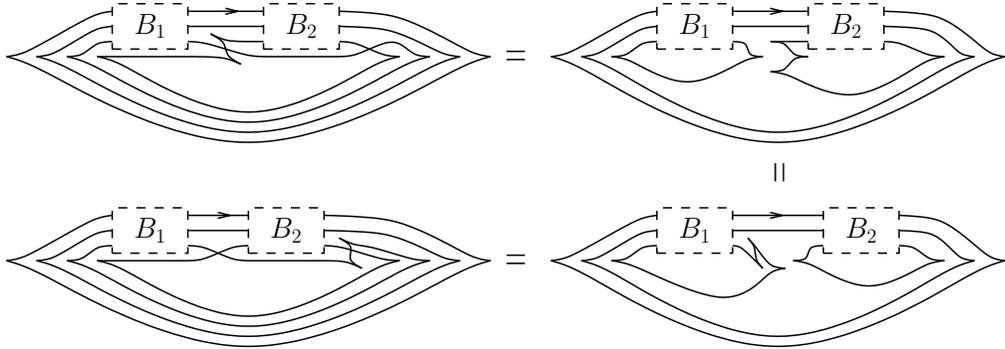
FIGURE 7. A braid exchange move produces a Legendrian-isotopic front. Equality denotes Legendrian isotopy.

Grid diagram	Braid	Legendrian knot	Transverse knot
torus translation	conjugation	Legendrian isotopy	transverse isotopy
vertical commutation	unchanged	Legendrian isotopy	transverse isotopy
horizontal commutation	conj, exchange	Legendrian isotopy	transverse isotopy
X:NE, O:SW stab	unchanged	Legendrian isotopy	transverse isotopy
X:SW, O:NE stab	conj, + braid stab	Legendrian isotopy	transverse isotopy
X:SE, O:NW stab	unchanged	− Legendrian stab	transverse isotopy
X:NW, O:SE stab	conj, − braid stab	+ Legendrian stab	transverse stab

TABLE 1. The effect of Cromwell moves on associated topological structures.

Table 1 has a summary of the effect of the Cromwell moves on grid diagrams correspond to changes in the associated braid, Legendrian knot, and transverse knot. The braid column is verified in Section 3, while the Legendrian and transverse columns were established in [OST]. For completeness, the table includes O as well as X stabilizations.

2.5. Symmetries and conventions

Here we discuss various symmetries of grid diagrams and how they relate the conventions for the maps in Formula (1) to other, sometimes conflicting, conventions in the literature. In this section, we will denote the maps $\mathcal{G} \to \mathcal{L}$, $\mathcal{G} \to \mathcal{T}$, $\mathcal{G} \to \mathcal{B}$ described in Section 2.4 by $G \mapsto L(G)$, $G \mapsto T(G)$, $G \mapsto B_\to(G)$, respectively.

Consider the symmetries S_1, S_2, S_3, and S_4 of grid diagrams defined as follows:

- S_1 rotates the grid diagram 180°;
- S_2 reflects the diagram about the NE-SW diagonal and interchanges X's and O's;
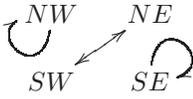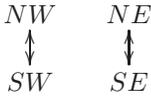- S_3 reflects the diagram across the horizontal axis; and

Symmetry	Knot	Braid	Legendrian	Transverse	X stabilizations
S_1	$K \mapsto K$	$B_\rightarrow \mapsto B_\leftarrow$	$L \mapsto \mu(L)$	—	$\begin{array}{cc} NW & NE \\ \times & \\ SW & SE \end{array}$
S_2	$K \mapsto K$	$B_\rightarrow \mapsto B_\uparrow$	$L \mapsto L$	$T \mapsto T$	$\begin{array}{cc} NW & NE \\ SW & SE \end{array}$
S_3	$K \mapsto m(K)$	$B_\rightarrow \mapsto m(B_\rightarrow)$	—	—	$\begin{array}{cc} NW & NE \\ \updownarrow & \updownarrow \\ SW & SE \end{array}$
S_4	$K \mapsto -K$	$B_\rightarrow \mapsto -B_\rightarrow$	$L \mapsto -\mu(L)$	$T \mapsto -\mu(T)$	$\begin{array}{cc} NW & NE \\ SW & SE \end{array}$

TABLE 2. The effect of symmetries of a grid diagram on associated topological structures.

