Math 262 Homework 2—due Thursday October 16

Fall 2008

1. If M is a smooth n-dimensional manifold, define the Euler characteristic of M by

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{R}} H^k(M).$$

   (a) Suppose that $M = U \cup V$ where U, V are open in M. Find a formula for $\chi(M)$ in terms of $\chi(U)$, $\chi(V)$, and $\chi(U \cap V)$.

   (b) Use (a) to calculate $\chi(\Sigma_g)$ where Σ_g is the closed genus g oriented surface. (Of course you know the answer in advance.)

2. Bott & Tu: exercises I.4.3 and I.4.3.1, pp. 36–37. For I.4.3.1(a), it may help to recall that the volume of the unit ball in $\mathbb{R}^n$ is $\pi^{n/2}/\Gamma(\frac{n}{2}+1)$, where Γ is the standard gamma function.

3. (a) Use Mayer–Vietoris to calculate $H^*(\mathbb{CP}^2)$ as a graded vector space.

   (b) Then use Poincaré duality to prove the isomorphism of graded rings

$$H^*(\mathbb{CP}^2) \cong \mathbb{R}[x]/(x^3),$$

   where x is a generator of $H^2(\mathbb{CP}^2)$.

4. (a) Calculate the degree of the following maps from S^n to S^n, where S^n is viewed as the unit sphere in $\mathbb{R}^{n+1}$: the reflection map

$$(x_1, x_2, x_3, \ldots, x_{n+1}) \mapsto (-x_1, x_2, x_3, \ldots, x_{n+1});$$

   the antipodal map

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, -x_2, \ldots, -x_{n+1}).$$

   (b) Let M be any connected closed oriented smooth n-manifold. Prove that there is a degree 1 map $M \to S^n$. (In the context of this class, one ought to prove that there is a smooth such map. To save time, however, I'll be satisfied with a continuous map, and you may then assume that the map you construct is smooth and proceed with the problem.)

   (c) Prove that the converse to (b) does not hold: in fact, prove that, for any $n \geq 2$, there is a connected closed oriented smooth n-manifold M for which all smooth maps $S^n \to M$ have degree 0.

5. Bott & Tu: exercise I.5.12, p. 50.