The augmentation polynomial and topological strings

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Partly based on joint work with: Mina Aganagic (Berkeley), Tobias Ekholm (Uppsala), and Cumrun Vafa (Harvard).

The conormal construction

$$M \text{ smooth manifold} \rightsquigarrow T^*M \text{ symplectic manifold}: \quad \omega = \sum dq_i \wedge dp_i$$

$$\rightsquigarrow ST^*M \text{ contact manifold}: \quad \alpha = \sum p_i dq_i$$

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$$K \subset M \text{ submanifold} \rightsquigarrow L_K \subset T^*M \text{ Lagrangian : } \quad \omega|_{L_K} \equiv 0$$

$$\rightsquigarrow \Lambda_K \subset ST^*M \text{ Legendrian : } \quad \alpha|_{\Lambda_K} \equiv 0.$$

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$L_K \subset T^*M$ is the *conormal bundle*

$$L_K = \{(q,p) \,|\, q \in K,\ \langle p, v\rangle = 0 \ \forall\ v \in T_qK\}.$$

$\Lambda_K \subset ST^*M$ is the *unit conormal bundle* $L_K \cap ST^*M$.

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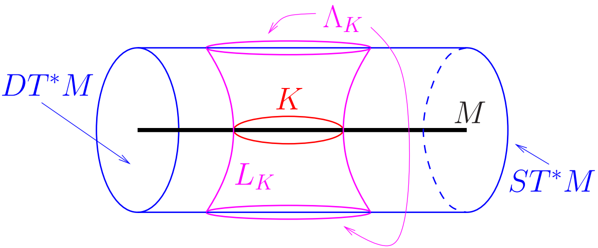
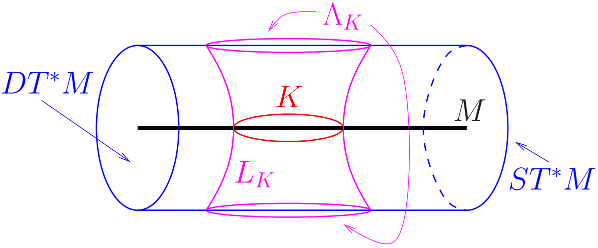


Symplectic invariants to smooth invariants

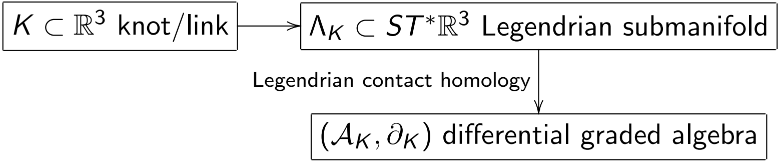


K, K' smoothly isotopic $\Longrightarrow \Lambda_K, \Lambda_{K'}$ Legendrian isotopic

Thus Legendrian-isotopy invariants of Λ_K give rise to smooth-isotopy invariants of K.

One such invariant is given by Legendrian contact homology, when defined (Eliashberg–Hofer, Chekanov, Ekholm–Etnyre–Sullivan).

Knot contact homology



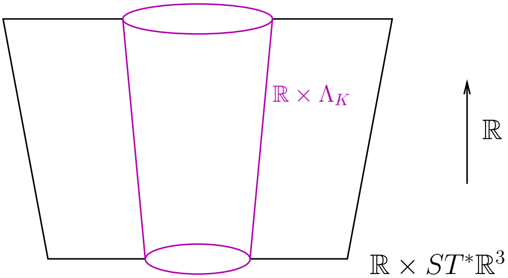
Definition

The knot contact homology of $K \subset \mathbb{R}^3$ is $HC_*(K) := H_*(\mathcal{A}_K, \partial_K)$. This is the LCH of $\Lambda_K \subset ST^*\mathbb{R}^3$.

Knot contact homology is an invariant of smooth knots.

Legendrian contact homology and the DGA $(\mathcal{A}_K, \partial_K)$

The algebra $\mathcal{A}_K$ is the free unital noncommutative algebra over $R := \mathbb{Z}[H_2(ST^*\mathbb{R}^3, \Lambda_K)] = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, Q^{\pm 1}]$ (when K is a knot) generated by Reeb chords of Λ_K.



The Lagrangian cylinder $\mathbb{R} \times \Lambda_K$ inside the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$.