- S_4 rotates the grid diagram $180°$ and interchanges X's and O's.

Both S_1 and S_2 preserve topological knot type, while S_3 produces the topological mirror knot $m(K)$ (with reversed orientation on $\mathbb{R}^3$), and S_4 produces the inverse (i.e., orientation-reversed) knot $-K$.

The symmetries descend to the quotient $\tilde{\mathcal{G}}$ of grid diagrams by translation and commutation. On $\tilde{\mathcal{G}}$, it is readily checked that the symmetries permute the four X stabilizations as shown in Table 2. We will use this information to examine the effect of the symmetries on Legendrian and transverse knots and braids, as shown in the table and explained below.

Since S_1 and S_2 send X:NE, X:SW stabilizations to themselves or each other, Proposition 2 implies that these symmetries descend to maps on $\mathcal{L}$. Indeed, it can be shown (see, e.g., [OST, Lemma 4.6]) that S_2 does not change Legendrian isotopy type: $L \circ S_2(G) = L(G)$. It follows also that $T \circ S_2(G) = T(G)$. On the other hand, we have $L \circ S_1(G) = \mu(L(G))$, where $\mu : \mathcal{L} \to \mathcal{L}$ is the Legendrian mirror operation, which reflects Legendrian front diagrams in the horizontal axis [FT, OST]. In general, the two maps lead to two distinct Legendrian knots [Ng]; note that Legendrian "mirroring" preserves topological type. We remark that S_3 does not descend to a map on $\mathcal{L}$ (there is no Legendrian version of the topological mirror construction), and Legendrian mirrors do not descend to the transverse category.

The map S_4 on Legendrian knots produces the orientation reverse of the Legendrian mirror: $L \mapsto -\mu(L)$. This operation descends to (oriented) transverse knots, in an operation that could be called the transverse mirror.

We next consider braids. Given a grid diagram, there are four equally valid ways to obtain a map $\mathcal{G} \to \mathcal{B}$ that preserves topological knot type. One can require that the braid goes from left to right, as we do in Section 2.4, but one could instead require that the braid go from bottom to top, right to left, or top to bottom. We write the resulting maps as $G \mapsto B_\rightarrow(G)$, $G \mapsto B_\uparrow(G)$, $G \mapsto B_\leftarrow(G)$, and $G \mapsto B_\downarrow(G)$, respectively. In general, these maps lead to four distinct braids, related by

$$B_\rightarrow \circ S_1(G) = B_\leftarrow(G) \qquad B_\rightarrow \circ S_2(G) = B_\uparrow(G) \qquad B_\rightarrow \circ S_1 \circ S_2(G) = B_\downarrow(G).$$

As noted in [KN], it follows from $L \circ S_2(G) = L(G)$ that the braids $B_\rightarrow(G)$ and $B_\uparrow(G)$ represent the same element of $\mathcal{T}$ even though they usually differ in $\mathcal{B}$, and the same is true of the pair $B_\leftarrow(G)$ and $B_\downarrow(G)$. In addition, if we define operations $B \mapsto m(B)$ and $B \mapsto -B$ on braids, where $m(B)$ replaces every letter in B by its inverse and $-B$ is the braid word B read backwards, then $B_\rightarrow \circ S_3(G) = m(B_\rightarrow(G))$ and $B_\rightarrow \circ S_4(G) = -B_\rightarrow(G)$.

All symmetries of the NW-NE-SE-SW square are generated by S_1, S_2, S_3. The following generalization of Proposition 2 is an immediate consequence of the symmetries and Proposition 2.

Corollary 6. *We have bijections*

$$\tilde{\mathcal{G}}/(X{:}NE, X{:}SE) \xrightarrow{B_\rightarrow} \mathcal{B} \qquad\qquad \tilde{\mathcal{G}}/(X{:}SW, X{:}SE) \xrightarrow{B_\uparrow} \mathcal{B}$$

$$\tilde{\mathcal{G}}/(X{:}NW, X{:}SW) \xrightarrow{B_\leftarrow} \mathcal{B} \qquad\qquad \tilde{\mathcal{G}}/(X{:}NW, X{:}NE) \xrightarrow{B_\downarrow} \mathcal{B}$$

