# From holomorphic curves to knot invariants via the cotangent bundle

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Based on joint work with: Tobias Ekholm (Uppsala University), John Etnyre (Georgia Tech), and Michael Sullivan (University of Massachusetts); and Mina Aganagic (UC Berkeley), Tobias Ekholm, and Cumrun Vafa (Harvard University).

These slides available at `http://www.math.duke.edu/~ng/math/tulane.pdf`.

# Outline

# Classification of manifolds

Motivating question in low-dimensional topology: classify or characterize topological/smooth manifolds in 3 and 4 dimensions, up to equivalence.

Three types of equivalence of manifolds:

- homotopy equivalence
- homeomorphism (topological equivalence)
- diffeomorphism (smooth equivalence).

We have

$$\text{diffeomorphic} \Rightarrow \text{homeomorphic} \Rightarrow \text{homotopy equivalent.}$$

In three dimensions, diffeomorphic $\Leftrightarrow$ homeomorphic.

# Poincaré conjecture

**Poincaré conjecture**

*Let $M$ be a closed topological 3-manifold such that*

$$\pi_1(M) = 1.$$

*Then $M$ is homeomorphic to $S^3$.*

Poincaré conjecture famously proven by Perelman about a decade ago.

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## $n$-dimensional Poincaré conjecture

*Any topological manifold homotopy equivalent to $S^n$ is homeomorphic to $S^n$.*

True in all dimensions (Smale $n \geq 5$; Freedman $n = 4$; Perelman $n = 3$).

# The smooth Poincaré conjecture

**Smooth $n$-dimensional Poincaré conjecture**

*Any smooth manifold homotopy equivalent to $S^n$ is diffeomorphic to $S^n$.*

True for $n \leq 3$; resolved for $n \geq 5$ (e.g., false for $n = 7$: Milnor's exotic $S^7$'s).

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Number of smooth structures on $S^n$:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # | 1 | 1 | | | | | | | | | | | | | |

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| # | 1 | 1 | | | 1 | 1 | 28 | 2 | 8 | 6 | 992 | 1 | 3 | 2 | 16256 |

Kervaire–Milnor (1963): count for $n \geq 5$ using homotopy theory.

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Perelman (2003): $n = 3$.

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # | 1 | 1 | 1 | ? | 1 | 1 | 28 | 2 | 8 | 6 | 992 | 1 | 3 | 2 | 16256 |

$n = 4$: open!

# Smooth 4-dimensional Poincaré

## Smooth 4-dimensional Poincaré conjecture

*If a smooth manifold $M$ is homotopy equivalent (or homeomorphic) to $S^4$, then it is diffeomorphic to $S^4$.*

There are a number of possible counterexamples to this conjecture: proposed "exotic $S^4$'s".

One stumbling block: a lack of good invariants of smooth 4-manifolds that apply to this setting.

# Smooth 4-dimensional Poincaré

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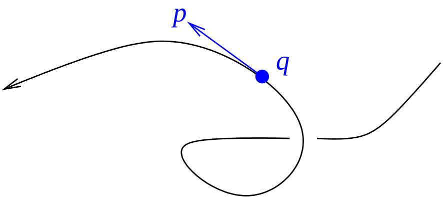
One stumbling block: a lack of good invariants of smooth 4-manifolds that apply to this setting.

Cotangent bundles to the rescue?

# Phase space

Particle in $\mathbb{R}^3$:

- position $q = (q_1, q_2, q_3)$
- momentum $p = (p_1, p_2, p_3)$




The *phase space* of the particle is

$$\mathbb{R}^6 = \mathbb{R}^3_{(q_1,q_2,q_3)} \times \mathbb{R}^3_{(p_1,p_2,p_3)}.$$

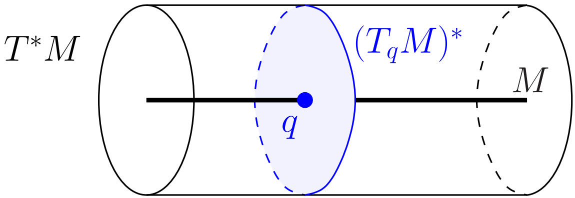
# The cotangent bundle

More generally, a particle in a manifold $M$ has a position $q \in M$ and a velocity vector $v \in T_q M$; for various reasons, it's more natural to consider the dual, momentum vector $p \in (T_q M)^*$.