Legendrian contact homology and the DGA $(\mathcal{A}_K, \partial_K)$

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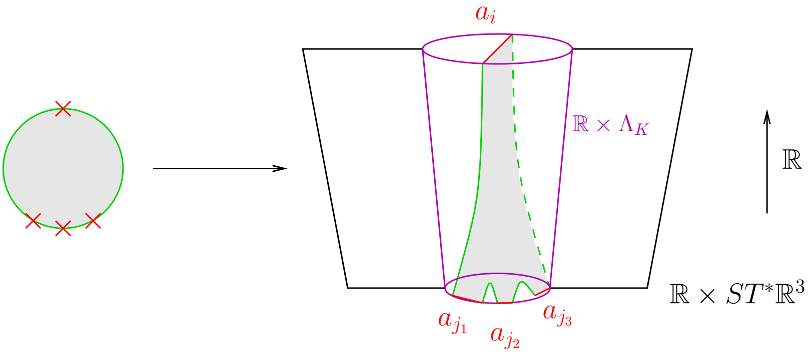

This holomorphic disk Δ contributes $e^{[\Delta]} a_{j_1} a_{j_2} a_{j_3}$ to $\partial_K(a_i)$, where a_i, a_{j_1}, a_{j_2}, a_{j_3} are Reeb chords, and $[\Delta] \in H_2(ST^*\mathbb{R}^3, \Lambda_K)$.

Properties of knot contact homology

Theorem

- *(Ekholm–Etnyre–N.–Sullivan, 2011) There is a combinatorial formulation for* $(\mathcal{A}_K, \partial_K)$*, associated to a braid whose closure is* K*. The algebra* $\mathcal{A}_K$ *is finitely generated and supported in degree* ≥ 0*.*
- *(N., 2005)* $(\mathcal{A}_K, \partial_K)$ *determines the Alexander polynomial* $\Delta_K(t)$*.*
- *(N., 2005) Knot contact homology is "relatively strong" as a knot invariant: it can distinguish mirrors, mutants, etc.*

Examples of the differential graded algebra $(\mathcal{A}_K, \partial_K)$

Recall the coefficient ring for knots is $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, Q^{\pm 1}]$.

Unknot: $\mathcal{A}_K = R\langle x_1, x_2\rangle$ with $|x_1| = 1$, $|x_2| = 2$,

$$\partial_K(x_1) = Q - \lambda - \mu + \lambda\mu$$
$$\partial_K(x_2) = 0,$$

and ∂_K extends to $\partial_K : \mathcal{A}_K \to \mathcal{A}_K$ by the Leibniz rule $\partial_K(xy) = (\partial_K x)y + (-1)^{|x|}x(\partial_K y)$.

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Right-handed trefoil: $\mathcal{A}_K = R\langle x_1, x_2, x_3, \dots\rangle$ with $|x_1| = 0$, $|x_2| = 1$, $|x_3| = 1$,

$$\partial_K(x_1) = 0$$
$$\partial_K(x_2) = Qx_1^2 - \mu Q x_1 + \lambda\mu^3(1-\mu)$$
$$\partial_K(x_3) = Qx_1^2 + \lambda\mu^2 x_1 + \lambda\mu^2(\mu - Q)$$
$$\vdots$$

A new polynomial knot invariant

Definition

The augmentation variety V_K of a knot K is (the highest-dimensional part of the closure of)

$$\{(\lambda, \mu, Q) \mid \exists \text{ algebra map } \epsilon : \mathcal{A}_K \to \mathbb{C} \text{ with } \epsilon \circ \partial_K = 0,$$
$$\epsilon(\lambda) = \lambda,\ \epsilon(\mu) = \mu,\ \epsilon(Q) = Q\}$$
$$\subset (\mathbb{C} \setminus \{0\})^3.$$

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This appears to be a codimension-1 variety for all knots K.

Definition

The augmentation polynomial of a knot K

$$\mathrm{Aug}_K(\lambda, \mu, Q) \in \mathbb{Z}[\lambda, \mu, Q]$$

is the reduced polynomial for which $V_K = \{\mathrm{Aug}_K(\lambda, \mu, Q) = 0\}$.