$$\tilde{\mathcal{G}}/(X{:}NE, X{:}SW) \xrightarrow{L} \mathcal{L} \qquad\qquad \tilde{\mathcal{G}}/(X{:}NW, X{:}SE) \xrightarrow{L \circ S_3} \mathcal{L}$$

$$\tilde{\mathcal{G}}/(X{:}NE, X{:}SW, X{:}SE) \xrightarrow{T} \mathcal{T} \qquad \tilde{\mathcal{G}}/(X{:}NW, X{:}NE, X{:}SW) \xrightarrow{T \circ S_1} \mathcal{T}$$

$$\tilde{\mathcal{G}}/(X{:}NW, X{:}SW, X{:}SE) \xrightarrow{T \circ S_3} \mathcal{T} \qquad \tilde{\mathcal{G}}/(X{:}NW, X{:}NE, X{:}SE) \xrightarrow{T \circ S_3 \circ S_2} \mathcal{T}$$

where L, T are induced from the maps $\mathcal{G} \to \mathcal{L}$, $\mathcal{G} \to \mathcal{T}$ described in Section 2.4.

Note that three of the bijections in Proposition 6 involve S_3 and thus topological mirroring.

We now discuss the conventions used in Section 2.4 in light of symmetries of grid diagrams. Our conventions are chosen to make the maps in Formula (1) always preserve topological knot type. This involves making several choices:

- vertical over horizontal line segments in grid diagrams (vs. horizontal over vertical), and Legendrian fronts obtained by $45°$ counterclockwise rotation (vs. clockwise);
- transverse knots given by positive pushoffs of Legendrian knots (vs. negative);
- braids going from left to right (vs. bottom to top, right to left, top to bottom).

These choices largely agree with the standard conventions in the literature [Cro, Dyn, EFM, Et, MOS, MOST]. One can obtain different conventions from ours by applying grid-diagram symmetries. For braids, this is discussed above, while for transverse

knots, positive pushoffs become negative pushoffs by applying the symmetry S_1: negative pushoffs are transversely isotopic under $X{:}NW, X{:}NE, X{:}SW$ (de)stabilization.

For the knot Floer homology invariant introduced in [OST] and subsequently used in [KN, NOT], a slightly different set of conventions is useful. Here an element λ^+ of combinatorial knot Floer homology HK^- is associated to any grid diagram, and λ^+ is shown to be invariant under translation, commutation, and $X{:}NW, X{:}SW, X{:}SE$ (de)stabilization. (Another element λ^- is also considered in [OST]; in our notation, $\lambda^- = \lambda^+ \circ S_1$.) If we apply symmetry $S_2 \circ S_3$ to a grid diagram G before calculating λ^+, then λ^+ becomes an invariant of the transverse knot $T(G)$.

In [KN, NOT, OST], the map $\mathcal{G} \to \mathcal{L}$ is thus given by $G \mapsto (L \circ S_2 \circ S_3)(G)$ rather than $G \mapsto L(G)$. More explicitly, given a grid diagram, one can use the horizontal-over-vertical convention and $45°$ clockwise rotation to obtain a Legendrian front, as is done in these papers. (In particular, to translate from our conventions to those of [KN], first apply $S_2 \circ S_3$ to all grid diagrams.) Note that due to the presence of S_3, λ^+ becomes an element of HK^- of the topological *mirror* of the transverse knot.

3. Proof of Proposition 2

Let $B(G)$ ($= B_\rightarrow(G)$ from Section 2.5) denote the braid associated to a grid diagram G as described in Section 2. Proposition 2 (or, more precisely, the braid statement of Proposition 2) is a direct consequence of the following stronger result.

Proposition 7. *Let G be a grid diagram.*
   (1) *Changing G by torus translation or $X{:}NE, X{:}SE$ (de)stabilization changes $B(G)$ by conjugation.*
   (2) *Changing G by commutation changes $B(G)$ by a combination of conjugation and exchange moves.*
   (3) *The map $G \mapsto B(G)$ induces a bijection between $\mathcal{G}/(\text{translation, commutation,} X{:}NE, X{:}SE \text{ (de)stabilization})$ and $\mathcal{B}/(\text{conjugation, exchange})$.*

Proof. We first check claims (1) and (2). A quick inspection of braid diagrams reveals that changing a grid diagram G by horizontal commutation or by $X{:}NE$ or $X{:}SE$ stabilization does not change the braid isotopy type of $B(G)$.

