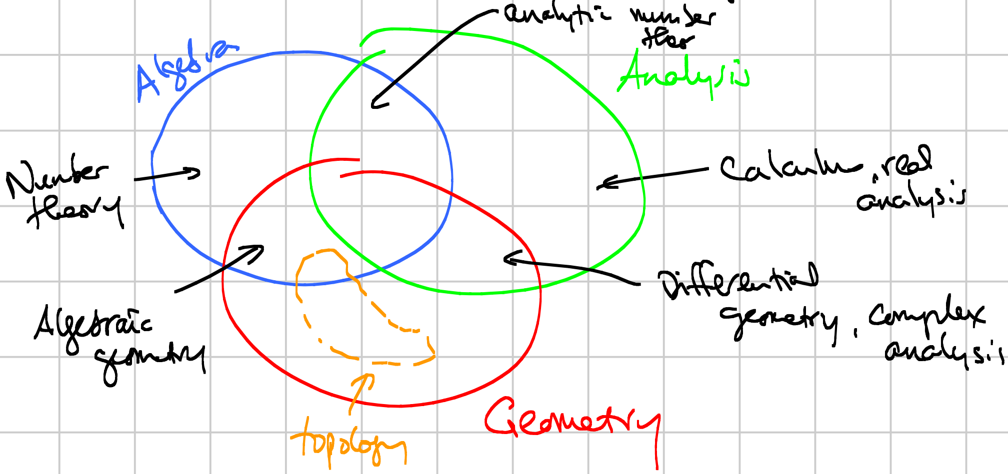
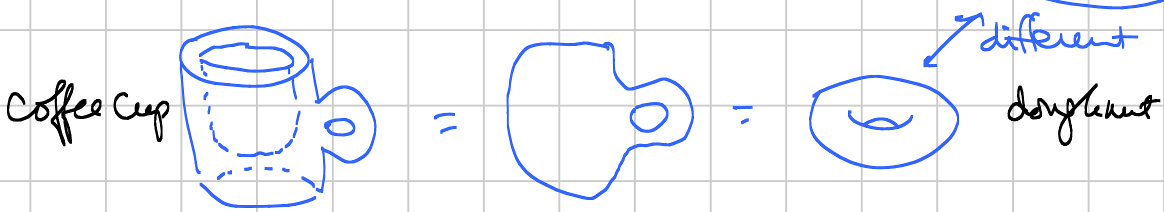
MATHEMATICS 411: TOPOLOGY

Note Title 8/24/2017

Topology — study of shapes and spaces and what remains if you allow arbitrary "continuous" deformation.



What does "continuous" mean? Deforming without breaking.



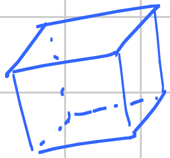
Knots:

One of the first results in topology: <u>Euler's Theorem/Formula.</u>

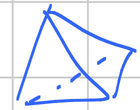
<u>Polyhedron</u> = object with <u>faces</u>, <u>edges</u>, <u>vertices</u>.

→ every edge is a side of 2 faces

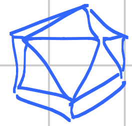
→ every edge has 2 vertices as endpoints.



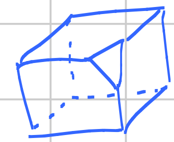
$(F, E, V) = (6, 12, 8)$



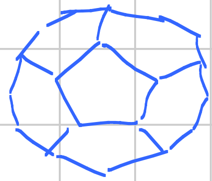
$(4, 6, 4)$



$(20, 30, 12)$



$(7, 15, 10)$



truncated icosahedron: 12 pentagons, 20 hexagons

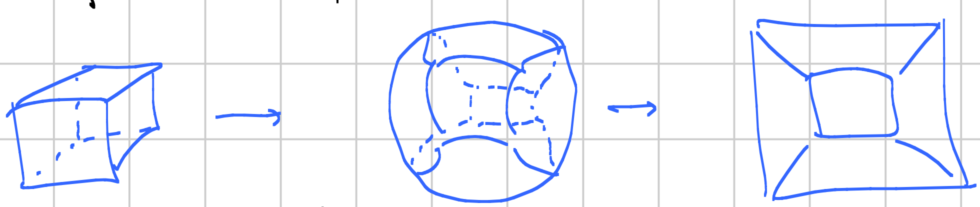
$(32, 60, 90)$

Euler's Thm.

$$F - E + V = 2$$

(# 2-dim) (# 1-dim) (# 0-dim)

Proof 1. Imagine inflating into a round ball; then puncture at a point and lay flat.



The result is a "graph" (fence) and we can count F, E, V as before (make sure to count outside face).

Now imagine flooding in such a way that the remaining fence never becomes disconnected.

$(6, 12, 8)$ $(5, 11, 8)$ $(4, 10, 8)$ $(4, 9, 7)$

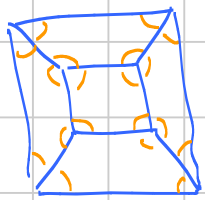
Until we get to $(1, 1, 2)$

Each time: $\Delta E = -1$ and either $\left\{ \begin{matrix} \Delta F = -1 \\ \Delta V = -1 \end{matrix} \right\}$

$\Rightarrow \Delta(F - E + V) = 0$ always. $\Rightarrow F - E + V = 1 - 1 + 2 = 2$.

2. (Only really works for convex polyhedra)

Do as in #1, and then straighten out edges. Say outside is m-gon.



Now add up all angles.

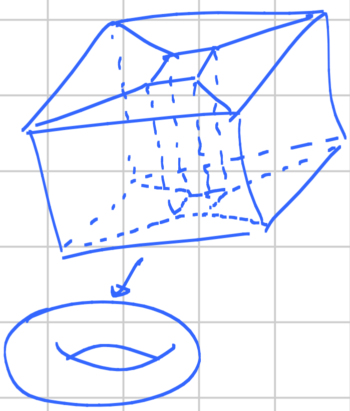
n-gon $\Rightarrow$ $(n-2)\pi$

So summing over all faces gives $\pi\Big[\underbrace{(\text{total \# of edges})}_{2E-m}-2(F-1)\Big]$

Summing over vertices? $(m-2)\pi + 2\pi(V-m)$

$\Rightarrow 2E-m-2F+2 = m-2+2V-2m \Rightarrow F-E+V=2.$ $\square$

What about this "polyhedron"?

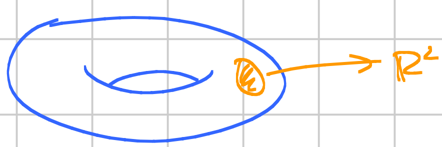


(16, 32, 16)

$F-E+V=0$

The problem? This can't be inflated into a ball without tearing.

Sphere, torus are both examples of surfaces: objects that locally look like $\mathbb{R}^2$.



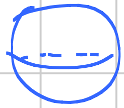
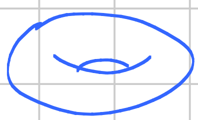
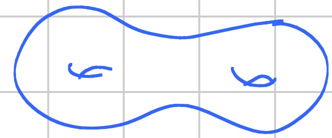
bijective, continuous map

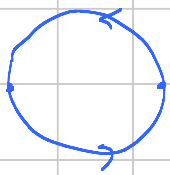
Break surface into (polygonal) faces, edges, vertices by cell decomposition.

Let F, E, V = # of faces, etc.

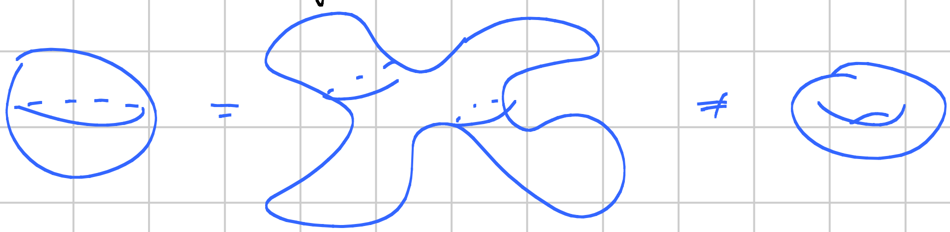
Thm Given a fixed surface, $F-E+V$ is the same for any cell decomposition.

This number is the Euler characteristic of the surface Σ, $\chi(\Sigma)$.

 $\chi = 2$  $\chi = 0$  $\chi = -2$

 $\mathbb{RP}^2$ $\chi = 1$ Klein bottle  $\chi = 0$

There's a notion of equivalence between surface: <u>homeomorphism</u>.
(essentially, can deform one into the other without tearing).



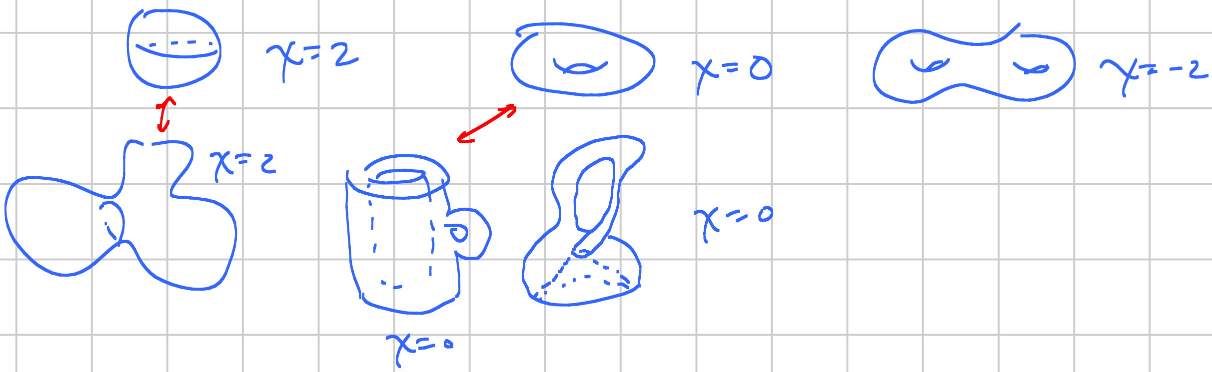
<u>Thm</u> If Σ_1, Σ_2 are homeomorphic, then $\chi(\Sigma_1) = \chi(\Sigma_2)$.
Converse isn't quite true (e.g. torus $\neq$ Klein bottle) but almost:
<u>classification of surfaces</u> (later).

χ is related to vector fields: ex: <u>hairy ball theorem</u>.
if Σ = surface, then one can comb a hairy Σ flat (without a cowlick)
if and only if $\chi(\Sigma) = 0$.

ϕ In another direction: only one surface, the sphere, has this property:
every closed loop can be deformed to a point. ("simply connected")
<u>Poincaré conjecture</u> (1904, proven by Perelman in ≈2003)
The 3-dimensional sphere $\{w^2 + x^2 + y^2 + z^2 = 1\}$ is the only 3-dim
object ("3-manifold") that is simply connected.

Last time: Surface $\Sigma \mapsto$ Euler characteristic $\chi(\Sigma)$.

Given a cell decomposition of Σ into faces, edges, vertices, $F-E+V$ is indep of cell decomp $=: \chi(\Sigma)$.



Say two surfaces are homeomorphic if we can deform one into the other continuously (without tearing).

Thm If Σ_1, Σ_2 are homeomorphic, then $\chi(\Sigma_1) = \chi(\Sigma_2)$.

Converse isn't quite true (eg. torus $\neq$ Klein bottle) but almost: classification of surfaces (later).

Starting point: notion of continuity: "maps nearby points to nearby pts".

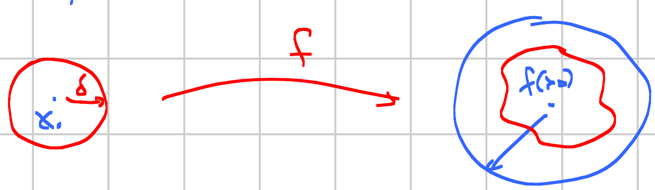
From calculus: $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous[1] at $\vec{x}_0 \in X$ if

$\forall \epsilon > 0 \; \exists \delta > 0$ st. $\|\vec{x} - \vec{x}_0\| < \delta \Rightarrow \|f(\vec{x}) - f(\vec{x}_0)\| < \epsilon$.

and f is continuous if it's continuous at all pts in X.

$x \in \mathbb{R}^n$, $\epsilon > 0$, $\vec{B}(x,\epsilon) = \{y \mid \|y-x\| < \epsilon\}$.

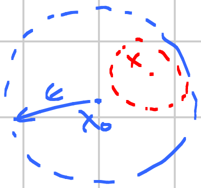
i.e.: for any small ball $B(f(\vec{x}_0),\epsilon)$, f maps some small ball $B(\vec{x}_0,\delta)$ into $B(f(\vec{x}_0),\epsilon)$.