Computing the augmentation polynomial

In practice, to a knot K, knot contact homology associates a finite, combinatorially defined collection of polynomials in some variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}[\lambda, \mu, Q]$:

$$K \rightsquigarrow \{p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n)\}.$$

The augmentation variety is the set of (λ, μ, Q) for which these polynomials have a common root in $x_1, \ldots, x_n$:

$$\begin{aligned} p_1(x_1, \ldots, x_n) &= 0 \\ p_2(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \ldots, x_n) &= 0. \end{aligned}$$

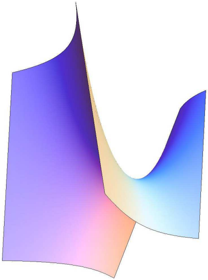
Augmentation polynomial: unknot

For $K = \bigcirc$, the unknot: the collection of polynomials in $n = 0$ variables is

$$\{Q - \lambda - \mu + \lambda\mu\}.$$

Thus

$$Aug_{\bigcirc}(\lambda, \mu, Q) = Q - \lambda - \mu + \lambda\mu.$$



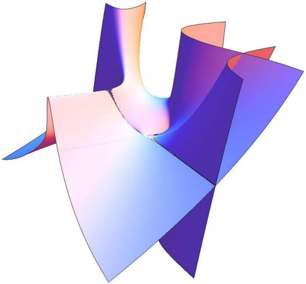
Augmentation polynomial: trefoil

For $K = T$, the right-handed trefoil: the collection of polynomials in $n = 1$ variable is

$$\{Qx_1^2 - \mu Q x_1 + \lambda\mu^3(1-\mu), Qx_1^2 + \lambda\mu^2 x_1 + \lambda\mu^2(\mu - Q)\}.$$

Then take the resultant of these two polynomials:

$$\mathrm{Aug}_T(\lambda,\mu,Q) = (Q^3 - \mu Q^2) + (-Q^3 + \mu Q^2 - 2\mu^2 Q + 2\mu^2 Q^2 + \mu^3 Q - \mu^4 Q)\lambda + (-\mu^3 + \mu^4)\lambda^2.$$



Two-variable augmentation polynomial

The 2-d augmentation variety of a knot K is (the highest-dimensional part of the closure of)

$$\begin{aligned} V_K^{Q=1} = \{(\lambda, \mu) \,|\, \exists \text{ algebra map } \epsilon : \mathcal{A}_K \to \mathbb{C} \text{ with } \epsilon \circ \partial = 0, \\ \epsilon(\lambda) = \lambda,\ \epsilon(\mu) = \mu,\ \epsilon(Q) = 1\} \\ \subset (\mathbb{C}^*)^2. \end{aligned}$$

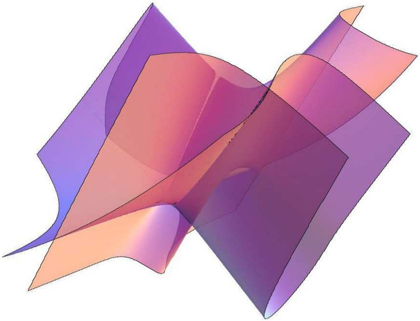
Definition

The 2-d augmentation variety is the vanishing set of the two-variable augmentation polynomial $\mathrm{Aug}_K(\lambda, \mu)$.

This is conjecturally $\mathrm{Aug}_K(\lambda, \mu, Q = 1)$ up to repeated factors.

Example: right-handed trefoil

$$\mathrm{Aug}_T(\lambda, \mu, Q) = (Q^3 - \mu Q^2) + (-Q^3 + \mu Q^2 - 2\mu^2 Q + 2\mu^2 Q^2 + \mu^3 Q - \mu^4 Q)\lambda + (-\mu^3 + \mu^4)\lambda^2$$



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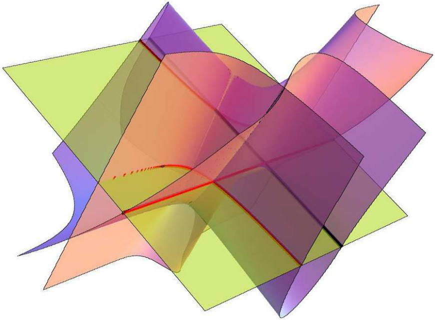
$$\mathrm{Aug}_T(\lambda,\mu) = \mathrm{Aug}_T(\lambda,\mu,1) = (\mu-1)(\lambda-1)(\lambda\mu^3+1).$$



A-polynomial

Let $m, l \in \pi_1(S^3 \setminus K)$ be the meridian and longitude of K. Consider a representation

$$\rho: \pi_1(S^3 \setminus K) \to SL(2, \mathbb{C})$$

Simultaneously diagonalize $\rho(m) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ and $\rho(l) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Definition

The (closure of the highest-dimensional part of)

$$\{(\lambda, \mu) \,|\, \rho = SL(2, \mathbb{C}) \text{ representation of } \pi_1(S^3 \setminus K)\}$$

is the vanishing set of the *A-polynomial* $A_K(\lambda, \mu)$.