Changing G by horizontal torus translation changes $B(G)$ by conjugation; some portion of the beginning of $B(G)$ is moved to the end, or vice versa. See Figure 8.

Next we claim that changing G by vertical torus translation also changes $B(G)$ by conjugation. Indeed, consider moving the topmost column of G to the bottom. By conjugating by a horizontal torus translation if necessary, we may assume that in the relevant row, the O lies to the left of the X. Then moving the column keeps the braid unchanged; see Figure 8 again.

Finally, we claim that changing G by a vertical commutation changes $B(G)$ by conjugation and/or exchange. Indeed, by conjugating with an appropriate torus translation if necessary, we may assume the following: the two relevant rows are the bottom two rows in the grid diagram; the X and O in the bottom row both lie to the right of the X and
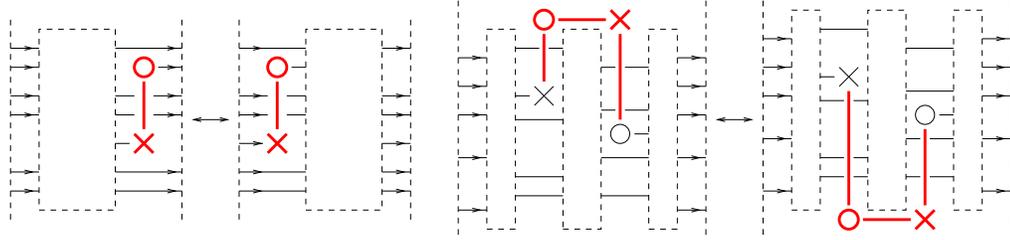
FIGURE 8. The effect on $B(G)$ of changing G by horizontal (left) and vertical (right) torus translation. The bold X and O represent the column/row being moved.
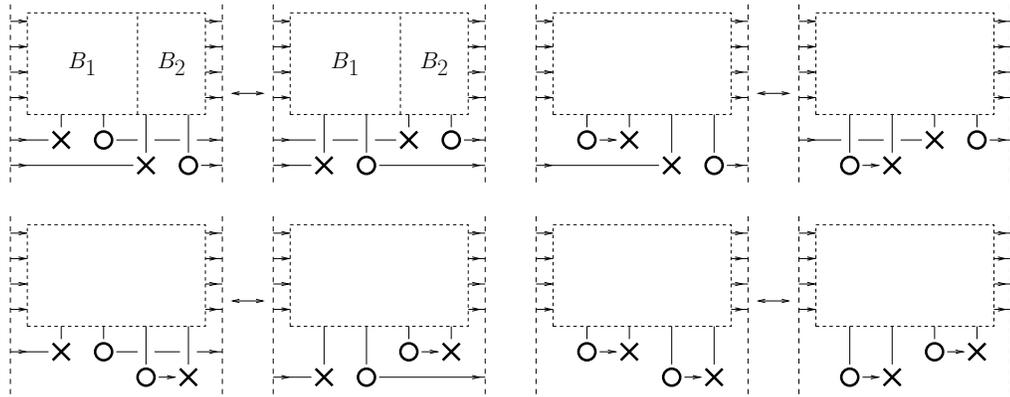

FIGURE 9. The effect on $B(G)$ of changing G by horizontal commutation. In three cases, $B(G)$ is unchanged. In the other case (upper left), the n-strand braid $B(G)$ changes from $B_1\sigma_{n-1}^{-1}B_2\sigma_{n-1}$ to $B_1\sigma_{n-1}B_2\sigma_{n-1}^{-1}$, an exchange move.

O in the row above it; and the bottom right corner of the grid diagram is occupied by an X or O. If X lies to the left of O in both rows, then the commutation changes $B(G)$ by exchange; otherwise, it does not change $B(G)$. See Figure 9.

We now establish claim (3). From claims (1) and (2), the map in (3) is well-defined. To prove bijectivity, we construct an inverse. Any braid B can be given a rectilinear braid diagram by replacing each crossing by an appropriate rectilinear version; see Figure 10.