$B(f(\vec{x}_0),\epsilon) = \{y \mid \|y - f(\vec{x}_0)\| < \epsilon\}$

More generally: <u>Def</u> An <u>open set</u> U in $\mathbb{R}^n$ is any subset in $\mathbb{R}^n$ such that if $\vec{x} \in U$ then there is ϵ with $B(\vec{x},\epsilon) \subset U$.

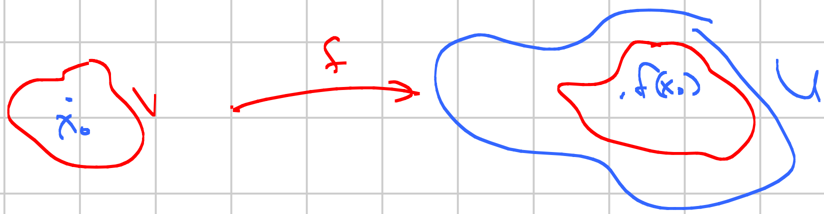
<u>Prop</u> $B(\vec{x}_0,\epsilon)$ is open.



Pf.: If $\vec{x} \in B(\vec{x}_0,\epsilon)$ then define $\delta = \epsilon - \|\vec{x}-\vec{x}_0\| > 0$,
then $B(\vec{x},\delta) \subset B(\vec{x}_0,\epsilon)$ since if
$y \in B(\vec{x},\delta) \Rightarrow \|y-x\| < \delta \Rightarrow \|y-x_0\| \leq \|y-x\| + \|x-x_0\| < \epsilon$.
$\square$

Def A <u>neighborhood</u> of $\vec{x}$ is an open set containing $\vec{x}$.

Def f is <u>continuous</u>(2) at $\vec{x}_0$ if for any neighborhood U of $f(\vec{x}_0)$, there is a nbd V of $\vec{x}_0$ such that $f(V) \subset U$.



$(1) \Rightarrow (2)$: given U, there is some $B(f(x_0),\epsilon) \subset U$
$\Rightarrow$ there is some δ with $\underbrace{f(B(x_0,\delta))}_{V} \subset B(f(x_0),\epsilon) \subset U$.

$(2) \Rightarrow (1)$: Choose $U = B(f(x_0),\epsilon) \Rightarrow \exists\ V$ nbd of x_0 with $f(V) \subset B(f(x_0),\epsilon)$.
Choose δ with $B(x_0,\delta) \subset V \Rightarrow f(B(x_0,\delta)) \subset$

More useful to generalize further.

Def X = set. A *topology* $\mathcal{T}$ on X is a (nonempty) collection of subsets of X ("open sets") satisfying:

1. $\phi \in \mathcal{T}$, $X \in \mathcal{T}$
2. any union of elements of $\mathcal{T}$ is also in $\mathcal{T}$
3. any *finite* intersection of elements of $\mathcal{T}$ is also in $\mathcal{T}$.

A set X with a specified topology $\mathcal{T}$ is called a *topological space*.

Ex. $X = \mathbb{R}^n$. $\mathcal{T} = \{\text{open sets in } \mathbb{R}^n\}$.

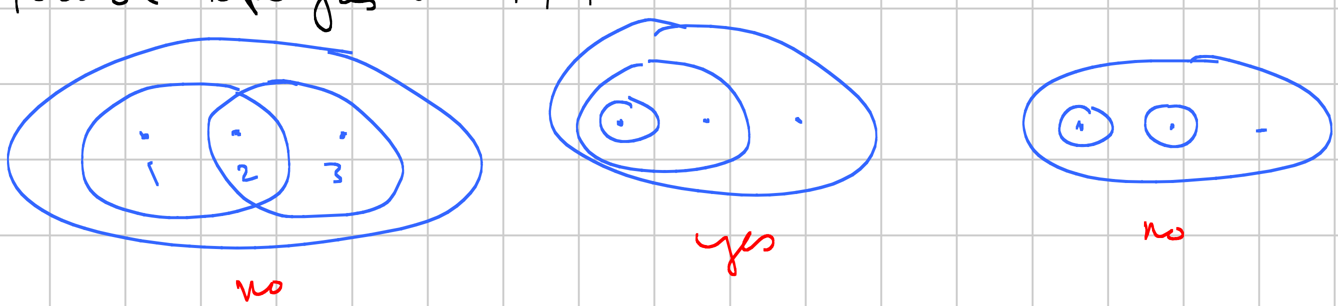
Check: ① ✓

② if $U_i \subset \mathbb{R}^n$ are open then $\bigcup_i U_i = U$ is open:
if $x \in U$ then $x \in U_i$ for some $i \Rightarrow$ for some $\epsilon > 0$, $B(x,\epsilon) \subset U_i \subset U$.

③ if $U_1, \ldots, U_m \subset \mathbb{R}^n$ are open then $\bigcap_{i=1}^m U_i = U$ is open:
if $x \in U$ then $x \in U_i$ for every $i \Rightarrow$ for some $\epsilon_i > 0$, $B(x,\epsilon_i) \subset U_i$.
Define $\epsilon = \min(\epsilon_1, \ldots, \epsilon_m) > 0$. Then $B(x,\epsilon) \subset B(x,\epsilon_i) \subset U_i \ \forall i$
$\Rightarrow B(x,\epsilon) \subset U$.

Rmk: why only finitely many $\cap$?
$X = \mathbb{R}$, $U_n = (-\frac{1}{n}, \frac{1}{n})$ for $n \geq 1$. $\bigcap_n U_n = \{0\}$ not open!

Possible topologies on $\{1,2,3\}$?



Examples of topologies

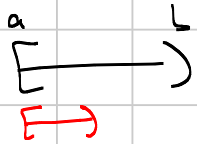
1. $X=\mathbb{R}^n$, $\mathcal{T}=\{\text{open sets}\}$. standard topology
2. $X=$ any set, $\mathcal{T}=\{\text{all subsets of } X\}$.
   This is called the discrete topology on X.
3. $X=$ any set, $\mathcal{T}=\{\emptyset, X\}$.
   This is the indiscrete topology on X.
4. Say X has a topology $\mathcal{T}$ and $Y\subset X$ is any subset.
   Then Y has a topology $\mathcal{T}'=\{Y\cap U \mid U\in\mathcal{T}\}$: the subspace topology for Y.

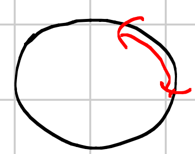
check: ① ✓ ②: $U_i\in\mathcal{T} \Rightarrow U_i' = Y\cap U_i \in \mathcal{T}'$

$$\Rightarrow \cup U_i' = \bigcup_i (Y\cap U_i) = Y\cap\Big(\overbrace{\bigcup_i U_i}^{\in\mathcal{T}}\Big) \in \mathcal{T}'$$

③: $$\cap U_i' = \bigcap_i (Y\cap U_i) = Y\cap\Big(\underbrace{\bigcap_i U_i}_{\in\mathcal{T}}\Big) \in \mathcal{T}'.$$

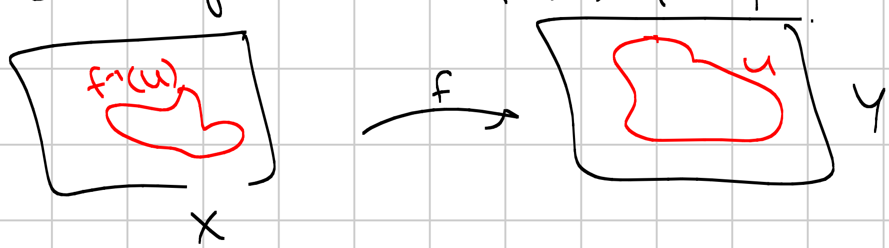
ex of open sets in subspace topology:

$X=[a,b)$ 

$X=S^1$
$=\{x^2+y^2=1\}$ 

Def X, Y topological spaces. A map $f: X\to Y$ is continuous if the inverse image of any open set in Y is open in X:

$$U\subset Y \text{ open} \Rightarrow f^{-1}(U)=\{x\in X \mid f(x)\in U\} \text{ open in } X.$$



<u>Prop</u> $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, both with standard topology.
This def agrees with the previous def of continuous.

<u>Pf</u> <u>Continuous ⇒ Continuous (2)</u>: given $x_0 \in \mathbb{R}^n$, let U be any nbd of $f(\vec{x}_0)$. Then $f^{-1}(U)$ is open in $\mathbb{R}^n$ and contains x_0
⇒ there is some ϵ with $B(x_0, \epsilon) \subset f^{-1}(U) \Rightarrow f^{-1}(U) =$ nbd of x_0.

<u>Continuous (2) ⇒ Continuous</u>: Say $U \subset \mathbb{R}^m$ is open; want $f^{-1}(U) \subset \mathbb{R}^n$ open.
Choose $x \in f^{-1}(U)$. Then U contains $f(x)$ and is open ⇒
$\exists\, \epsilon > 0$ with $B(f(x), \epsilon) \subset U \Rightarrow \exists\, \delta > 0$ with $f(B(x,\delta)) \subset B(f(x), \epsilon) \subset U$
$\Rightarrow \quad B(x, \delta) \subset f^{-1}(U)$. □

8/31 ②

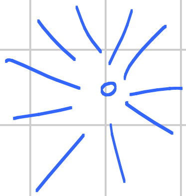
<u>Def</u> A map $f: X \to Y$ is a <u>homeomorphism</u> if
- f is bijective → can define $f^{-1}: Y \to X$
- f is continuous
- f^{-1} is continuous.

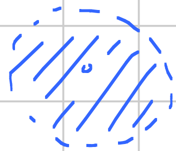
<u>Note</u>: f continuous ⇒ if $V \subset Y$ open then $f^{-1}(V) \subset X$ open
$g = f^{-1}$ continuous ⇒ if $U \subset X$ open then $g^{-1}(U) = f(U) \subset Y$ open.

So these combined imply
$$(U \subset X \text{ is open}) \iff f(U) \subset Y \text{ is open}$$
and f gives a 1-1 correspondence between open sets in X and open sets in Y.
T_X = topology on X, T_Y = topology on Y: $f: X \to Y$ induces bijection $f: T_X \to T_Y$.

<u>Ex</u> T, T' two topologies on Y. Then $id: (X, T) \to (X, T')$ is a homeo
$\iff T = T'$.

Ex $X = \mathbb{R}^2 \setminus \{0\}$, $\quad Y = \{(x,y) \mid 0 < x^2+y^2 < 1\}$, standard topology

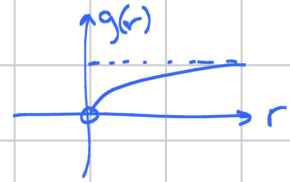




$f: X \to Y$ defined by $f(x,y) = \frac{h(r)}{r}(x,y)$

$r = \sqrt{x^2+y^2}$, $\quad h(r) = \frac{2}{\pi}\tan^{-1}(r)$

or $\frac{r}{r+1}$



inverse is $g(x,y) = \frac{j(r)}{r}(x,y)$ $\qquad j = h^{-1}$.

Basis

Often difficult to specify all open sets; easier to give a few that specify the topology.

Def $\mathcal{T}$ = topology on X. A basis for the topology $\mathcal{T}$ is a collection $\mathcal{B} \subset \mathcal{T}$ of open sets such that every open set is a (possibly empty) union of elts of $\mathcal{B}$. Elements of $\mathcal{B}$ are basic open sets.

Ex $\mathcal{T}$ = standard topology on $\mathbb{R}^n$

$\mathcal{B} = \{\text{open balls}\} = \{B(x_0,r) \mid x_0 \in \mathbb{R}^n, r > 0\}$.

Note: $U \subset \mathbb{R}^n$ open $\Rightarrow$ for each $x \in U$, $\exists\, r(x) \subset U$ s.t. $B(x, r(x)) \subset U$

$\Rightarrow U = \bigcup_{x \in U} B(x, r(x))$.

→ In particular, for $\mathbb{R}$, a basis is given by {open intervals}.

→ Could also use (for any $\epsilon > 0$) $\mathcal{B}_\epsilon = \{\text{open balls with radius} < \epsilon\}$.

Useful criterion for being a basis.

Thm $\mathcal{T}$ = topology on X, $\mathcal{B}$ = collection of subsets. TFAE:

1. $\mathcal{B}$ is a basis for $\mathcal{T}$
2. $\mathcal{B} \subset \mathcal{T}$ and for any open set $U \in \mathcal{T}$ and $x \in U$, $\exists$ element $B \in \mathcal{B}$ with $x \in B \subset U$.