Curious observation: $A_K(\lambda, \mu)$ always divides $\mathrm{Aug}_K(\lambda, \mu^2)$.

KCH representations

Definition

A KCH representation is $\rho : \pi_1(S^3 \setminus K) \to GL(n, \mathbb{C})$ for some $n \geq 1$, with

$$\rho(m) = \left(\begin{array}{c|ccc} \mu & 0 & \cdots & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right), \qquad \rho(l) = \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right).$$

Definition

The (closure of the highest-dimensional part of)

$$\{(\lambda, \mu) \,|\, \rho = \text{KCH representation of } \pi_1(S^3 \setminus K)\}$$

is the vanishing set of the stable A-polynomial $\tilde{A}_K(\lambda, \mu)$.

Ranks of KCH representations

Observation: if ρ is an $SL(2,\mathbb{C})$ representation of $\pi_1(S^3 \setminus K)$, then $\tilde{\rho}$ is a rank-2 KCH representation (with $\mu \mapsto \mu^2$), where

$$\tilde{\rho}(\gamma) = \mu^{\mathrm{lk}(K,\gamma)}\rho(\gamma):$$

$\rho(m) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \Rightarrow \tilde{\rho}(m) = \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore

$$A_K(\lambda, \mu^{1/2}) \,\Big|\, \tilde{A}_K(\lambda, \mu).$$

Theorem (C. Cornwell, 2013)

For a fixed knot K, the rank of an irreducible KCH representation is bounded above by the meridional rank of the knot. Thus the set of (λ, μ) for KCH representations of rank $\leq n$ stabilizes as $n \to \infty$, and the stable A-polynomial $\tilde{A}_K(\lambda, \mu)$ is well-defined.

Augmentation polynomial and the A-polynomial

Theorem (N., 2012)

Any KCH representation ρ induces an augmentation ϵ of $(\mathcal{A}_K, \partial_K)$ with $\epsilon(Q) = 1$, $\epsilon(\lambda) = \lambda$, and $\epsilon(\mu) = \mu$. Thus

$$A_K(\lambda, \mu^{1/2}) \,\Big|\, \tilde{A}_K(\lambda, \mu) \,\Big|\, \mathrm{Aug}_K(\lambda, \mu).$$

Dunfield–Garoufalidis, Boyer–Zhang, 2004 (based on Kronheimer–Mrowka 2003): the A-polynomial detects the unknot.

Corollary (N., 2005)

The two-variable augmentation polynomial $\mathrm{Aug}_K(\lambda, \mu)$, and thus knot contact homology, detects the unknot:

$$\mathrm{Aug}_K(\lambda, \mu) = \mathrm{Aug}_{\bigcirc}(\lambda, \mu) \Rightarrow K = \bigcirc.$$

Augmentation polynomial and the stable A-polynomial

Theorem (Cornwell, 2013)

Any augmentation of $(\mathcal{A}_K, \partial_K)$ induces a KCH representation. Thus the two-variable augmentation polynomial is equal to the stable A-polynomial:

$$\tilde{A}_K(\lambda, \mu) = \mathrm{Aug}_K(\lambda, \mu).$$

Corollary (Cornwell, 2013)

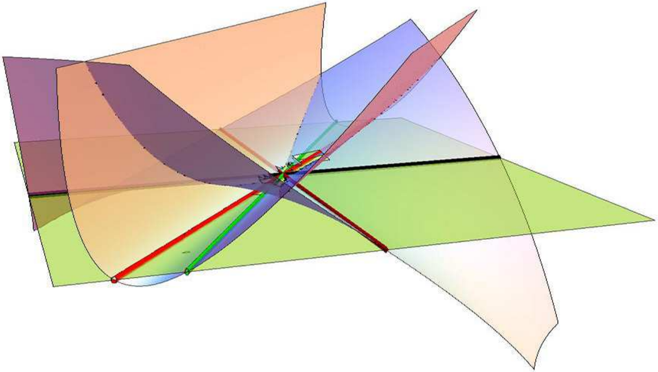
If K is two-bridge, then $\mathrm{Aug}_K(\lambda, \mu) = A_K(\lambda, \mu^{1/2}) = \tilde{A}_K(\lambda, \mu)$.