Perturb the resulting rectilinear diagram slightly to another rectilinear diagram for which no vertical line segments have the same x-coordinate (i.e., are collinear), and no horizontal line segments have the same y-coordinate except for those that are identified when the ends of the braid are identified. The perturbed diagram is oriented (from left

FIGURE 10. Turning a braid diagram into a rectilinear braid diagram.
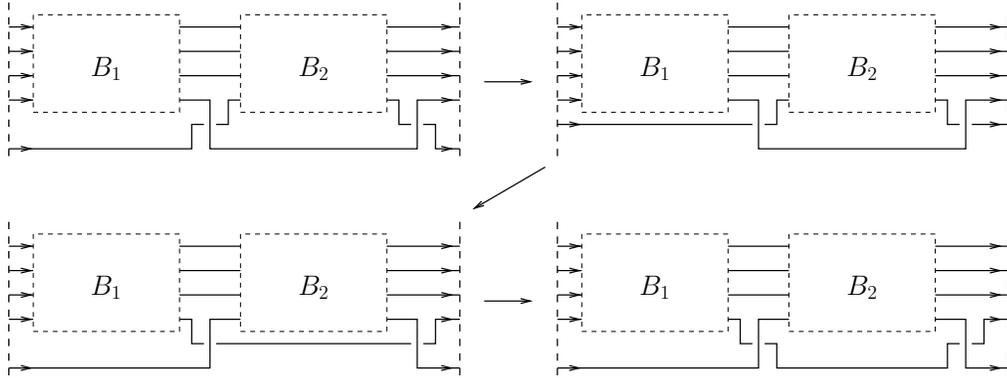

FIGURE 11. Accomplishing an exchange move through a sequence of commutation and (de)stabilization moves. The first arrow is given by commutations, one *X:NE* destabilization, and one *X:SE* destabilization; the second is a horizontal commutation; the third is commutations, one *X:NE* stabilization, and one *X:SE* destabilization. See also Figure 12 for the moves corresponding to the first and third arrows.

to right), and each corner can be assigned an X or O in the usual way. The collection of X's and O's forms a grid diagram $G(B)$, and by construction we have $B = B(G(B))$.

Note that $G(B)$ depends on the choice of perturbation from rectilinear braid diagram to grid diagram, but a different perturbation simply changes $G(B)$ by commutation. In fact, up to commutation and *X:SW*,*X:SE* (de)stabilization, $G(B)$ is well-defined for an isotopy class of braids B. This fact is readily established by examining how $G(B)$ changes when the braid word for B changes by one of the relations $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$, $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$, and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. See [Cro] for details.

In addition, changing B by conjugation changes $G(B)$ by horizontal torus translation, while changing B by an exchange move changes $G(B)$ by a combination of horizontal commutations and *X:NE*,*X:SE* (de)stabilizations; see Figures 11 and 12. Thus B induces a map from $\mathcal{B}/(\text{conjugation, exchange})$ to $\mathcal{G}/(\text{translation, commutation, } X:NE, X:SE$ (de)stabilization).
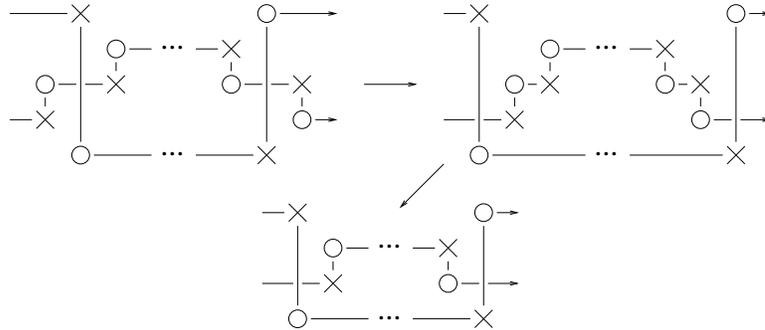
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FIGURE 12. Detail of local moves in the first step of Figure 11. A vertical commutation move is followed by *X:NE* and *X:SE* destabilization.

If we consider G and B as maps between $\mathcal{G}/$(translation, commutation, *X:NE*, *X:SE* (de)stabilization) and $\mathcal{B}/$(conjugation, exchange), then as noted earlier, $B \circ G$ is the identity, and one readily checks that $G \circ B$ is the identity as well. Claim (3) follows, and the proof of Proposition 7 is complete. □

Acknowledgments

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