Pf $1 \Rightarrow 2$: $U \in \mathcal{T} \implies U = \cup B_i,\ B_i \in \mathcal{B}$

$x \in U \implies \exists i$ with $x \in B_i,\ B_i \subset U$.

$2 \Rightarrow 1$: $U \in \mathcal{T}$. For $x \in U$, say $x \in B_x \subset U$, $B_x \in \mathcal{B}$. Then $U = \bigcup_x B_x$. $\square$

In particular, immediate that $\mathcal{B} = \{\text{open balls}\}$ is a basis for std topology on $\mathbb{R}^n$

or $\mathcal{B}_\epsilon = \{\text{open balls, radius} < \epsilon\}$

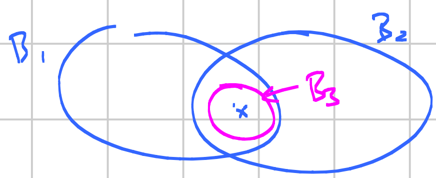
Another basis for $\mathbb{R}$: {open intervals with rational endpts}.

Another ex: $\mathcal{T}$ = discrete topology on X, $\mathcal{B} = \{\text{points}\}$.

What if we don't start with a topology?

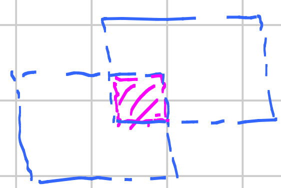
Def A basis (for some topology on X) is a collection $\mathcal{B}$ of subsets of X such that:

1. $\bigcup_{U \in \mathcal{B}} U = X$ i.e. $\forall x \in X\ \exists\ U \in \mathcal{B}$ with $x \in U$.
2. if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then $\exists\ B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.



Note: if $\mathcal{T}$ = topology and $\mathcal{B}$ = basis for $\mathcal{T}$, then $\mathcal{B}$ satisfies these. (check!)

Ex of a basis: $X = \mathbb{R}^2$, $\mathcal{B} = \{\text{open rectangles}\}$.



$\boxed{\text{Topology } \mathcal{T}} \rightleftarrows \boxed{\text{basis } \mathcal{B}}$

Def Given a basis $\mathcal{B}$, the topology determined by $\mathcal{B}$ is

$\mathcal{T} = \{(\text{possibly empty}) \text{ unions of basic open sets}\}$. Note $\mathcal{B} \subset \mathcal{T}$.

Prop $\mathcal{B}$ = basis, $\mathcal{T}$ = topology determined by $\mathcal{B}$, $U \subset X$. TFAE:

1. $U \in \mathcal{T}$
2. $\forall x \in U$, $\exists$ $B \in \mathcal{B}$ with $x \in B \subset U$.

Pf $1 \Rightarrow 2$: $U = \cup B_i \Rightarrow$ done.

$2 \Rightarrow 1$: $x \in B_x \subset U \Rightarrow U = \bigcup_{x \in U} B_x$. $\square$

Two properties.

1. $\mathcal{B}$ = basis, $\mathcal{T}$ = topology determined by $\mathcal{B}$. Then $\mathcal{T}$ is a topology.

$$\left.\begin{array}{l} \cdot\ \phi, X \in \mathcal{T} \\ \cdot\ U_i \in \mathcal{T} \Rightarrow \bigcup_i U_i \in \mathcal{T}. \end{array}\right\} \text{clear}$$

· $U_1, \ldots, U_n \in \mathcal{T} \Rightarrow U_1 \cap \cdots \cap U_n \in \mathcal{T}$. Assume $n=2$ (then use induction).

$U_1, U_2 \in \mathcal{T} \Rightarrow \forall x \in U_1 \cap U_2$ $\exists$ $B_1, B_2 \in \mathcal{B}$, $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$ by prop.

9/5 $\Rightarrow$ $\exists$ $B_3 \in \mathcal{B}$, $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$.

2. (a) $\mathcal{B}$ = basis, $\mathcal{T}$ = topology determined by $\mathcal{B} \Rightarrow \mathcal{B}$ = basis for $\mathcal{T}$

(b) $\mathcal{T}$ = topology, $\mathcal{B}$ = basis for $\mathcal{T} \Rightarrow \mathcal{T}$ = topology determined by $\mathcal{B}$.

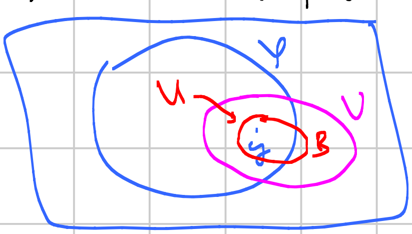
Subspace topology relation:

Prop $Y \subset X$, $\mathcal{T}$ topology on X, $\mathcal{B}$ basis. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology $\mathcal{T}'$ on Y.

Pf $\mathcal{B}' \subset \mathcal{T}'$ ✓

For any $U \in \mathcal{T}'$, $y \in U$, need $B' \in \mathcal{B}'$ with $y \in B' \subset U$.

$U \in \mathcal{T}' \Rightarrow U = V \cap Y$, $V \in \mathcal{T} \Rightarrow \exists$ $B \in \mathcal{B}$ with $y \in B \subset V$. Then $B' = B \cap Y$ works.



Different topologies can sometimes be compared.

<u>Def</u> $\mathcal{T}, \mathcal{T}'$ topologies on X. If $\mathcal{T} \subseteq \mathcal{T}'$ then say $\mathcal{T}'$ is <u>finer</u> than $\mathcal{T}$, $\mathcal{T}$ is <u>coarser</u> than $\mathcal{T}'$. (if $\mathcal{T} \neq \mathcal{T}'$, <u>strictly</u> finer/coarser).

(analogy to pebbles)

<u>Ex</u> on $\mathbb{R}^n$: indiscrete $\subset$ standard $\subset$ discrete.

note: if $\mathcal{T}'$ finer than $\mathcal{T}$ and $\mathcal{T}$ finer than $\mathcal{T}'$ then $\mathcal{T} = \mathcal{T}'$.

<u>Prop</u> $\mathcal{B}$ basis for $\mathcal{T}$, $\mathcal{B}'$ basis for $\mathcal{T}'$. TFAE:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$
2. for all $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, $\exists\, B' \in \mathcal{B}'$ with $x \in B' \subset B$.

<u>Pf</u> $1 \Rightarrow 2$: $B \in \mathcal{B} \Rightarrow B \in \mathcal{B} \subset \mathcal{T} \subset \mathcal{T}'$ so $B = \bigcup_i B_i'$, $B_i' \in \mathcal{B}'$.

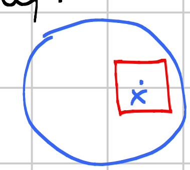
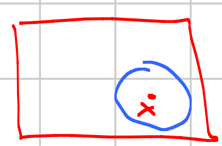
$x \in B \Rightarrow x \in B_i' \subset B$ for some $B_i' \in \mathcal{B}'$.

$2 \Rightarrow 1$: if $U \in \mathcal{T}$, need $U \in \mathcal{T}'$. By previous prop, suffices to show that if $x \in U$ then $\exists\, B' \in \mathcal{B}'$ with $x \in B' \subset U$; but $\exists\, B \in \mathcal{B}$ with $x \in B \subset U \Rightarrow \exists\, B'$ with $x \in B' \subset B \subset U$. $\square$

<u>Ex</u> $\mathcal{T}$ = standard topology on $\mathbb{R}^2$, $\mathcal{B}$ = {open balls}

$\mathcal{T}'$ = "rectangular topology" generated by $\mathcal{B}'$ = {open rectangles}
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " $(a,b) \times (c,d)$

Then $\mathcal{T} = \mathcal{T}'$. Why?

$\mathcal{T}'$ finer than $\mathcal{T}$:  $\mathcal{T}$ finer than $\mathcal{T}'$: 

More topologies

- Product topology

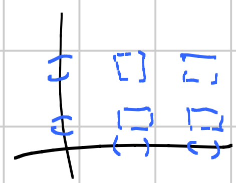
  X, Y topological spaces. Define the collection of subsets
  $\mathcal{B} = \{U \times V \mid U \subset X \text{ open and } V \subset Y \text{ open}\}$.
  Then $\mathcal{B}$ is a basis:
  - $\cup (U \times V) = X \times Y$
  - $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$.

  The topology determined by $\mathcal{B}$ is the <u>product topology</u> on $X \times Y$.

  <u>Prop</u> If $\mathcal{B}_X$ = basis for X and $\mathcal{B}_Y$ = basis for Y then
  $\{B \times C \mid B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$ is a basis for the product topology.

  <u>Pf</u> Let Ω = open set in product topology, $(x,y) \in U$. Then
  $\exists\, U, V$ open such that $(x,y) \in U \times V \subset \Omega$
  $\Rightarrow \exists\, B, C$ such that $x \in B \subset U$, $y \in C \subset V \Rightarrow (x,y) \in B \times C \subset U \times V$. $\square$

  <u>Ex</u> $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. original basis $U \times V$

  new basis: $\{\text{open intervals}\}$ = basis for $\mathbb{R}$
  $\Rightarrow$ $\{\text{open rectangles}\}$ = basis for $\mathbb{R}^2$.

  So product topology = standard topology.

- <u>Lower limit topology</u> $\mathcal{T}'$ on $\mathbb{R}$ is the topology generated by
  $\mathcal{B}' = \{[a,b) \mid a < b\}$. Compare to standard topology $\mathcal{T}$:
  open intervals are unions of these, but not vice versa; guess
  $\mathcal{T}'$ is strictly finer than $\mathcal{T}$. $\quad \mathcal{T} \subset \mathcal{T}'$.
  - $\mathcal{T}'$ is finer than $\mathcal{T}$: $x \in (a,b) \subset \mathbb{R} \Rightarrow x \in [\frac{a+x}{2}, b) \in \mathcal{B}'$.
  - $\mathcal{T}$ is not finer than $\mathcal{T}'$: $x \in [a,b) \subset \mathbb{R}$, choose $x = a$. Then there's no open interval (c,d) with $x \in (c,d) \subset [a,b)$.

- Order topology

X = set. An order relation ("simple order") on X is $<$ satisfying:

① for all $x, y \in X$ with $x \neq y$, $x<y$ or $y<x$ (note by ② and ③, these can't both hold)

② for all x, $x \nless x$

③ if $x<y$ and $y<z$ then $x<z$.

Given $(X,<)$, for $a,b \in X$ with $a<b$, define:

$(a,b) = \{x \in X \mid a<x<b\}$

$[a,b)$ $\quad a \leq x < b$

$(a,b]$ $\quad a<x\leq b$

Def Given $(X,<)$. Assume X has more than 1 element. The order topology on X is the topology determined by the basis $\mathcal{B}$, where $\mathcal{B}$ is the collection of:

$$\left\{\begin{array}{ll} (a,b) & \text{for any } a<b \\ [a_0,b) & \text{if } X \text{ has a least element } a_0 \\ (a,b_0] & \text{if } X \text{ has a greatest element } b_0. \end{array}\right.$$

Check that this is a basis:

1. $x \in X \Rightarrow$ if x is neither least nor greatest, choose $a<x<b$; then $x \in (a,b)$.
2. $B_1, B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2 \overset{?}{\Rightarrow} x \in B_3 \subset B_1 \cap B_2$.

   if $B_1 = (a_1,b_1)$, $B_2=(a_2,b_2)$ then chose $a=\max(a_1,a_2)$, $b=\min(b_1,b_2)$

   $\Rightarrow x \in (a,b)$. Other cases similar.

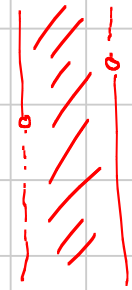
Ex • ($\mathbb{R}$, order topology) is the same as ($\mathbb{R}$, standard).

• $\mathbb{R}^2$: has an order relation, lexicographic (dictionary) order:

$(a_1, b_1) < (a_2,b_2) \Leftrightarrow$ either $a_1<a_2$, or $a_1=a_2$ and $b_1<b_2$.

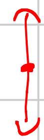
Write $a_1 \times b_1$ $\quad a_2 \times b_2$.

What does $(a_1 \times b_1, a_2 \times b_2)$ look like?



$a_1 = a_2$:

This is <u>strictly finer</u> than $(\mathbb{R}^2, \text{standard})$.



<u>Closed sets; Limit points</u>

<u>Def</u> X topological space. A subset $A \subset X$ is <u>closed</u> if the complement $X \setminus A$ is open.

<u>Ex</u>. $\emptyset, X$ always closed.

$[a,b], [a,\infty), (-\infty,a] \in \mathbb{R}$ ,  $\subset \mathbb{R}^2$

discrete topology: everything.

More interesting: $(0,1) \cup (2,3)$: $(0,1)$ and $(2,3)$ are closed (!)