In general, $\mathrm{Aug}_K(\lambda, \mu)$ can have factors not in A_K, coming from higher-rank KCH representations.

Example: $K = T(3,4)$

When K is the $(3,4)$ torus knot, we have:

$$A_K(\lambda, \mu^{1/2}) = (\lambda - 1)(\lambda\mu^6 - 1)(\lambda\mu^6 + 1)$$
$$\mathrm{Aug}_K(\lambda, \mu) = (\mu - 1)(\lambda - 1)(\lambda\mu^6 - 1)(\lambda\mu^6 + 1)(\lambda\mu^8 - 1).$$



Colored HOMFLY polynomials

For $n \geq 1$, let $P_{K;n}(a,q)$ be the colored HOMFLY polynomials of K colored by the n-th symmetric power of the fundamental representation.

Theorem (Garoufalidis, 2012)

The polynomials $\{P_{K;n}(a,q)\}_{n=1}^{\infty}$ are q-holonomic.

That is: define operators L, M by

$$L(P_{K;n}(a,q)) = P_{K;n+1}(a,q), \qquad M(P_{K;n}(a,q)) = q^n P_{K;n}(a,q).$$

Then there is a (minimal) recurrence relation of the form

$$\widehat{A}_K(a,q,M,L)P_{K;n}(a,q) = 0$$

where $\widehat{A}_K$ is a polynomial in commuting variables a, q and noncommuting variables M, L with $ML = qLM$.

Colored HOMFLY and the 3-variable augmentation polynomial

One can consider the "classical limit" $\widehat{A}_K|_{q=1}(a, M, L)$, which is an honest polynomial in a, M, L.

Conjecture

Under the identification $a = Q$, $M = \mu^{-1}$, $L = \frac{\mu-1}{\mu-Q}\lambda$,

$$\widehat{A}_K|_{q=1}(a, M, L) = \mathrm{Aug}_K(\lambda, \mu, Q).$$

Compare this to the AJ conjecture:

$$A\text{-poly} = \text{classical limit of recursion for colored Jones}$$
$$\text{aug poly} = \text{classical limit of recursion for colored HOMFLY.}$$

HOMFLY and the 3-variable augmentation polynomial

There is also a conjectured relation between Aug_K and the usual HOMFLY polynomial. The line $\{(\lambda, \mu, Q) = (0, Q, Q)\} \subset \mathbb{C}^3$ lies in the closure of the augmentation variety of K. Near this line, points (λ, μ, Q) on the variety satisfy

$$\mu = Q + f(Q)\lambda + O(\lambda^2)$$

for some polynomial $f(Q)$ determined by $\mathrm{Aug}_K(\lambda, \mu, Q)$.

Conjecture

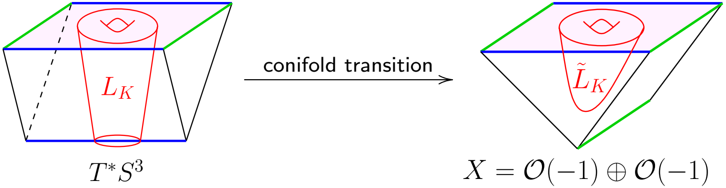
If $P_K(a, q)$ is the HOMFLY polynomial of K, then

$$\frac{f(Q)}{Q-1} = P_K(Q^{-1/2}, 1).$$

This has been verified for all computed examples of the augmentation polynomial.

Physical motivation

Let X be the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$; this has the same geometry at infinity (ST^*S^3) as T^*S^3. The conormal $L_K \subset T^*S^3$ passes through to a Lagrangian $\tilde{L}_K \subset X$ whose boundary is also the Legendrian Λ_K.



Using topological strings, Aganagic–Vafa (2012) propose a generalized SYZ conjecture that associates a mirror Calabi–Yau 3-fold X_K to the pair $(X, \tilde{L}_K)$.

Physical motivation, continued

The mirror Calabi–Yau X_K is of the form

$$X_K = \{uv = \mathbf{A}_K(e^x, e^p, Q)\} \subset \mathbb{C}^4_{uvxp}$$

for some 3-variable polynomial $\mathbf{A}_K$ ("Q-deformed A-polynomial"). For physical reasons, $\mathbf{A}_K$ satisfies the relations to HOMFLY stated previously.

Conjecture (Aganagic, Vafa, Ekholm, N.)