The phase space of the particle is the cotangent bundle

$$T^* M = \{(q, p) \,|\, q \in M,\ p \in (T_q M)^*\}.$$

If $\dim_{\mathbb{R}} M = n$, then $\dim_{\mathbb{R}} T^* M = 2n$.



# Symplectic manifolds

Cotangent bundles $T^*M$ are examples of symplectic manifolds.

**Definition**

A 2-form $\omega$ on a $2n$-dim'l manifold $W$ is a symplectic form if

- $d\omega = 0$ ($\omega$ is closed)
- $\omega^n$ is a nowhere zero $2n$-form ($\omega$ is nondegenerate).

**Definition**

An even-dimensional manifold is a symplectic manifold if it has a symplectic form.

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**Definition**

An even-dimensional manifold is a symplectic manifold if it has a symplectic form.

The "prototypical" symplectic manifold is $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ and symplectic form

$$\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n.$$

# Cotangent bundles are symplectic

More generally, on a cotangent bundle $T^*M$ with local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, we can define a 2-form $\omega \in \Omega^2(T^*M)$ by

$$\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n.$$

**Theorem**

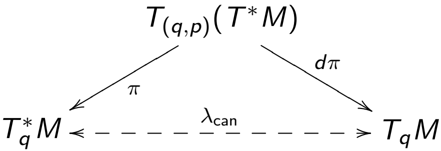
- *For any smooth manifold $M$, $\omega$ is independent of coordinates, and $(T^*M, \omega)$ is a symplectic manifold.*
- *If $M$ and $M'$ are diffeomorphic (equivalent as smooth manifolds), then the symplectic manifolds $T^*M$ and $T^*M'$ are symplectomorphic (equivalent as symplectic manifolds).*

# The symplectic form on $T^*M$

Coordinate-free definition of $\omega \in \Omega^2(T^*M)$:

There is a canonical 1-form $\lambda_{\text{can}} \in \Omega^1(T^*M)$, the Liouville form: for $v \in T_{(q,p)}(T^*M)$,

$$\lambda_{\text{can}}(v) = \langle \pi(v), d\pi(v) \rangle.$$




Then

$$\omega = -d\lambda_{\text{can}}.$$

## Arnol‘d's strategy

V. I. Arnol‘d: study the smooth topology of $M$ via the symplectic topology of $T^*M$.

**Question**

*If $M, M'$ are closed smooth manifolds such that $T^*M$ and $T^*M'$ are symplectomorphic, are $M$ and $M'$ necessarily diffeomorphic?*

Note: recent result of Adam Knapp (2012) shows that this is not necessarily true without the closed condition: exotic $\mathbb{R}^4$'s have symplectomorphic cotangent bundles.

# Smooth invariants from symplectic geometry

One way to produce invariants of smooth manifolds:

| Smooth topology | Symplectic topology |
| --- | --- |
| $M$ smooth manifold → | $T^*M$ symplectic manifold |
| | ↓ ?? |
| smooth invariant of $M$ ←(:=) | symplectic invariant of $T^*M$ |

# Smooth invariants from symplectic geometry

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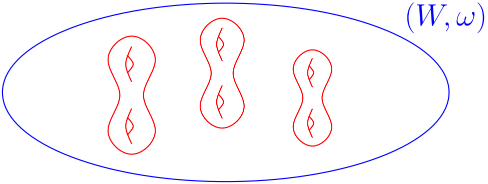
| | $\downarrow$ ?? |
| smooth invariant of $M$ $\xleftarrow{:=}$ | symplectic invariant of $T^*M$ |

The symplectic invariants are often given by counts of holomorphic curves.