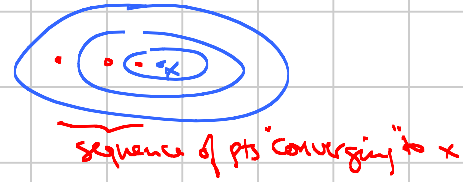
<u>Prop</u> 1. $\emptyset, X$ closed

2. Any intersection of closed sets is closed

3. Any finite union of closed sets is closed.

<u>Pf</u> For 2, 3, use DeMorgan's Laws. Ex: $X \setminus \left(\bigcap_i A_i\right) = \bigcup_i (X \setminus A_i)$. $\square$

Def $A\subset X$ subset. A limit point of A is $x\in X$ such that every neighborhood of x contains some point of $A\setminus\{x\}$.



Ex. subsets of $\mathbb{R}$: $A=(0,1]$ $\quad\{0,1\}$

$A=\mathbb{Z}$ $\quad$ no limit pts

$A=\{\frac{1}{n} \mid n\in\mathbb{N}\}$ $\quad\{0\}$

$A=\mathbb{Q}$ $\quad\mathbb{R}$

$A=\{1\le x^2+y^2<4\}\subset\mathbb{R}^2$

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Prop A is closed $\Leftrightarrow$ it contains all of its limit points.

Pf $\Rightarrow$: x = limit pt. If $x\notin A$ then $x\in X\setminus A$ open $\Rightarrow$ $X\setminus A$ contains a pt of $A\setminus\{x\}$.

$\Leftarrow$: want $X\setminus A$ open. Suppose $x\in X\setminus A$. Then x isn't a limit pt of A

$\Rightarrow$ there is $U\ni x$ with $U\cap A=U\cap(A\setminus\{x\})=\emptyset \Rightarrow U\subset X\setminus A$. $\square$

Fun example: Cantor set.

$C_0=[0,1]$

$C_1=C_0-$ middle third $=C_0\setminus(\frac{1}{3},\frac{2}{3})=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$.

C_2: $C_1\setminus$ middle third $=[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$

in general, $C_n=C_{n-1}\setminus\bigcup_{k=1}^{3^{n-1}}\left(\frac{3k-2}{3^n},\frac{3k-1}{3^n}\right)$.

$\emptyset$

Def The Cantor set $C=\bigcap_{n=0}^{\infty}C_n$.

Note: not empty: any endpt of one of the intervals in C_n is in C.

also other things: $\frac{1}{4}=\frac{1}{3}-\frac{1}{3^2}+\frac{1}{3^3}-\cdots$ in $(0,\frac{1}{3}),(\frac{2}{9},\frac{3}{9}),(\frac{6}{27},\frac{7}{27}),\ldots$

C is:
- closed
- perfect: every point is a limit pt. — $x\in C$, $U\ni x\Rightarrow x\in$ some interval in C_n for each $n\Rightarrow$ for large n, one of these must lie in U.
- nowhere dense: every open set in $\mathbb{R}$ contains an open subset that doesn't intersect C.

Def The closure $\overline{A}$ of a subset $A \subset X$ is the intersection of all closed sets containing A. (note: $\overline{A}$ is closed).

Prop A is closed $\Leftrightarrow$ $\overline{A} = A$.

Prop $\overline{A} = A \cup \{\text{limit points of } A\}$.

Pf $\subset$: suppose $x \in \overline{A} \setminus A$. Let U = nbd of x: want $U \cap A$ to be nonempty.
If $U \cap A = \phi$ then $X \setminus U$ is closed and contains A, so $x \in \overline{A} \subset X \setminus U$ ⨳

$\supset$: clearly $A \subset \overline{A}$. Suppose x = limit point of A. If K is a closed set containing A, want $x \in K$. If $x \notin K$, then $x \in X \setminus K$ = open and thus $(X \setminus K) \cap A \neq \phi$, so $K \not\supset A$ ⨳ . $\square$

Another way to say this: $x \in \overline{A} \Leftrightarrow$ every nbd of x contains a point of A.

Def Given $A \subset X$, the interior $\operatorname{int} A$ is the union of all open sets contained in A.

ex: 

Notes:
1. $\operatorname{int} A$ is open
2. A is open $\Leftrightarrow$ $A = \operatorname{int} A$
3. $\operatorname{int} A = X \setminus \overline{(X \setminus A)}$ (DeMorgan: $X \setminus (\cup U_i) = \cap (X \setminus U_i)$)

Def The boundary of A is $\operatorname{Bd} A = \overline{A} \cap \overline{X \setminus A}$.
$x \in \operatorname{Bd} A \Leftrightarrow$ every nbd of x intersects both A and $X \setminus A$.
(for more: see next HW).

Convergence, Hausdorff

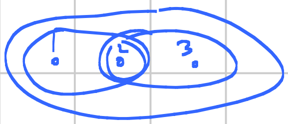
Limit in $\mathbb{R}^n$: $\lim_{n\to\infty} x_n = x$: $\forall \epsilon>0\ \exists N$ s.t. $n\ge N \Rightarrow \|x_n - x\| < \epsilon$.

Def A sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x\in X$ if for any nbd U of x, $\exists N$ st. $n \ge N \to x_n \in U$.

Ex. $X=(\mathbb{R}^n, \text{std})$: this is the usual notion.

X with discrete topology: must have $x_n = x$ for all $n\ge N$

X with indiscrete topology: anything converges to x.

$X =$ 

: $2, 2, 2, \ldots \to 2, \to 1, \to 3$ simultaneously.

Omit: is this true??

Claim x is a limit pt of $A \Leftrightarrow \exists$ sequence $x_1, x_2, \ldots \in A\setminus\{x\}$ converging to x.

one of "separation axioms"

Def A topological space X is Hausdorff ("T_2") if for any $x_1\ne x_2$ in X, $\exists$ nbds U_1, U_2 of x_1, x_2 with $U_1\cap U_2=\emptyset$.

Prop X Hausdorff. Then every sequence $\{x_n\}$ in X converges to at most one limit.

Pf Say $x_n\to x, x'$, $x\in U_1$, $x'\in U_2$, $U_1\cap U_2=\emptyset$. Then $\exists N_1, N_2$ st $n\ge N_1 \Rightarrow x_n\in U_1$, $n\ge N_2\Rightarrow x_n\in U_2$ $\Rightarrow\Leftarrow$. □

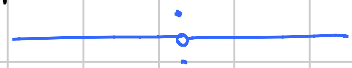
Ex $\mathbb{R}^n$ is Hausdorff, as is any $X\subset\mathbb{R}^n$.

X, discrete: Hausdorff.

X, indiscrete: not Hausdorff unless $X=\emptyset$ or $\{pt\}$.

X, Finite complement topology : not Hausdorff unless X is finite.

Usual example of non-deranged, non-Hausdorff space:



$X = (\mathbb{R} \setminus \{0\}) \cup \{0_1, 0_2\}$.

$\pi : X \to \mathbb{R}$ projection, $\mathcal{T} = \{\pi^{-1}(U) \mid U \text{ open in } \mathbb{R}\}$

basis : (a,b) $\quad 0<a<b$ or $a<b<0$

along with $(a,0)\cup(0,b)\cup\{0_1, 0_2\}$ $\quad a<0<b$.

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Def X is $\underline{T_1}$ if each single point $\{x\} \subset X$ is closed.

(Note $\Leftrightarrow$ topology on X is finer than finite-complement topology)

Prop $T_2 \Rightarrow T_1$.

Pf Want $\{x_0\}$ closed. Suppose $x \neq x_0$; then $U_1 \ni x$, $U_2 \ni x_0$ with $U_1 \cap U_2 = \emptyset$

so $x \neq$ limit point of $\{x_0\}$. Thus $\{x_0\}$ = its own closure. $\square$

Note: Converse not true: finite complement topology.

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Continuous functions, again

Prop $f: X \to Y$ map of topological spaces. TFAE:

1. f is continuous: $f^{-1}(\text{open})$ is open
2. $\mathcal{B}$ = basis for topology of Y $\Rightarrow$ $f^{-1}(B)$ is open in X $\forall\, B \in \mathcal{B}$
3. for any $A \subset X$, $\quad f(\overline{A}) \subset \overline{f(A)}$
4. for any $B \subset Y$, $\quad \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$
5. for any $B \subset Y$ closed, $\quad f^{-1}(B)$ is closed.

PF $1 \overset{\text{obvious}}{\Rightarrow} 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$.

$2 \Rightarrow 3$: need: if $x \in \overline{A}$ the $f(x) \in \overline{f(A)}$. If V = nbd of $f(x)$ then $x \in f^{-1}(V)$ = open $\Rightarrow \exists\, y \neq x$ with $y \in f^{-1}(V) \cap A \Rightarrow f(y) \in V \cap f(A)$.

$3 \Rightarrow 4$: Set $A := f^{-1}(B) \Rightarrow f(A) \subset B \Rightarrow \overline{f(A)} \subset \overline{B}$

$f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} \Rightarrow \overline{A} \subset f^{-1}(\overline{B})$.

$4 \Rightarrow 5$: B closed $\Rightarrow f^{-1}(B) \subset \overline{f^{-1}(B)} \underset{\text{by } 4}{\subset} f^{-1}(B)$

$5 \Rightarrow 1$: $V \subset Y$ open $\Rightarrow f^{-1}(\underbrace{Y \setminus V}_{\text{closed}}) = \{x \mid f(x) \notin V\} = X \setminus f^{-1}(V)$ is closed

$\Rightarrow f^{-1}(V)$ open. $\square$

Other useful properties of continuous fns.

Prop 1. Constant fns, identity fn 1_X are continuous.

2. $A \subset X$ subset (with subspace topology), $f: X \to Y$ continuous. The $f|_A : A \to Y$ is continuous. (ex: $i: A \to X$)

3. $f: X \to Y$, $g: Y \to Z$ continuous $\Rightarrow g \circ f : X \to Z$ continuous.

PF 1. Clear. 2. $U \subset Y$ open $\Rightarrow f^{-1}(U)$ open $\Rightarrow (f|_A)^{-1}(U) = f^{-1}(U) \cap A$ open in A.

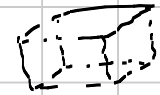
3. $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. $\square$

Product topology, revisited.

Recall: X, Y top. spaces with bases $\mathcal{B}_X, \mathcal{B}_Y \rightsquigarrow$ product topology on $X \times Y$ with basis $\{B \times C \mid B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$.

More generally, given top. spaces $X_1, \dots, X_n$, we can define the product topology on $X_1 \times \dots \times X_n$: $\mathcal{B}_1, \dots, \mathcal{B}_n$ bases for the topologies of $X_1, \dots, X_n$

$\rightarrow$ basis for product topology $\{B_1 \times \dots \times B_n \mid B_i \in \mathcal{B}_i\}$.

ex $\mathbb{R}^n$: basis is given by "boxes" $(a_1,b_1)\times\cdots\times(a_n,b_n) =$ 

As before, product topology = standard topology on $\mathbb{R}^n$

Let's restrict to products $X_1 \times X_2$. Projection maps
$$\begin{tikzcd} & X_1\times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Prop 1. π_1, π_2 are continuous, where $X_1 \times X_2$ has product topology.

2. The product topology is the <u>coarsest</u> topology on $X_1 \times X_2$ for which π_1, π_2 are continuous.

Pf 1. $\pi_1^{-1}(U) = U \times X_2$ is open

2. $U_1 \subset X_1$, $U_2 \subset X_2$ open $\Rightarrow \pi_1^{-1}(U_1), \pi_2^{-1}(U_2)$ open $\Rightarrow U_1 \times U_2 = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2)$ open. □

Prop $f_1 : Y \to X_1$, $f_2 : Y \to X_2$ maps $\rightsquigarrow$ define $f : Y \to X_1 \times X_2$ by $f(y) = (f_1(y), f_2(y))$. Then f is continuous $\Leftrightarrow$ f_1 and f_2 are continuous.

Pf $f_i = f \circ \pi_i$ so if f is continuous then so are f_1, f_2.
Conversely, if f_1, f_2 are continuous, $U_1 \times U_2$ = basic open set in $X_1 \times X_2$, suffices to show $f^{-1}(U_1 \times U_2)$ open: but this is $f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$. □

Prop $X_1 \times X_2$ Hausdorff $\Leftrightarrow$ X_1 and X_2 Hausdorff.
Pf easy.

<u>Note</u> Can generalize to infinite products. $\{X_\alpha\}_{\alpha\in J}$ top. spaces.
Define $\prod_{\alpha\in J} X_\alpha = \{$"$(x_\alpha)_{\alpha\in J}$": for each $\alpha\in J$, a choice of $x_\alpha\in X_\alpha$.

