Math 262 Homework 1 (revised)—due September 25

Fall 2008

1. Prove directly (without using the proof of the Poincaré Lemma) that $H^*(\mathbb{R}^2) = \mathbb{R}$ if $* = 0$, 0 if $* \neq 0$; and that $H^*_c(\mathbb{R}^2) = \mathbb{R}$ if $* = 2$, 0 if $* \neq 2$.
2. Bott & Tu, Exercise 1.7, p. 19.
3. Let ω be the 1-form $(x\,dy - y\,dx)/(x^2+y^2)$ in $\Omega^1(\mathbb{R}^2 \setminus \{0\})$.
   (a) Check that $d\omega = 0$.
   (b) Prove that $[\omega] \neq 0$ as an element of $H^1(\mathbb{R}^2 \setminus \{0\})$. (You may want to consider the unit circle.)
4. Write $\mathbb{CP}^1$, as usual, as the set of points $(z_0 : z_1)$ with $z_0, z_1 \in \mathbb{C}$ not both 0, modulo the equivalence relation $(z_0 : z_1) \sim (\lambda z_0 : \lambda z_1)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. An atlas for $\mathbb{CP}^1$ is given by $\mathbb{CP}^1 = U_0 \cup U_1$ with $U_0 = \{(z_0 : z_1) | z_0 \neq 0\}$ and $U_1 = \{(z_0 : z_1) | z_1 \neq 0\}$. Both U_0 and U_1 are homeomorphic to $\mathbb{C} = \mathbb{R}^2$ via the maps $(z_0 : z_1) \mapsto z_1/z_0$ (for U_0) and $(z_0 : z_1) \mapsto z_0/z_1$ (for U_1).
   (a) If we identify $U_0 \cong \mathbb{C}$ with $\mathbb{R}^2$ via the map $z = x + iy \mapsto (x, y)$, then we can define a 2-form ω_{U_0} on U_0 by
   $$\omega_{U_0} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}.$$
   Show that ω_{U_0} can be extended to a smooth 2-form $\omega \in \Omega^2(\mathbb{CP}^1)$. This is (up to a constant multiple) the *standard Kähler* (or *symplectic*) *form* on $\mathbb{CP}^1$, and generates $H^2(\mathbb{CP}^1) \cong \mathbb{R}$.
   (b) Let a be any fixed complex number. The map $(z_0, z_1) \mapsto (z_0 + \overline{a} z_1, z_1 - a z_0)$ induces a smooth map $f_a : \mathbb{CP}^1 \to \mathbb{CP}^1$ sending $(z_0 : z_1)$ to $(z_0 + \overline{a} z_1 : z_1 - a z_0)$. (On U_0, this is the Möbius transformation $z \mapsto \frac{z-a}{1+\overline{a}z}$.) Prove that $f_a^*(\omega) = \omega$.
   (In the language of symplectic geometry, f_a is a *symplectomorphism* of $(\mathbb{CP}^1, \omega)$; in the language of Riemannian geometry, f_a is an *area-preserving map* for the metric corresponding to ω.)
5. Let $M = \mathbb{R}^n \setminus \{0\}$, $N = S^{n-1} = \{\|\mathbf{x}\| = 1\} \subset M$ for some $n \geq 1$. Show that the map $r : M \to N$ defined by $r(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ is a deformation retraction.

Not to be handed in, but important nevertheless:

- Understand (or derive on your own!) the proof of Proposition 4.6, pp. 38–39.