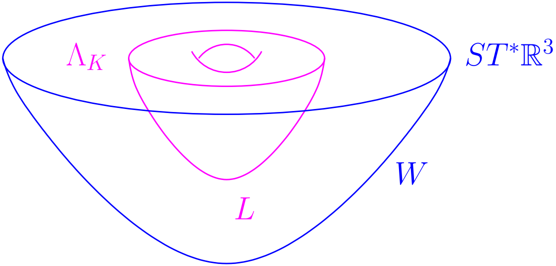
$\mathrm{Aug}_K(\lambda, \mu, Q) = \mathbf{A}_K(\lambda, \mu, Q)$.

This has been proven in some cases, and we have a pathway to a proof for all knots where the augmentation variety is irreducible.

Lagrangian fillings

Mathematically, the connection between the two sides is given by considering Lagrangian fillings of Λ_K: these are $L \subset W$ such that

- W is exact symplectic with positive boundary $ST^*\mathbb{R}^3$
- L is Lagrangian with boundary Λ_K.



Lagrangian fillings of Λ_K

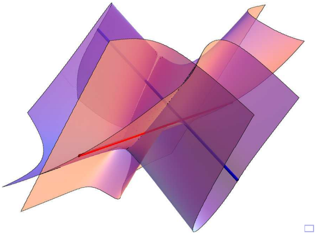
Exact Lagrangian fillings L of Λ_K give rise to augmentations of $(\mathcal{A}_K, \partial_K)$. Two exact fillings in $W = T^*S^3$:

- L_K (topologically $S^1 \times D^2$): this fills in the meridian of the torus Λ_K and gives augmentations ϵ with $\epsilon(\mu) = \epsilon(Q) = 1$:

$$L_K \rightsquigarrow \{\mu = Q = 1\} \subset V_K$$

- N_K (topologically $S^3 \setminus K$): this fills in the longitude of Λ_K and gives augmentations ϵ with $\epsilon(\lambda) = \epsilon(Q) = 1$:

$$N_K \rightsquigarrow \{\lambda = Q = 1\} \subset V_K.$$



Nonexact Lagrangian fillings and augmentations

Nonexact Lagrangian fillings L of Λ_K, such as $\tilde{L}_K \subset X$, do not directly give augmentations. Instead: count holomorphic disks with boundary on L to construct a Gromov–Witten potential function

$$W(x, Q).$$

Considering obstruction chains, à la Fukaya–Oh–Ohta–Ono, shows that parts of the augmentation variety V_K of Λ_K satisfy

$$p = \partial W / \partial x$$

where $\lambda = e^x$, $\mu = e^p$.
The same potential also appears in the physical argument.

Augmentation variety

If K is an n-component link, knot contact homology produces an augmentation variety

$$V_K \subset (\mathbb{C}^*)^{2n+1},$$

where $(\mathbb{C}^*)^{2n+1}$ has coordinates $\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n, Q$.
For fixed Q, $V_K \subset (\mathbb{C}^*)^{2n}$ appears to always be Lagrangian with respect to the symplectic form

$$\omega = \sum_{i=1}^{n} \frac{d\lambda_i \wedge d\mu_i}{\lambda_i \mu_i}.$$

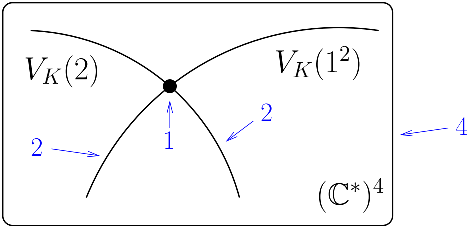
In general, V_K is not irreducible, and its irreducible components intersect in highly non-generic ways.

Example: Hopf link

When K is the Hopf link, $V_K = V_K(1^2) \cup V_K(2) \subset (\mathbb{C}^*)^5$ has two components:

- $V_K(1^2) = \{Q - \lambda_1 - \mu_1 + \lambda_1\mu_1 = Q - \lambda_2 - \mu_2 + \lambda_2\mu_2 = 0\}$, given by "split" augmentations and corresponding to the partition $\{\{1\},\{2\}\}$ of $\{1,2\}$
- $V_K(2) = \{\lambda_1 - \mu_2 = \lambda_2 - \mu_1 = 0\}$, given by "non-split" augmentations, corresponding to the partition $\{\{1,2\}\}$.

For fixed Q, these intersect non-generically along a curve.



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