$B_\alpha \subset X_\alpha$ basic open set $\Rightarrow$ $\prod B_\alpha = \{(x_\alpha) \mid x_\alpha\in B_\alpha \ \forall\alpha\}$ basic open set.
This gives the <u>box topology</u>.
Note: for J infinite, this is <u>not</u> the "best" choice of topology on the product: instead, <u>product topology</u> (basis $= \{\pi_\alpha^{-1}(B_\alpha)\}$).

Weird ex: $\mathbb{R}^\infty = \prod_{n\in\mathbb{N}} \mathbb{R} = \{(x_1, x_2, \dots) \mid x_i\in\mathbb{R}\}$.
Then $f: \mathbb{R}\to\mathbb{R}^\infty$ defined by $f(x)=(x,x,x,\dots)$ should be continuous.
But: $U=(-1,1)\times(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{3},\frac{1}{3})\times\cdots$ is open in $\mathbb{R}^\infty$
and $f^{-1}(U)=\{0\}$ isn't open in $\mathbb{R}$.

<u>Metric topology</u>

<u>Def</u> A <u>metric</u> (distance function) on a set X is a function $d: X\times X\to\mathbb{R}$ with:
1. $d(x,y)\geq 0 \ \forall x,y$; $d(x,y)=0 \Leftrightarrow x=y$
2. $d(x,y)=d(y,x)$
3. $d(x,z)\leq d(x,y)+d(y,z)$.

(X,d) is called a <u>metric space</u>.

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<u>Ex.</u> - $X=\mathbb{R}^n$, $d(x,y)=\|x-y\|$; $d(x,y)=\max(|x_1-y_1|,\dots,|x_n-y_n|)$
- $X=C^0([0,1])$, continuous fns $[0,1]\to\mathbb{R}$. $d(f,g)=\sup_{t\in[0,1]}|f(t)-g(t)|$
- (X,d) metric space $\Rightarrow$ $(X,\bar{d})$ also a metric space, $\bar{d}(x,y)=\min(d(x,y),1)$
also can use this for $X=C^0(\mathbb{R})$ or $C^0(\text{anything})$.

Given a metric space (X,d), define

or $B_d(x,\epsilon)$ for clarity $\rightarrow$ $B(x,\epsilon) = \{ y \in X \mid d(x,y) < \epsilon \}$.

Prop $\{B(x,\epsilon) \mid x \in X, \epsilon > 0\}$ forms a basis.

Lemma $y \in B(x,r)$; then there is r' with $B(y,r') \subset B(x,r)$.

Pf Choose $r' = r - d(x,y)$. Then $z \in B(y,r') \Rightarrow d(x,z) \leq d(x,y) + d(y,z) < r$. $\square$

Main pf Clearly $X = \cup B(x,\epsilon)$. Suppose B_1, B_2 are basic open sets, $y \in B_1 \cap B_2$. Then $\exists$ r_1', r_2' with $B(y,r_1') \subset B_1$, $B(y,r_2') \subset B_2$.
Write $\epsilon = \min(r_1', r_2')$; then $y \in B(y,\epsilon) \subset B_1 \cap B_2$. $\square$

Def The metric topology on X induced by d is the topology determined by $\{B(x,\epsilon)\}$.

Observations • $(X$, metric topology$)$ is always Hausdorff:
if $x,y \in X$ and $\epsilon = d(x,y)/2$ then $B(x,\epsilon) \cap B(y,\epsilon) = \phi$.

Say a topological space is metrizable if there is some metric giving this topology; then X must be Hausdorff for it to be metrizable.

• If X has metric d and $Y \subset X$ is any subset, then d is also a metric on Y.
The metric topology on Y is the subspace topology of the metric topology on X.

• Lemma The metric topology for (X,d) is finer than the metric topology for (X,d')
$\Leftrightarrow$ for any $x \in X$ and $\epsilon > 0$, there is δ such that $B_d(x,\delta) \subset B_{d'}(x,\epsilon)$.

Pf. Finer $\Rightarrow$ for any $x \in X$ and $\epsilon > 0$, there are y, δ such that $x \in B_d(y,\delta) \subset B_{d'}(x,\epsilon)$.
So $\Leftarrow$ is clear. $\Rightarrow$: if $x \in B_d(y,\delta)$ and we define $\delta' = \delta - d(x,y)$ then
$B_d(x,\delta') \subset B_d(y,\delta)$ since $d(x,z) < \delta' \Rightarrow d(y,z) \leq d(y,x) + d(x,z) < \delta$. $\square$

Special case: (X,d) and $(X,\bar{d})$ have same metric topology.

Prop The following topologies on $\mathbb{R}^n$ are the same:

- the metric topology for $d_1 = \|x-y\|$ (standard topology)
- " " " $d_2 = \max(|x_1-y_1|, \dots, |x_n-y_n|)$
- the product topology on $\mathbb{R}^n$.

Pf Two metric topologies: note $d_2(x,y) \le d_1(x,y) \le \sqrt{n}\, d_2(x,y)$

$\Rightarrow B_{d_1}(x,\epsilon) \subset B_{d_2}(x,\epsilon)$, $B_{d_2}(x, \frac{\epsilon}{\sqrt{n}}) \subset B_{d_1}(x,\epsilon)$.

Now use lemma.

Product and d_2 topologies: basis for product is $\mathcal{B}_3 = \{(a_1,b_1)\times\dots\times(a_n,b_n)\}$.

Basis for d_2 is $\mathcal{B}_2 = \{(x_1-\epsilon, x_1+\epsilon)\times\dots\times(x_n-\epsilon, x_n+\epsilon)\}$.

Note $\mathcal{B}_2 \subset \mathcal{B}_3$. If $x=(x_1,\dots,x_n) \in (a_1,b_1)\times\dots\times(a_n,b_n) = B$ then

choose $\epsilon_1,\dots,\epsilon_n$ such that $(x_i-\epsilon_i, x_i+\epsilon_i) \subset (a_i,b_i)$, $\epsilon = \min(\epsilon_i)$

$\Rightarrow (x_1-\epsilon, x_1+\epsilon)\times\dots\times(x_n-\epsilon, x_n+\epsilon) \subset B$. $\square$

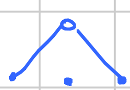
Compactness

Motivating question: when does a function $f: X \to \mathbb{R}$ have a max/min?

(max: $\exists\, x_0 \in X$ with $f(x_0) \ge f(x)\ \forall x$)

Extreme value thm: any continuous $f: [a,b] \to \mathbb{R}$ has a max/min.

How to generalize? replace this by X

Can't drop continuity:  Can't replace domain by (a,b) or $\mathbb{R}$.

Def $X \subset \mathbb{R}^n$ is *bounded* if $\exists\, R$ with $X \subset B(0,R)$.

We'll see: if $X \subset \mathbb{R}^n$ is bounded and closed, then EVT holds.

What if X isn't in $\mathbb{R}^n$?

Bolzano-Weierstrass property: every infinite subset of X has a limit point.

BW $\Rightarrow$ EVT. Now: more "modern" and useful notion: compactness.

Def X = top. space. An <u>open cover</u> of X is a collection $\mathcal{F}$ of open sets with $\bigcup_{U\in\mathcal{F}} U = X$.

Def X is <u>compact</u> if every open cover of X has a finite subcover: $\exists\, U_1,\dots,U_n \in \mathcal{F}$ with $\bigcup_{i=1}^{n} U_i = X$.

<u>Ex</u> not compact: $\mathbb{R}$ open cover $\{(-n,n) \mid n\in\mathbb{N}\}$ or $\{(n-1,n+1) \mid n\in\mathbb{N}\}$

$(0,1)$ open cover $\{(\frac{1}{n}, 1-\frac{1}{n})\}$

$(0,1]$ $\{(\frac{1}{n},1]\}$

but $[0,1]$ or $[a,b]$ <u>is</u> compact. $(\frac{1}{n},1]$ isn't an open cover: whatever covers 0 will be part of a finite subcover.

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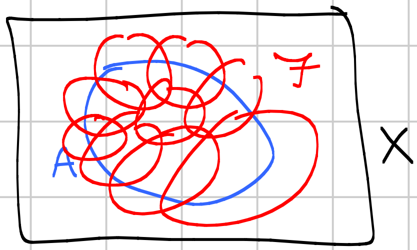
Def $A\subset X$ is <u>compact</u> if (A, subspace topology) is compact.

Equivalently:

An <u>open cover</u> of A ($\subset X$) is a collection $\mathcal{F}$ of open sets in X such that $A \subset \bigcup_{U\in\mathcal{F}} U$.



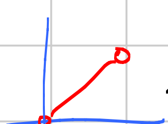
Note: then $\{U\cap A \mid U\in\mathcal{F}\}$ is an open cover of (A, subspace top)

Then $A\subset X$ is compact $\Leftrightarrow$ any open cover of A has a finite subcover.

<u>Motivation</u>: what topological spaces X satisfy:

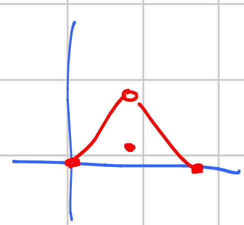
<u>Extreme Value Theorem</u> Any continuous $f: X\to\mathbb{R}$ has a max and min: $\exists\, x_{max}, x_{min} \in X$ such that for all $x\in X$, $f(x_{min}) \le f(x) \le f(x_{max})$.

Not always true: ex $X=(0,1)$



$f(x)=x$

also if we drop continuity: $X=[0,1]$

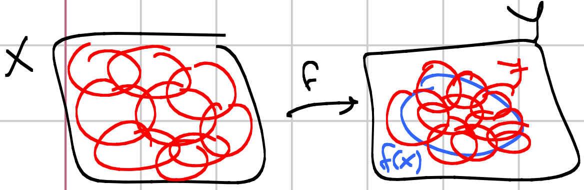


but EVT holds for $X=[a,b]$.

Thm If X is compact, EVT holds: any continuous $f: X \to \mathbb{R}$ has a max + min.

Prop $f: X \to Y$ continuous, X compact. Then $f(X)$ ($\subset Y$) is compact.

Pf Say $\mathcal{F}$ = open cover of $f(X)$: $f(X) \subset \bigcup_{U \in \mathcal{F}} U$. Then



$\{f^{-1}(U) \mid U \in \mathcal{F}\}$ is an open cover of X:

if $x \in X$ then $f(x) \in f(X) \to \exists U \in \mathcal{F}$ with $f(x) \in U \Rightarrow x \in f^{-1}(U)$.

So this has a finite subcover: $X = f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n)$.

But then $f(X) \subset U_1 \cup \dots \cup U_n$. $\square$

Cor If X, Y are homeomorphic, then X compact $\Leftrightarrow$ Y compact. (this is actually clear)

Need least upper bound property of $\mathbb{R}$ ("Completeness"):

if $S \subset \mathbb{R}$ is nonempty and bounded above, then it has a supremum.

(x = upper bound if $x \geq y$ $\forall y \in S$; x = supremum if x = upper bound and $x \geq z$ $\forall z$ = upper bound)

Pf of EVT. We'll show f has a max. Let $A = f(X) \subset \mathbb{R}$.

Claim 1 A is bounded above: there is M such that $f(x) \leq M$ for all $x \in X$.

Pf: if not, then $\forall N$, $A \not\subset (-\infty, N)$ (or else $M = N$ works).

Then $\{(-\infty, N) \mid N \in \mathbb{Z}\}$ is an open cover of A. $A = f(X)$ compact $\Rightarrow$

it's covered by a finite subcover $(-\infty, N_1), \dots, (-\infty, N_n)$: but $\cup(-\infty, N_i) = (-\infty, N)$

where $N = \max(N_1, \dots, N_n)$.

Write $M = \sup\{x \mid x \in A\}$. Then M is an upper bound for A ($x_n \to x$, x_n = upper bound $\Rightarrow$ x = upper bound)

i.e. $A \subset (-\infty, M]$: $M \geq f(x)$ for all $x \in A$.

Claim 2 $M \in A$.

Pf: if not, then write $U_n = (-\infty, M - \frac{1}{n})$: then $A \subset (-\infty, M) = \bigcup_{n=1}^{\infty} U_n$

so $\{U_n\}$ is an open cover of $A \Rightarrow$ finite subcover $U_{n_1}, \dots, U_{n_k}$.

If $n = \max(n_1, \dots, n_k)$ then $U_{n_1} \cup \dots \cup U_{n_k} = U_n \Rightarrow A \subset U_n$

$\Rightarrow M - \frac{1}{n}$ is an upper bound for A. $\Rightarrow\Leftarrow$ $\square$

<u>Next</u>: how to construct compact spaces.