# Holomorphic curves

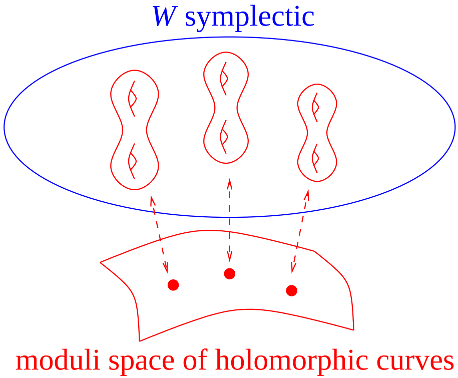
Gromov, 1980s: one can create interesting invariants of symplectic manifolds $(W, \omega)$ by studying holomorphic curves in $W$: Riemann surfaces in $W$ satisfying a certain compatibility condition with $\omega$ (involving an almost complex structure on $W$ tamed by $\omega$).




Gromov's insight: in many cases, there are only *finitely many* holomorphic curves, and counting them yields symplectic invariants (cf. algebraic geometry).

# Holomorphic curves continued

More generally, the *moduli space* of holomorphic curves is often well-behaved (e.g., a manifold with corners) and studying this moduli space yields symplectic invariants.



# Hamiltonian Floer Homology

One invariant of (certain) symplectic manifolds: Hamiltonian Floer homology (based on Floer, 1988).

**Theorem (Viterbo 1996, Salamon–Weber 2003, Abbondandolo–Schwarz 2004)**

*The Hamiltonian Floer homology of the symplectic manifold $T^*M$ is isomorphic to the singular homology of the free loop space $\mathcal{L}M$:*

$$HF_*(T^*M) \cong H_*(\mathcal{L}M).$$

Thus the symplectic structure on $T^*M$ remembers at least some homotopic data about $M$.

# Exotic spheres and cotangent bundles

Recently, Mohammed Abouzaid has shown that the symplectic structure on $T^*M$ can encode more than the homotopic/topological structure of $T^*M$: it can encode smooth information.

**Theorem (Abouzaid, 2008)**

*If $\Sigma$ is an exotic $S^{4k+1}$ that does not bound a parallelizable manifold, then $T^*\Sigma$ is not symplectomorphic to $T^*S^{4k+1}$.*

Kervaire–Milnor: there are 8 different smooth structures on $S^9$; this shows that 6 of them are distinct from the standard smooth structure.

Abouzaid's argument studies certain moduli spaces of holomorphic curves on $T^*\Sigma$.

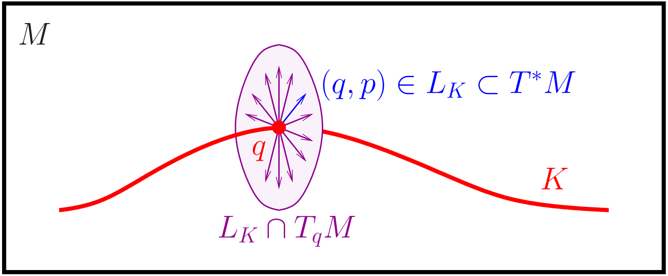
# Conormal bundles

We will focus on a relative of the cotangent construction.

## Definition

Let $K \subset M$ be a submanifold. The conormal bundle to $K$ is

$$L_K := \{(q,p) \mid q \in K \text{ and } \langle p, v\rangle = 0 \text{ for all } v \in T_qK\} \subset T^*M.$$



# Conormal bundle and the symplectic structure

If $\dim(M) = n$, then $\dim(T^*M) = 2n$ and dimension counting shows that $\dim(L_K) = n$ regardless of the dimension of $K$.

**Theorem**

*For any submanifold $K \subset M$,*

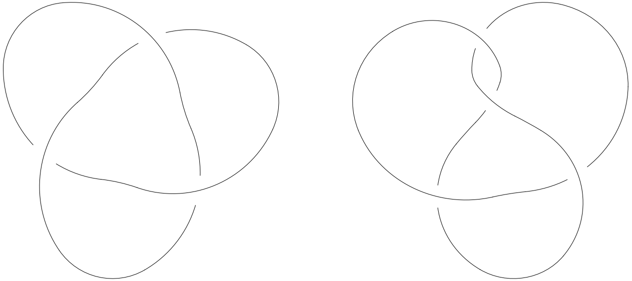
$$L_K \subset T^*M$$

*is Lagrangian: a maximal-dimensional submanifold of $T^*M$ on which the symplectic form $\omega$ is identically 0.*

We will be interested in the case where $M = \mathbb{R}^3$ and $K \subset \mathbb{R}^3$ is a knot: a smooth embedding of $S^1$ in $\mathbb{R}^3$. In this case, $L_K \cong S^1 \times \mathbb{R}^2$ is a Lagrangian submanifold of $T^*\mathbb{R}^3 \cong \mathbb{R}^6$.

