Math 262 Homework 4—due Tuesday December 2

Fall 2008

1. (a) Let E be a line bundle over a manifold M, that is, a vector bundle with fiber $\mathbb{R}$. Say that E is *trivial* if it is isomorphic to the trivial line bundle $M \times \mathbb{R}$ over M. Prove that E is trivial if and only if there is a section $s : M \to E$ that is nowhere zero (i.e., for all $x \in M$, $s(x) \neq 0$ in $\mathbb{R} \cong E_x$).

   (b) Let E be the "Möbius line bundle" over S^1 discussed in class: if we cover S^1 by two open sets U_0, U_1 as usual, then E is the vector bundle over S^1 with fiber $\mathbb{R}$ for which the transition function g_{01} is 1 on one component of $U_0 \cap U_1$, -1 on the other. Prove that E is not isomorphic (as a vector bundle) to the trivial line bundle over S^1.

2. (This is a fact that we've used several times in class.) Let $(K^{*,*}, \delta, d)$ be a double complex; as usual, this produces a singly graded complex (K^*, D) with $D = \delta + (-1)^i d$ on $K^{i,j}$. Define the mirror complex $(\overline{K}^{*,*}, \overline{\delta}, \overline{d})$ by $\overline{K}^{i,j} = K^{j,i}$ for all i, j and $\overline{\delta}(x) = d(x)$ and $\overline{d}(x) = \delta(x)$ for all $x \in \overline{K}^{*,*} = K^{*,*}$. This also produces a singly graded complex $(\overline{K}^*, \overline{D})$ with $\overline{D} = \overline{\delta} + (-1)^i \, \overline{d}$ on $\overline{K}^{i,j}$. Prove that $H^*_D(K) \cong H^*_{\overline{D}}(\overline{K})$.

3. Bott & Tu Exercise III.14.11, page 163. (To clarify, the exercise ends before the paragraph "We say that...".)

4. (Hands-on spectral sequences.) Let $(K^{*,*}, \delta, d)$ be the double complex of $\mathbb{Z}/2$-vector spaces defined as follows. A basis for $K^{*,*}$ is given by 12 generators $v^{0,1}$, $v^{0,2}$, $v_1^{1,0}$, $v_2^{1,0}$, $v_1^{1,1}$, $v_2^{1,1}$, $v_3^{1,1}$, $v^{1,2}$, $v^{2,0}$, $v_1^{2,1}$, $v_2^{2,1}$, $v^{2,2}$, where the superscript denotes the bigrading; e.g., $K^{2,1}$ is the 2-dimensional vector space over $\mathbb{Z}/2$ generated by $v_1^{2,1}$ and $v_2^{2,1}$. The differentials are given by

$$\delta(v^{0,1}) = v_3^{1,1}, \quad \delta(v_2^{1,0}) = v^{2,0}, \quad \delta(v_1^{1,1}) = \delta(v_2^{1,1}) = v_1^{2,1}, \quad \delta(v^{1,2}) = v^{2,2}$$

   and $\delta = 0$ on all other generators;

$$d(v^{0,1}) = v^{0,2}, \quad d(v_1^{1,0}) = v_3^{1,1}, \quad d(v_2^{1,0}) = v_1^{1,1}, \quad d(v^{2,0}) = v_1^{2,1}, \quad d(v_2^{2,1}) = v^{2,2}$$

   and $d = 0$ on all other generators. (You can check that $d\delta = \delta d$ if you like.)

   (a) Calculate $H^*_D(K)$ directly from the definition $D = \delta + (-1)^i d$.

   (b) Calculate the spectral sequence for K; that is, if the spectral sequence degenerates at the E_r term, then calculate $(E_1^{*,*}, d_1), (E_2^{*,*}, d_2), \ldots, (E_r^{*,*} = E_\infty^{*,*}, d_r = 0)$. Use this to recalculate $H^*_D(K)$.

   (c) Calculate the spectral sequence $\{(E'_r, d'_r)\}$ for K. (Recall that this is the "alternate" spectral sequence for K, obtained by using the spectral sequence for $\overline{K}$.) Hint: to compute d_2, use your previous calculation of $H^*_D(K)$.

5. Bott & Tu exercise III.14.22.1, page 173. (Of course, use spectral sequences here, not other techniques.)