<u>Prop</u> $[a,b] \subset \mathbb{R}$ is compact.

<u>Pf</u> Say $\mathcal{F}$ = open cover of $[a,b]$. Define $S \subset [a,b]$ as
$S = \{x \in [a,b] \mid [a,x] \text{ has a finite subcover}\}$.
Then S is nonempty ($a \in S$) and bounded above; want $b \in S$.
LUB $\Rightarrow \exists$ least upper bound $s = \sup S$.

<u>Claim 1</u> $s \in S$.

<u>Pf</u>: choose $U \in \mathcal{F}$ open containing s. Then for some ϵ, $(s-\epsilon, s] \subset (s-\epsilon, s+\epsilon) \cap [a,b] \subset U$.
Now $s-\epsilon$ isn't an upper bound for S so $\exists\, x > s-\epsilon$ s.t. $[a,x]$ has a finite subcover, $[a,x] \subset \bigcup_{i=1}^{n} U_i$. Then $[a,s] \subset \left(\bigcup_{i=1}^{n} U_i\right) \cup U$.

<u>Claim 2</u> $s = b$.

<u>Pf</u>: otherwise $s < b$, $[a,s] \subset \bigcup_{i=1}^{n} U_i$. Say $s \in U_k$. Then $[s, s+\epsilon) \subset U_k$ for small ϵ, so $[a, s+\frac{\epsilon}{2}] \subset \bigcup_{i=1}^{n} U_i$ and $s + \frac{\epsilon}{2} \in S$ $\Rightarrow\Leftarrow$ □

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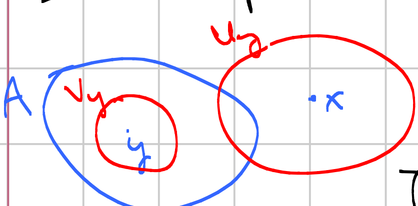
<u>Prop</u> Any closed subset of a compact space is compact.

<u>Pf</u> C closed $\subset X$ compact, $\mathcal{F}$ = open cover of C. Then $\mathcal{F}$ along with $X \setminus C$ is an open cover of $X \Rightarrow$ finite subcover $U_1, \dots, U_n$, maybe $X \setminus C$.
Then $C \subset U_1 \cup \dots \cup U_n$. □

<u>Prop</u> Any compact subset of a Hausdorff space is closed.

<u>Pf</u> A compact $\subset X$ Hausdorff; want $X \setminus A$ = open. Fix $x \in X \setminus A$. For $y \in A$, $\exists$ disjoint nbds U_y, V_y of x, y. Then $\{V_y \mid y \in A\}$ is an open cover of $A \Rightarrow$ finite subcover $V_{y_1}, \dots, V_{y_n}$.



Then $U_{y_1} \cap \dots \cap U_{y_n}$ is open and disjoint from $V_{y_1} \cup \dots \cup V_{y_n} \supset A$, so $x \in U_{y_1} \cap \dots \cap U_{y_n} \subset X \setminus A$. □

Cor $f: X \to Y$ continuous, bijective. If Y is Hausdorff and X is compact, then f is a homeomorphism.

Pf Want $g = f^{-1}$ continuous; suffices to show $g^{-1}(\text{closed})$ is closed.
Let $C \subset X$ be closed; then C is compact, so $f(C) = g^{-1}(C)$ is compact $\Rightarrow$ closed. $\square$

So: $[a,b]$ cpt; any finite union of cpt spaces is cpt.
What else?

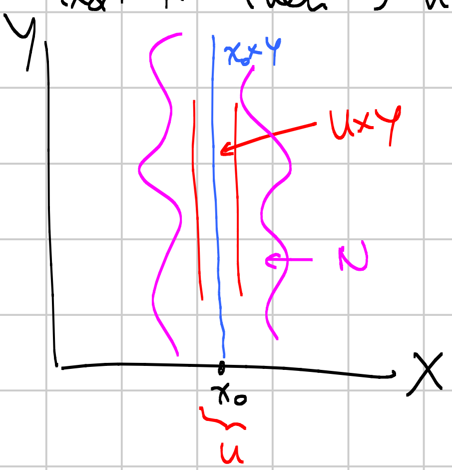
Prop X, Y compact $\Rightarrow X \times Y$ compact (with product topology).

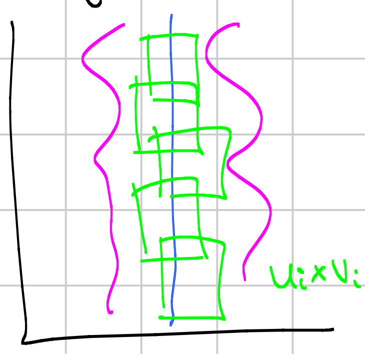
So: $[a_1,b_1] \times \dots \times [a_n,b_n] \subset \mathbb{R}^n$ is compact, as is any closed subset of this.

Lemma X, Y spaces, $x_0 \in X$. Then $\{x_0\} \times Y \overset{\subset X \times Y}{}$ is homeomorphic to Y.

Pf. Just check $Y \underset{g}{\overset{f}{\rightleftarrows}} \{x_0\} \times Y$ are continuous: $f(y) = (x_0, y)$; $g(x_0, y) = y$.
Note $g = \pi_2|_{\{x_0\} \times Y}$ ✓ f: a basic open set in $\{x_0\} \times Y$ is $(\{x_0\} \times Y) \cap (U \times V) = \{x_0\} \times V$
and $f^{-1}(\{x_0\} \times V) = V$ is open. $\square$

Tube Lemma X, Y spaces, Y compact, $x_0 \in X$, N = open set containing $\{x_0\} \times Y$. Then $\exists$ nbd U of x_0 with $U \times Y \subset N$.





Note: not true if $Y \neq$ compact.

Pf Since $\{U \times V\}$ = basis for topology on $X \times Y$, the subset $\{U \times V \mid U \times V \subset N\} = \mathcal{F}$ is an open cover of $\{x_0\} \times Y$: $\forall (x_0, y)\ \exists\ (x_0, y) \in U \times V \subset N$.

Note $\{x_0\}\times Y$ is cpt since it's homeo to Y, so $\mathcal{F}$ has a finite subcover $U_1\times V_1, \dots, U_n\times V_n$. Can assume $x_0 \in U_i$ for each i; otherwise $U_i\times V_i$ is disjoint from $\{x_0\}\times Y$ so we could just drop $U_i \times V_i$.

Write $U = U_1\cap\dots\cap U_n$. Then $x\in U$ and $U\times Y$ is covered by $\{U_i\times V_i\}_{i=1}^n$: if $(x,y)\in U\times Y$ then $(x_0,y)\in\{x_0\}\times Y \Rightarrow (x_0,y)\in U_i\times V_i$ for some i $\Rightarrow y\in V_i \Rightarrow (x,y)\in U_i\times V_i$.

Finally: $U\times Y \subset \bigcup_{i=1}^n (U_i\times V_i) \subset N$. $\square$

<u>Pf of Prop</u> Suppose $\mathcal{F}$ = open cover of $X\times Y$.

For each $x_0\in X$, $\{x_0\}\times Y$ is compact so $\exists$ finite subcover $N_1,\dots,N_k \in \mathcal{F}$ covering $\{x_0\}\times Y$: $\cup N_i \supset \{x_0\}\times Y$. Tube Lemma $\Rightarrow \exists$ nbd U_{x_0} of x_0 with $U_{x_0}\times Y \subset \cup N_i$.

So: for any $x_0\in X$, can construct U_{x_0} = nbd of x_0 s.t. $U_{x_0}\times Y$ is covered by finitely many of $\mathcal{F}$. Now X is compact and $\{U_{x_0} \mid x_0\in X\}$ = open cover so $\exists$ finitely many $x_1,\dots,x_m$ with $\bigcup_{i=1}^m U_{x_i} = X$.

Each $U_{x_i}\times Y$ is covered by finitely many sets in $\mathcal{F}$, so that's also true of $\bigcup_{i=1}^m (U_{x_i}\times Y) = X\times Y$. $\square$

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By induction: $X_1,\dots,X_n$ compact $\Rightarrow X_1\times\dots\times X_n$ compact.

<u>Tychonoff Thm</u> Any (possibly infinite) product of compact spaces is compact. (uses Axiom of Choice)

<u>Heine-Borel Thm</u> K = subset of $\mathbb{R}^n$. Then K is compact $\Leftrightarrow$ K is closed and bounded.

there is R such that $K\subset B(0,R)$.

<u>Note</u>: here $\mathbb{R}^n$ has standard metric d (cf. any subset of $(\mathbb{R}^n,\bar{d})$ is bounded).

Pf. $\boxed{\Rightarrow}$: Compact subset of Hausdorff is closed.
Bounded: $\{B(0,n)\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}^n$ $\to$ open cover of K
$\to K \subset$ finite subcover $\bigcup_{i=1}^m B(0,n_i) = B(0,n)$ where $n=\max(n_1,\dots,n_m)$.
$\boxed{\Leftarrow}$: K bounded $\Rightarrow K \subset B(0,R) \subset [-R,R]\times\dots\times[-R,R]$ compact.
Closed $\subset$ compact is compact. $\square$

MIDTERM 1
MATERIAL ENDS HERE

Other major theorem involving compactness..

Bolzano-Weierstrass Thm flavor 1.
X compact. Then any infinite subset $A \subset X$ has a limit point.

Pf Suppose A has no limit point. Then $\overline{A} = A$ so A is closed $\Rightarrow$ compact.
For any $x \in A$ there is a nbd U_x of x with $U_x \cap A = \{x\}$.
Compactness $\Rightarrow \exists\, x_1,\dots,x_n \in A$ with $A \subset \bigcup_{i=1}^n U_{x_i} \Rightarrow A \subset \left(\bigcup_{i=1}^n U_{x_i}\right) \cap A = \bigcup_{i=1}^n (U_{x_i}\cap A)$
$= \{x_1,\dots,x_n\} \Rightarrow\Leftarrow$ $\square$

Bolzano-Weierstrass Thm flavor 2
X compact, metric space. Then any sequence $\{x_n\}_{n=1}^\infty$ has a convergent subsequence: $\exists\, n_1 < n_2 < \cdots$ with $\{x_{n_i}\} \to x$.

Pf Choose $A = \cup\{x_n\}$. If A is finite then there are ∞ many n with $x_n = x$, done. If A is infinite then it has a limit pt x.
Choose n_1 with $x_{n_1} \in B(x,1)$, $x_{n_1} \neq x$.
Then inductively choose n_i with $x_{n_i} \in B(x,\frac{1}{i})$, $x_{n_i} \neq x$, $n_i > n_{i-1}$.
(Why does this exist? Otherwise $B(x,\frac{1}{i}) \cap A \subset \{x_1,\dots,x_{n_{i-1}},x\}$
so $B(x,\epsilon) \cap (A\setminus\{x\}) = \emptyset$ where $\epsilon = \min(\frac{1}{i}, d(x,x_1),\dots,d(x,x_{n_{i-1}}))$.
$\Rightarrow\Leftarrow$ $\square$

Note Converse of Bolzano-Weierstrass 1 isn't true.

$Y = \{p_1, p_2\}$ with indiscrete topology, $\mathbb{N}$ with discrete topology.

$X = \mathbb{N} \times Y$. Any nonempty $A \subset X$ has a limit pt: if $(p_1, y) \in A$ then (p_2, y) is a limit pt of A.

But X isn't compact: $\{ \{1, \dots, n\} \times Y \mid n \in \mathbb{N} \}$ is an open cover.

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For later use:

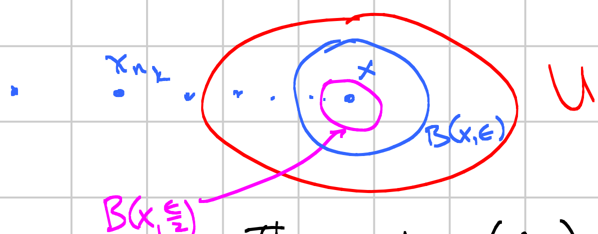
Lebesgue Lemma X compact metric space, $\mathcal{F}$ open cover. Then there exists $\delta > 0$ such that any subset of X of diameter $< \delta$ is contained in some $U \in \mathcal{F}$.

$\rightarrow$ δ = "Lebesgue number"

$\rightarrow$ recall $\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y)$.

Pf Suppose not. Then $\exists$ subsets $A_1, A_2, \dots$ with $\operatorname{diam}(A_n) < \frac{1}{n}$ such that $A_n \not\subset U$, for any $U \in \mathcal{F}$. Choose $x_n \in A_n \Rightarrow$ sequence $\{x_n\}$

$\Rightarrow$ convergent subsequence $x_{n_k} \to x$. Say $x \in U$, $U \in \mathcal{F}$.