# Knots in $\mathbb{R}^3$

We consider knots in $\mathbb{R}^3$ up to smooth isotopy: two knots $K_0$ and $K_1$ are smoothly isotopic if there is a 1-parameter family of knots $K_t$ for $0 \leq t \leq 1$.



Smoothly isotopic knots (here, the right-handed trefoil).

# The conormal bundle as a knot invariant

If knots $K_0, K_1 \subset \mathbb{R}^3$ are smoothly isotopic, then there is a 1-parameter family of Lagrangian submanifolds $L_{K_t} \subset T^*\mathbb{R}^3$: $L_{K_0}, L_{K_1}$ are Lagrangian isotopic.

**Question**

*How much of the topology of the knot $K \subset \mathbb{R}^3$ is encoded in the symplectic/Lagrangian structure of $L_K \subset T^*\mathbb{R}^3$?*

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**Conjecture?**

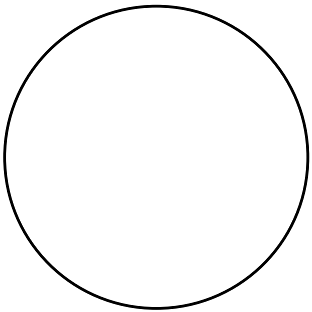
*The Lagrangian submanifold $L_K$ is a complete knot invariant: if $K_0, K_1$ are knots such that $L_{K_0}$ and $L_{K_1}$ are Lagrangian isotopic, then $K_0$ and $K_1$ are smoothly isotopic.*

(More precise conjecture involves "Legendrian isotopy" in the contact manifold $ST^*\mathbb{R}^3$ of $\Lambda_K := L_K \cap ST^*\mathbb{R}^3$.)

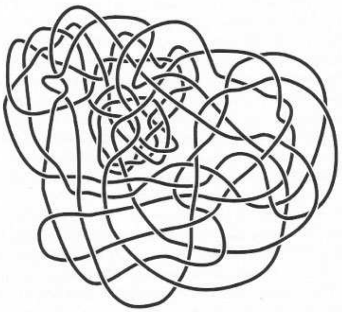
# Conormal bundle detects the unknot

**Theorem (N., 2005)**

*$L_K$ detects the unknot $O$: if $K \subset \mathbb{R}^3$ is a knot such that $\Lambda_K$ and $\Lambda_O$ are Legendrian isotopic, then $K$ is unknotted: $K = O$.*



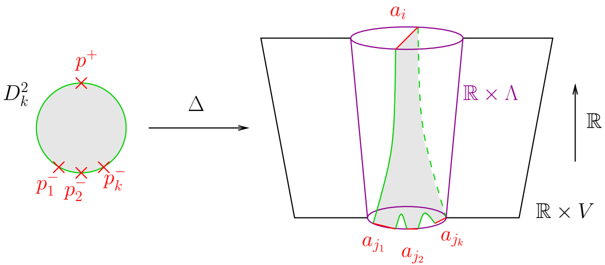
the unknot $O$



$K = O$?

# Legendrian contact homology

To distinguish between Lagrangians $L_K$ for different knots $K$, need good invariants of Lagrangian submanifolds in symplectic manifolds.




One is given by Legendrian contact homology (LCH) (Eliashberg–Hofer, 1990s; Etnyre–Ekholm–Sullivan, 2005). LCH inputs a Legendrian submanifold $\Lambda$ of a contact manifold $V$, and outputs a count of holomorphic curves in the symplectization $\mathbb{R} \times V$ with boundary on $\mathbb{R} \times \Lambda$ and certain asymptotic behavior.

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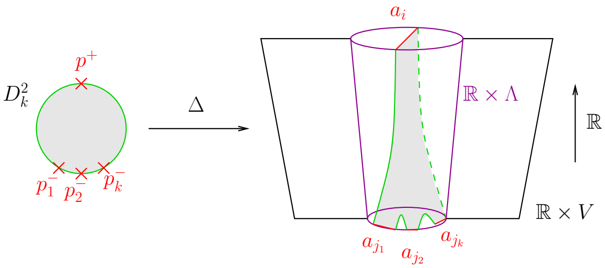


In our setting, LCH counts certain holomorphic disks in $T^*M$ with boundary on $L_K$.

# Knot contact homology

$K \subset \mathbb{R}^3$ knot $\longrightarrow$ $L_K \subset T^*\mathbb{R}^3$ Lagrangian

$\downarrow$ $LCH$

$\longleftarrow$ $HC_*(L_K)$, symplectic invariant

# Knot contact homology

$$
\begin{array}{ccc}
K \subset \mathbb{R}^3 \text{ knot} & \longrightarrow & L_K \subset T^*\mathbb{R}^3 \text{ Lagrangian} \\
 & & \downarrow LCH \\
HC_*(K), \text{ knot invariant} & \overset{:=}{\longleftarrow} & HC_*(L_K), \text{ symplectic invariant}
\end{array}
$$

**Definition**

Let $K \subset \mathbb{R}^3$ be a knot. The knot contact homology $HC_*(K)$ is the LCH associated to $L_K \subset T^*\mathbb{R}^3$. This is a knot invariant (an invariant of knots up to smooth isotopy).

# Knot contact homology, continued

**Theorem (N. 2003, 2005, 2010; Ekholm–Etnyre–N.–Sullivan 2011)**

*There is a combinatorially-defined differential graded algebra $(\mathcal{A}, \partial)$ associated to a knot $K$, for which*

$$H_*(\mathcal{A}, \partial) = HC_*(K).$$

The algebra $\mathcal{A}$ is a finitely-generated noncommutative algebra over the ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U^{\pm 1}]$.

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**Conjecture?**

*Knot contact homology is a complete knot invariant: if knots $K_1, K_2$ satisfy*

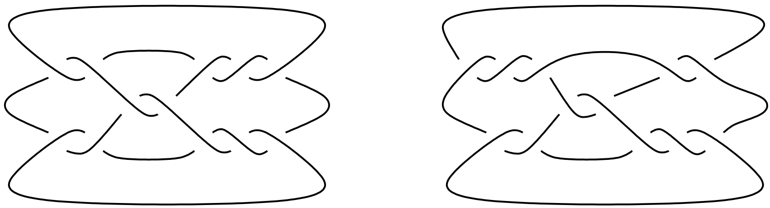
$$HC_*(K_1) \cong HC_*(K_2)$$

*then $K_1 = K_2$.*

# Properties of knot contact homology

**Theorem (N., 2005)**

- *Knot contact homology $HC_*(K)$ determines the Alexander polynomial $\Delta_K(t)$.*
- *Knot contact homology is "relatively strong" as a knot invariant: it can distinguish mirrors, mutants, etc.*



Two famous "mutant" knots: the Kinoshita–Terasaka knot and the Conway knot.

# A new polynomial knot invariant

## Definition

The augmentation variety of a knot $K$ (with DGA $(\mathcal{A}, \partial)$) is

$$\{(\lambda, \mu, U) \in (\mathbb{C} \setminus \{0\})^3 \mid \text{there is an algebra map } \epsilon : \mathcal{A} \to \mathbb{C} \text{ with } \epsilon \circ \partial = 0\} \subset (\mathbb{C} \setminus \{0\})^3.$$

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This appears to be a codimension-1 algebraic set for all knots $K$.

**Definition**

The augmentation polynomial of a knot $K$

$$\mathrm{Aug}_K(\lambda, \mu, U) \in \mathbb{Z}[\lambda, \mu, U]$$

is the polynomial for which the augmentation variety is $\{\mathrm{Aug}_K(\lambda, \mu, U) = 0\}$.