Choose $\epsilon > 0$ with $B(x,\epsilon) \subset U$ and choose N large enough that $\frac{1}{N} < \frac{\epsilon}{2}$ and $x_N \in B(x, \frac{\epsilon}{2})$.

Then $\operatorname{diam}(A_N) < \frac{\epsilon}{2}$ and $d(x, x_N) < \frac{\epsilon}{2}$

$\Rightarrow$ for any $y \in A_N$, $d(x,y) \le d(x,x_N) + d(x_N, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

So $A_N \subset B(x,\epsilon) \subset U$, contradiction. $\square$

Connectedness

<u>Def</u> A top space X is <u>connected</u> if

$X = U \sqcup V$, U, V open $\Rightarrow$ $U = X$, $V = \phi$ or vice versa.
$(X = U \cup V,\ U \cap V = \phi)$

<u>Prop</u> TFAE:

1. X is connected
2. $A \subset X$ is both open and closed $\Leftrightarrow$ $A = X$ or $A = \phi$.
3. Whenever $X = A \sqcup B$ with $A, B \neq \phi$, one of A, B contains a limit point of the other.

<u>Pf</u> <u>$1 \Rightarrow 2$</u>: $X = A \sqcup (X \setminus A)$

<u>$2 \Rightarrow 3$</u>: Suppose $X = A \sqcup B$, neither contains a limit pt of the other. Then $\bar{A} \cap B = A \cap \bar{B} = \phi$ so $A = \bar{A}$ $(\bar{A} \subset A)$ and $B = \bar{B}$

$\Rightarrow A$ is closed and open $(X \setminus A = B$ is closed$)$ $\rightarrow$ $A = X$ or $A = \phi$.

<u>$3 \Rightarrow 1$</u>: Suppose $X = U \sqcup V \Rightarrow U = \bar{U}, V = \bar{V}$. If U, V nonempty then say $U \cap \bar{V} \neq \phi \Rightarrow U \cap V \neq \phi$ $\Rightarrow \Leftarrow$

<u>Def</u> A <u>separation</u> of X is $X = A \sqcup B$, $A \cap \bar{B} = \bar{A} \cap B = \phi$, $A, B \neq \phi$.

<u>Note</u>: If $X = A \sqcup B$ is a separation then as above $A = \bar{A}$, $B = \bar{B}$ so A, B are open. So X is connected $\Leftrightarrow$ it has no separation.

Def $Y \subset X$ is connected if it's connected in subspace topology.
Equivalently: there is no separation of Y: $Y = A \amalg B$, $A, B \neq \emptyset$, neither contains a limit point of the other.
(exercise: closure of A in Y is $\bar{A} \cap Y$: so limit pt of A in Y = limit point of A in X that's contained in Y.)

Connectedness implies the following: if U, V are disjoint open in X and $Y \subset U \cup V$ then either $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$.

What sets are connected?

Def $X \subset \mathbb{R}$ is an interval if whenever $x, y \in X$, $[x,y] \subset X$.

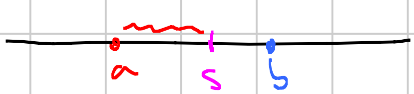
Prop The intervals are: $\mathbb{R}$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$, $[a,b]$, (a,b), $[a,b)$, $(a,b]$.

Pf Write $a = \inf X \in \mathbb{R} \cup \{-\infty\}$, $b = \sup X \in \mathbb{R} \cup \{+\infty\}$.
We'll consider the case $a, b \in \mathbb{R}$. Then $X \subset [a,b]$. Also if $z \in (a,b)$ then $z \neq$ lower bound for $X \Rightarrow \exists\ x < z$ with $x \in X$; similarly $\exists\ y > z$ with $y \in X$ $\Rightarrow z \in X$. Thus $(a,b) \subset X$. $\Rightarrow X = [a,b]$, $[a,b)$, $(a,b]$, or (a,b). $\square$

Prop $X \subset \mathbb{R}$ is connected $\Leftrightarrow$ X is an interval (or $X = \emptyset$).

Pf $\Rightarrow$: if X isn't an interval then $\exists\ x < z < y$ with $x, y \in X$, $z \notin X$; then $X \subset (-\infty, z) \cup (z, \infty)$ and X intersects both $(-\infty, z)$ and (z, ∞).

$\Leftarrow$: Suppose $X = A \amalg B$, $A, B \neq \emptyset$. Choose $a \in A$, $b \in B$; say $a < b$ so $[a,b] \subset X$. Let $s = \sup\{a' \in A \mid a' < b\} \in [a,b] \subset X$.



Case 1 $s \in A$. Then $(s, b] \subset B$ so s = limit pt of B.
Case 2 $s \in B$. Then since $s = \sup$, s = limit pt of A. $\square$

10/12 2

Prop $f: X \to Y$ continuous, X connected $\Rightarrow f(X)$ connected.

Pf $f(X) = A \amalg B \Rightarrow X = f^{-1}(A) \amalg f^{-1}(B) \Rightarrow$ say $\overline{f^{-1}(A)} \cap f^{-1}(B) \neq \emptyset$
$\Rightarrow \bar{A} \cap B \neq \emptyset$. $\square$

$\overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$

Cor If X, Y are homeomorphic, then X is connected $\Leftrightarrow Y$ is connected.

Intermediate Value Thm $f : [a,b] \to \mathbb{R}$ continuous.
If c lies between $f(a)$ and $f(b)$ then $\exists\, x \in [a,b]$ with $f(x) = c$.

Pf If not then $\overbrace{f([a,b])}^{S} \subset \mathbb{R} \setminus \{c\} = (-\infty, c) \cup (c, \infty)$ and S intersects both $(-\infty,c)$ and (c,∞). But $[a,b]$ is connected $\Rightarrow$ so is S. $\square$

Prop Suppose $\mathcal{F} = \{A_\alpha\}$ = collection of subsets of X s.t. $A_\alpha, A_\beta \in \mathcal{F} \Rightarrow A_\alpha \cap A_\beta \neq \emptyset$.
If all $A_\alpha \in \mathcal{F}$ are connected, then $\bigcup_\alpha A_\alpha$ is connected.

Pf Write $Y = \bigcup_\alpha A_\alpha$. Suppose we have a separation $Y = A \amalg B$.
For each α, $A_\alpha \subset A$ or $A_\alpha \subset B$, else $A_\alpha = (A_\alpha \cap A) \amalg (A_\alpha \cap B)$ is a separation of A_α.
Can't have $A_\alpha \subset A$ and $A_\beta \subset B$, so all $A_\alpha \subset A$ or all $A_\alpha \subset B$
$\Rightarrow Y \subset A$ or $Y \subset B \Rightarrow B = \emptyset$ or $A = \emptyset$. $\square$

Prop X, Y connected $\Rightarrow X \times Y$ connected. ($\Rightarrow$ Same holds for finite products)

Pf $\{x\} \times Y \cong Y$, $X \times \{y_0\} \cong X$ are connected

$\Rightarrow Z(x_0, y_0) = (\{x_0\} \times Y) \cup (X \times \{y_0\})$ is connected.

Now use $\mathcal{F} = \{Z(x,y)\}$. $\square$

Cor $\mathbb{R}^n$ is connected.

Prop $A \subset X$ connected. If $A \subset B \subset \overline{A}$ (i.e. $B = A \cup$ some limit pts)
then B is connected.

Pf Suppose $B = B_1 \amalg B_2$ separation. Then $A \subset B_1$ or $A \subset B_2$, say $A \subset B_1$.
($\overline{B_1} \cap B_2 = \emptyset \Rightarrow \overline{A \cap B_1} \cap (A \cap B_2) = \emptyset$ etc.). Then $B \subset \overline{A} \subset \overline{B_1}$ is disjoint from B_2. $\Rightarrow\Leftarrow$

Ex $\cdot$ $S^n \subset \mathbb{R}^{n+1}$ is connected: $S^n \setminus pt$ is connected (it's homeo to $\mathbb{R}^n$) and $S^n = \overline{S^n \setminus pt}$.

or: $S^n = (S^n \setminus N) \cup (S^n \setminus S)$

- $S^1 \times S^1 =$ torus is connected
- $B^n = \{\|x\| < 1\}$ is connected (homeo to $\mathbb{R}^n$);
  $\{\|x\| > 1\}$ is connected (homeo to $\mathbb{R} \times S^{n-1}$).

10/17

Connected Components : "maximal" connected subsets of a set.

Define an equivalence relation $\sim$ on X by

$x \sim y \iff \exists$ connected $A \subset X$ with $x, y \in A$.

(This is an equiv relation: $x \sim x$; $x \sim y \Leftrightarrow y \sim x$; $\underset{A}{x \sim y} \underset{B}{\sim z} \Rightarrow \underset{A \cup B}{x \sim z}$.)

Def Equivalence classes are components of X.

$\hookrightarrow$ $x \in X \rightsquigarrow [x]$, write $[x] = [y]$ if $x \sim y$.

Ex $\cdot$ Connected spaces have 1 component

- $(-1,1) \setminus \{0\}$ has two components.

- $\mathbb{R}^{n+1} \setminus S^n$ has two components: $\{\|x\| > 1\}$, $\{\|x\| < 1\}$.
- $\mathbb{Z}$ or $\mathbb{Q}$: each point is in its own connected component:
  for $\mathbb{Q}$, if $r_1 < r_2$, then there is an irrational number $\alpha \in (r_1, r_2)$
  So if $A =$ connected subset of $\mathbb{Q}$, $r_1, r_2 \in A$, then $A \subset (-\infty, \alpha) \cup (\alpha, \infty)$.

A space where all connected components are single points is called totally disconnected.

Prop The components of X are closed, connected, and disjoint, and their union is X. Any nonempty connected subset of X must lie in one of the components.

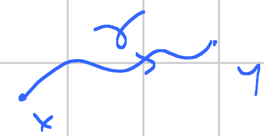
Pf Disjoint, union $= X$ ✓

Nonempty connected can't intersect two components ✓

Component = connected: Say $A =$ component, $x_0 \in A$. Then for any $x \in A$, there is connected A_x containing x_0 and x $\Rightarrow A_x \subset A \Rightarrow \bigcup_{x \in A} A_x = A$ is connected since $x_0 \in A_x \ \forall x$.

Component = closed: $A =$ connected $\Rightarrow \bar{A} =$ connected $\Rightarrow \bar{A} \subset$ some component $\Rightarrow \bar{A} \subset A \Rightarrow \bar{A} = A$. $\square$

Path Connectedness

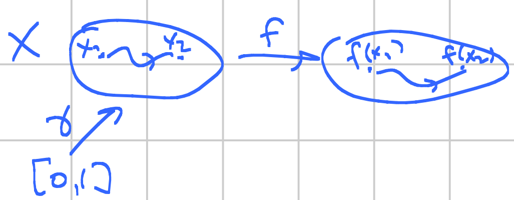


Def A *path* in X from x to y is a continuous function $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$.

Def X is *path connected* if $\forall x, y \in X$, $\exists$ path from x to y.

Prop If X is homeo to Y, then X is path connected $\Leftrightarrow$ Y is.

Pf.



If γ is a path from x_1 to x_2 then $f \circ \gamma$ is a path from $f(x_1)$ to $f(x_2)$. $\square$

Prop Path connected $\Rightarrow$ Connected.

Pf Suppose X = path connected, $X = A \amalg B$, A, B open, $a \in A$, $b \in B$. There is $\gamma : [0,1] \to X$ from a to b. Then $\gamma^{-1}(A), \gamma^{-1}(B)$ are open and nonempty in $[0,1]$

10/19 and $[0,1] = \gamma^{-1}(A) \amalg \gamma^{-1}(B)$, contradicting connectedness of $[0,1]$. $\square$

Converse does not hold: topologist's sine curve.

$Y = \{0\} \times [-1,1]$

$Z = \{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\}$ $\subset \mathbb{R}^2$

$X = Y \cup Z$.

Z is path connected $\Rightarrow$ connected. Straightforward to check $\overline{Z} = Z \cup Y = X$, so X is connected.

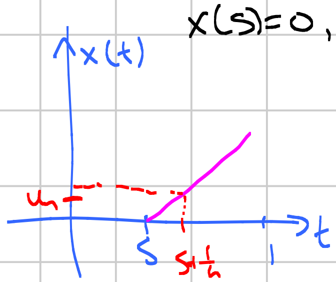
Prop: X isn't path connected.

Pf Suppose $\gamma : [0,1] \to X$, $\gamma(0) \in Y$, $\gamma(1) \in Z$. Since Y is closed in $\mathbb{R}^2$, $\gamma^{-1}(Y) \subset [0,1]$ is closed. Write $s = \sup \gamma^{-1}(Y)$: note $s \in \gamma^{-1}(Y)$ so $s < 1$.