# Computing the augmentation polynomial

In practice, to a knot $K$, knot contact homology associates a finite, combinatorially defined collection of polynomials in some variables $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}[\lambda, \mu, U]$:

$$K \rightsquigarrow \{p_1(x_1, \dots, x_n), \dots, p_m(x_1, \dots, x_n)\}.$$

The augmentation variety is the set of $(\lambda, \mu, U)$ for which these polynomials have a common root in $x_1, \dots, x_n$:

$$\begin{aligned} p_1(x_1, \dots, x_n) &= 0 \\ p_2(x_1, \dots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \dots, x_n) &= 0. \end{aligned}$$

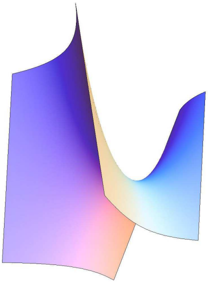
# Augmentation polynomial: unknot

For $K = O$, the unknot: the collection of polynomials in $n = 0$ variables is

$$\{U - \lambda - \mu + \lambda\mu\}.$$

Thus

$$Aug_O(\lambda, \mu, U) = U - \lambda - \mu + \lambda\mu.$$



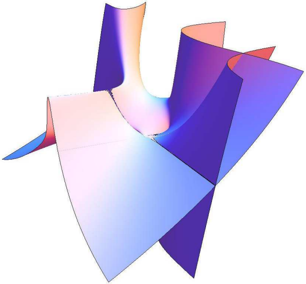
# Augmentation polynomial: trefoil

For $K = T$, the right-handed trefoil: the collection of polynomials in $n = 1$ variable is

$$\{Ux_1^2 - \mu U x_1 + \lambda\mu^3(1-\mu),\ Ux_1^2 + \lambda\mu^2 x_1 + \lambda\mu^2(\mu - U)\}.$$

Then take the resultant of these two polynomials:

$$\mathrm{Aug}_T(\lambda,\mu,U) = (U^3 - \mu U^2) + (-U^3 + \mu U^2 - 2\mu^2 U + 2\mu^2 U^2 + \mu^3 U - \mu^4 U)\lambda + (-\mu^3 + \mu^4)\lambda^2.$$



# Relation to other knot invariants

**Theorem (N. 2005)**

*A specialization of the augmentation polynomial,*

$$\mathrm{Aug}_K(\lambda, \mu, 1),$$

*contains the A-polynomial* $A_K(\lambda, \mu^2)$ *as a factor.*

Here the $A$-polynomial is a knot invariant related to $SL_2\mathbb{C}$-representations of the knot complement and hyperbolic structures.

**Corollary (N. 2005)**

*The augmentation polynomial* $\mathrm{Aug}_K(\lambda, \mu, U)$*, and thus knot contact homology, detects the unknot: if* $\mathrm{Aug}_K = \mathrm{Aug}_O$ *then* $K = O$.

# Relation to other knot invariants, continued

It appears that knot contact homology in general is intimately related with the topology of the knot complement.

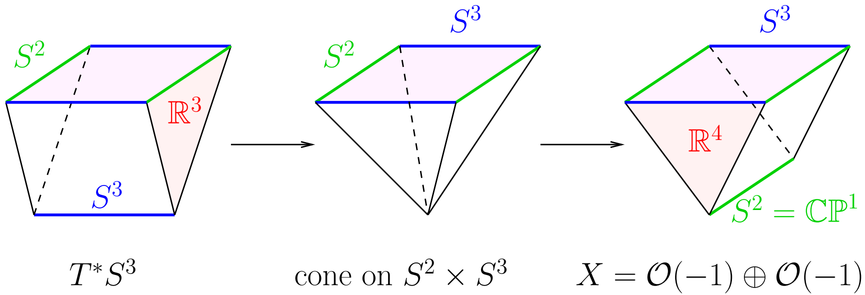
In a different direction, knot contact homology is also related to the HOMFLY-PT polynomial, a two-variable knot polynomial that generalizes the Alexander and Jones polynomials:

**Conjecture**

*The augmentation polynomial encodes a specialization of the HOMFLY-PT polynomial, $P_K(a, 1)$.*

The motivation for this conjecture comes from physics.

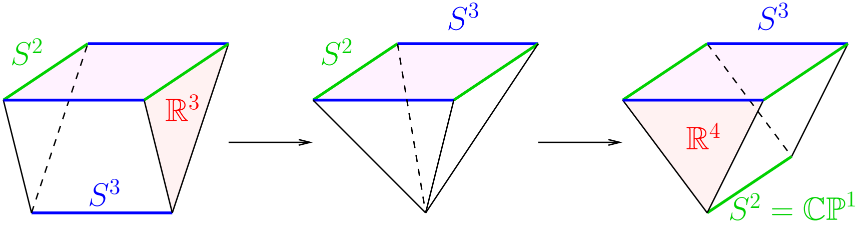
# Conifold transition




Gopakumar–Vafa (1998), building on work of Witten: starting with $T^*S^3$, pass through the “conifold transition” to obtain a 6-manifold $X$, the total space of the rank 2 complex vector bundle

$$\begin{array}{c} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{CP}^1. \end{array}$$

# Conifold transition




$T^*S^3$ cone on $S^2 \times S^3$ $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

## Conjecture (Gopakumar–Vafa)

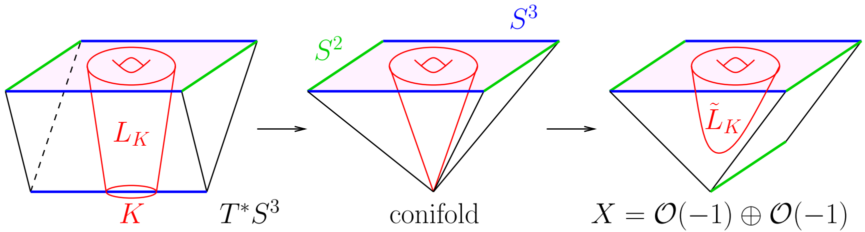
*In the large $N$ limit:*

*$SU(N)$ Chern–Simons theory on $S^3$*

$\updownarrow$

*closed topological string theory on $X$.*

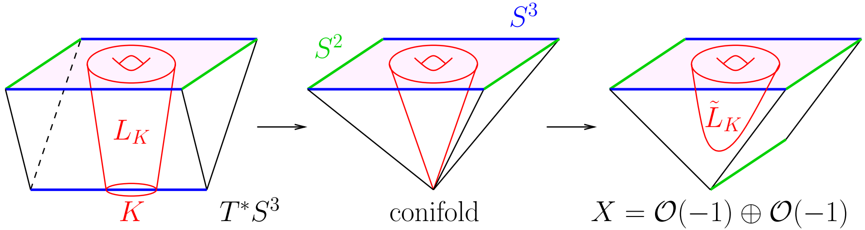
# Conifold transition and $L_K$




Ooguri–Vafa (1999): given a knot $K \subset S^3$, follow the Lagrangian $L_K$ through the conifold transition to obtain a Lagrangian

$$\tilde{L}_K \subset X.$$

# Conifold transition and $L_K$




## Conjecture (Ooguri–Vafa)

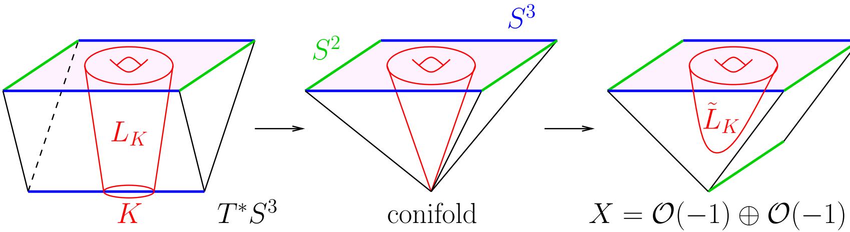
*In the large $N$ limit:*

*$SU(N)$ Chern–Simons theory for $K \subset S^3$*

$$\updownarrow$$

*open topological string theory for $\tilde{L}_K \subset X$.*

# Conifold transition and $L_K$




Checked for unknot, some torus knots.

Slightly more mathematical statement:

Chern–Simons knot invariants for $K \subset S^3$
(e.g. Jones polynomial)

$\updownarrow$

open Gromov–Witten invariants for $\tilde{L}_K \subset X$.

# Mirror manifold

Aganagic–Vafa (2012) propose a "generalized Strominger–Yau–Zaslow conjecture" that uses $\tilde{L}_K \subset X$ to produce a mirror to $X$.

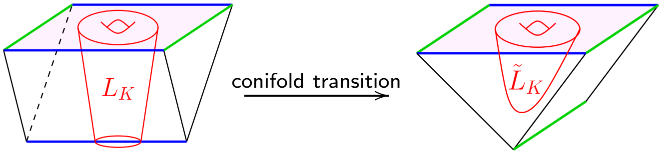
**Conjecture (Aganagic–Vafa)**

*The pair $(X, \tilde{L}_K)$ produces a mirror Calabi–Yau 3-fold to $X$,*

$$X_K = \{(u, v, x, p) \mid uv = \mathbf{A}_K(e^x, e^p, Q)\} \subset \mathbb{C}^4.$$

Here $Q$ is a parameter measuring the complexified Kähler class of $\mathbb{CP}^1$ and $\mathbf{A}_K$ is a three-variable polynomial.

# The mirror and knot invariants




$T^*S^3$ $\longrightarrow$ $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

mirror ↓

$$X_K = \{uv = \mathbf{A}_K(e^x, e^p, Q)\} \subset \mathbb{C}^4$$

HOMFLY-PT polynomial $P_K(a, q)$, colored HOMFLY-PT

Khovanov homology, Khovanov–Rozansky homologies?

The dashed arrows use string-theoretic arguments of Gukov–Schwarz–Vafa (2004) and others.

# Physics and the augmentation polynomial

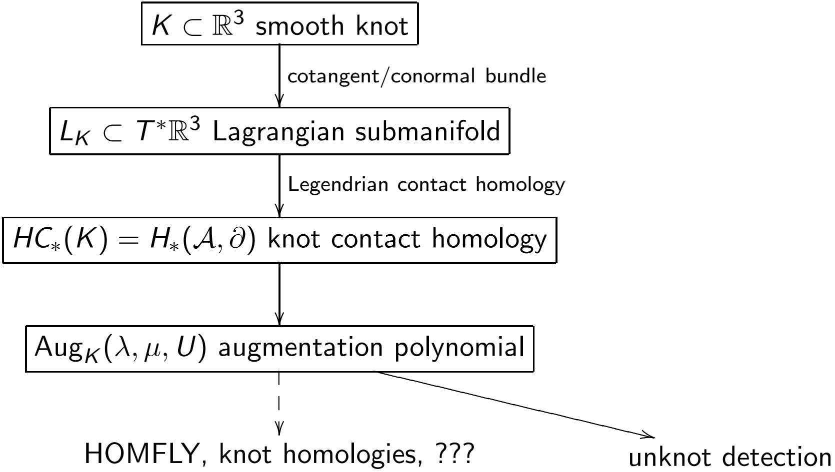
**Conjecture (Aganagic–Ekholm–N.–Vafa 2012)**

*The two polynomials $\mathbf{A}_K$ and $\mathrm{Aug}_K$ are equal for all knots $K$.*

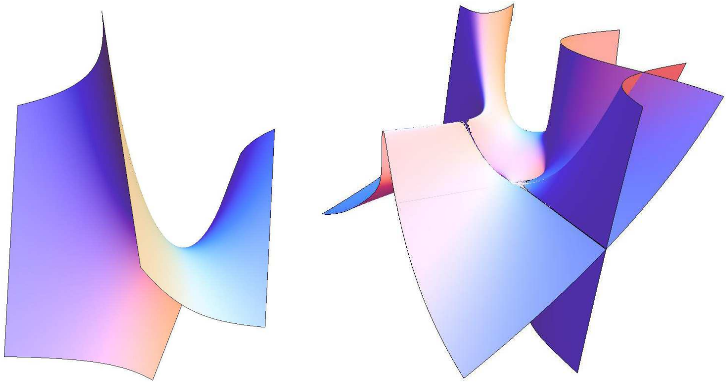
This would imply that the augmentation polynomial $\mathrm{Aug}_K(\lambda, \mu, U)$ is at least as strong as many other known knot invariants.

Currently: a great deal of circumstantial evidence for this conjecture, but no proof.

# Summary of knot invariants



# Thanks!



For further reading:

- T. Perutz, The symplectic topology of cotangent bundles, article in the March 2010 EMS Newsletter
- L. Ng, Conormal bundles, contact homology, and knot invariants, math/0412330
- T. Ekholm and J. Etnyre, Invariants of knots, embeddings and immersions via contact geometry, math/0412517
- L. Ng, A topological introduction to knot contact homology, forthcoming
- Another forthcoming survey paper?