So we have $\gamma : [s,1] \to X$, $\gamma(s) \in Y$, $\gamma(t) \in Z$ for $t > s$.

Write $\gamma(t) = (x(t), y(t))$ where $x(t), y(t)$ are continuous $[s,1] \to \mathbb{R}$,

$x(s) = 0$, $x(t) > 0$ for $t > s$, $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for $t > s$.



For $n > 0$, choose u_n with $0 < u_n < x(s + \frac{1}{n})$ and $\sin\left(\frac{1}{u_n}\right) = (-1)^n$. (note $\frac{1}{x} = \begin{cases} \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots \\ \frac{3\pi}{2}, \frac{7\pi}{2}, \dots \end{cases} \Rightarrow \sin\frac{1}{x} = \begin{cases} +1 \\ -1 \end{cases}$)

IVT $\Rightarrow$ $\exists\, t_n$ with $s < t < s + \frac{1}{n}$ and $x(t_n) = u_n$

$\Rightarrow y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = (-1)^n$.

So $\{t_n\}$ is a convergent sequence (to s) but $\{y(t_n) = (-1)^n\}$ doesn't converge, contradicting the fact that y is continuous. $\square$

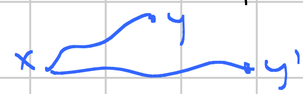
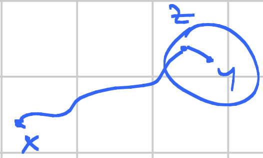
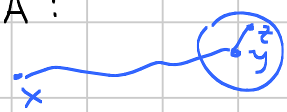
Def A path component of X is an equivalence class of points in X where $x \sim y \Leftrightarrow \exists$ path from x to y. (transitivity)

Each path component lies in a component but not necessarily conversely; also path components aren't necessarily closed.

Prop If $A \subset \mathbb{R}^n$ is connected <u>and open</u> then A is path connected.

<u>Pf</u> $x \in A$. Write $U(x)$ = path component of A containing x

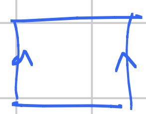
$= \{ y \in A \mid \exists \text{ path from } x \text{ to } y \}$.

- $U(x)$ is path connected: 
- $U(x)$ is closed: if $y \in A \setminus U(x)$ then choose ϵ with $B(y,\epsilon) \subset A$: then $B(y,\epsilon) \cap U(x) = \phi$. 
- $U(x)$ is open: if $y \in U(x)$ then choose ϵ with $B(y,\epsilon) \subset A$: then $B(y,\epsilon) \subset U(x)$. 

$\Rightarrow U(x) = A$ is path connected. $\square$

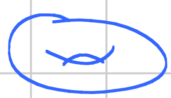
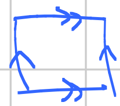
Quotient Spaces, Quotient Topology

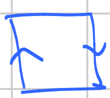
Motivation: describe surfaces as "identification spaces" by identifying points.

Cylinder:  $\cong$ 

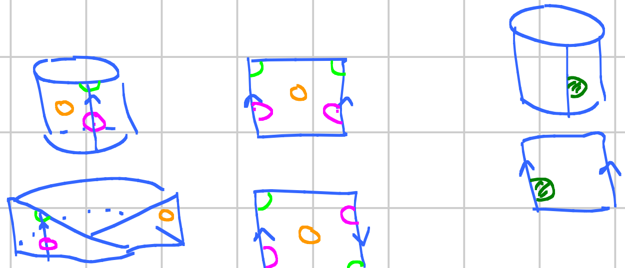
$x^2+y^2=1,\ 0 \le z \le 1$ $\qquad I \times I / \sim$

equivalence relation: $(0,y) \sim (1,y)$

Torus:  $(r-2)^2+z^2=1$ $(r = \sqrt{x^2+y^2})$ $\cong$  $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$

Möbius strip:  $\cong$  $(0,y) \sim (1,1-y)$

10/24

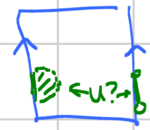
What does a nbd of a point look like? 

These are all open sets in $I \times I$

But not <u>all</u> open sets in $I \times I$ correspond to open sets in cylinder etc. $\rightarrow$ eg

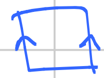
Each of these examples can be written $\pi : X \to Y$

and open sets in Y are $U \subset Y$ such that $\pi^{-1}(U)$ is open in X.



Suppose we have an equivalence relation on a top. space X.

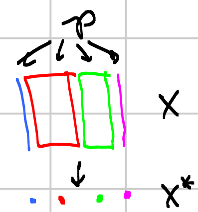
The collection of equiv classes is a "partition" $\mathcal{P}$ of X: a set of disjoint nonempty subsets of X whose union is X. $\bigcup_{P \in \mathcal{P}} P = X$.

ex: 

$$\mathcal{P} = \left\{ \begin{array}{l} \{(x,y)\},\ 0 < x < 1,\ 0 \le y \le 1 \\ \{(0,y),(1,y)\},\ 0 \le y \le 1. \end{array} \right\}$$

Conversely, a partition $\mathcal{P}$ of X gives rise to an equivalence relation:

$x \sim y \iff x, y$ are in same set in $\mathcal{P}$.



~ or $\mathcal{P}$ gives rise to a quotient space or identification space X^*:

elements of X^* are equivalence classes = sets in $\mathcal{P}$.

This comes equipped with a "projection map" (surjective)

$X \to X^*$

$x \mapsto [x] = P$ where $P \in \mathcal{P}$ and $x \in P$.

Def The quotient topology on X^* is defined by:

$U \subset X^*$ is open $\iff \pi^{-1}(U) \subset X$ is open.

→ Check this is a topology: e.g. if U_α are open in X^* then so is $\pi^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha \pi^{-1}(U_\alpha)$

$(= \{x \in X \mid \pi(x) \in U_\alpha$ for some $\alpha\})$

Rmk Then $\pi : X \to X^*$ is continuous; in fact the quotient topology is the finest topology on X^* such that π is continuous.

(π continuous $\Rightarrow$ if U is open in X^* then $\pi^{-1}(U)$ is open in $X \Rightarrow U$ is open in quotient topology)

More generally:

Prop X^* = quotient space of X, $\pi : X \to X^*$, Z = top. space. Then

$f : X^* \to Z$ is continuous $\iff \underbrace{f \circ \pi}_{g} : X \to Z$ is continuous.

Pf $U \subset Z$ open. Then $f^{-1}(Z)$ is open in $X^* \iff g^{-1}(Z) = \pi^{-1}(f^{-1}(Z))$ is open in X. $\square$

Alternate approach to quotient space:

<u>Def</u> A map $p: X \to Y$ is a <u>quotient map</u> if it's surjective and

$U \subset Y$ is open $\Leftrightarrow p^{-1}(U) \subset X$ is open.

As before:

<u>Prop</u> $p: X \to Y$ quotient map, $f: Y \to Z$. Then f is continuous $\Leftrightarrow f \circ p$ is continuous.

<u>Natural example</u>: $\pi: X \to X^*$ is a quotient map.

Conversely: given a quotient map $p: X \to Y$, define a partition $\mathcal{P}$ of X by

$\mathcal{P} = \{p^{-1}(y) \mid y \in Y\} \rightsquigarrow \mathcal{P}$ produces X^*, quotient space of X.

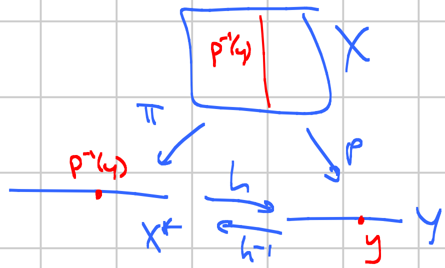
<u>Prop</u> X^* is homeomorphic to Y.

<u>Pf</u> Define $h: X^* \to Y$ by $h(p^{-1}(y)) = y$.

This is bijective (since p is surjective) and $h \circ \pi = p$.

Now: h is continuous $\Leftrightarrow h \circ \pi$ is continuous (which it is)

and h^{-1} is continuous $\Leftrightarrow h^{-1} \circ p$ is continuous (which it is). $\square$



Reversing this:

<u>Def</u> Given a top. space X and a surjective map $p: X \to Y$, define the <u>quotient topology</u> on Y to be the topology such that p is a quotient map.

10/26

<u>Summary</u>: 3 equivalent approaches.

1. X top space, $\mathcal{P}$ partition of $X \rightsquigarrow X^*$ = quotient space of X where $X^* = \{\text{sets in } \mathcal{P}\}$, $\pi: X \to X^*$, top. on X^* = finest for which π is cont.
2. $p: X \to Y$ quotient map (surj., $U \subset Y$ open $\Leftrightarrow p^{-1}(U) \subset X$ open). Then Y is a quotient space of X where $\mathcal{P} = \{p^{-1}(y)\}$.
3. $p: X \to Y$ surjective, X top space. Equip Y with finest topology for which p is cont. Then Y is a quotient space of X.

Special kinds of quotient maps

Def $f: X \to Y$ is <u>open</u> if $f(\text{open sets})$ are open sets;
<u>closed</u> closed closed.

Prop If $f: X \to Y$ is surjective and either open or closed, then f is a quotient map.

Pf Open: U open in $Y \Rightarrow f^{-1}(U)$ open in $X \Rightarrow U = f(f^{-1}(U))$ open in Y

Closed: C closed in $Y \to f^{-1}(C)$ closed in $X \Rightarrow C = f(f^{-1}(C))$ closed in Y.

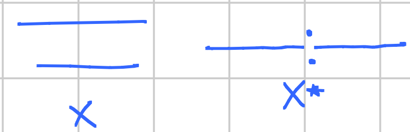
Then U open in $Y \Leftrightarrow Y \setminus U$ closed $\Leftrightarrow Y \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$ closed. $\square$

Cor $f: X \to Y$ surjective, X compact, Y Hausdorff. Then f is a quotient map.

Pf C closed in $X \Rightarrow f(C)$ compact $\Rightarrow f(C)$ closed. $\square$

Rmk If Y = quotient space of X then there is a continuous surjective map $p: X \to Y$, so if X is {compact, connected, path connected} then so is Y. <u>Not true</u> for Hausdorff.

Ex $X = \{0,1\} \times \mathbb{R}$, partition $\{(0,x),(1,x)\}$ $x \neq 0$
$\{(0,0)\}$
$\{(1,0)\}$



Open sets in X^* look like

Examples

1. Cylinder. $X = [0,1] \times [0,1]$, $Y = \{x^2+y^2=1, 0 \le z \le 1\} \subset \mathbb{R}^3$,

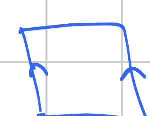
Define $f: X \to Y$ by $f(x,y) = (\cos(2\pi x), \sin(2\pi x), y)$.

This is a quotient map because it's surjective.

Corresponding partition of X: $f^{-1}((1,0,z)) = \{(0,z),(1,z)\}$

$f^{-1}(\text{anything else}) = 1$ point.

$\rightsquigarrow$ quotient space X^* defined by equiv. relation $(0,y) \sim (1,y)$,



and X^* homeo to Y.

2. torus. $X = [0,1]\times[0,1]$ $\quad Y = S^1\times S^1$ $\quad S^1 = \{|z|=1\} \subset \mathbb{C} = \mathbb{R}^2$.

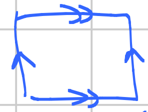
$f: X \to Y \quad f(x,y) = (e^{2\pi i x}, e^{2\pi i y})$

Partition is $\{(0,0),(0,1),(1,0),(1,1)\}$

$\{(x,0),(x,1)\} \quad 0<x<1$

$\{(0,y),(1,y)\} \quad 0<y<1$

$\{(x,y)\} \quad 0<x,y<1$



equiv relation: $(0,y)\sim(1,y)$, $(x,0)\sim(x,1)$

3. If $A\subset X$ is a subset, can define a partition of X by A and $\{$single pts not in $A\}$.

The resulting quotient space is $\underline{X/A}$. ("collapse A to a point")

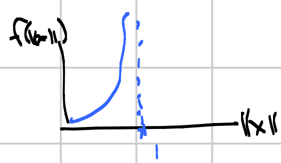
Key ex.: $X = \overline{B}^n$ closed unit ball in $\mathbb{R}^n$, $A = S^{n-1}$ unit sphere.

Claim: $\overline{B}^n/S^{n-1}$ is homeomorphic to S^n.

Want: Surjective map $p: \overline{B}^n \to S^n$ ($\Rightarrow$ quotient map) such that all of S^{n-1} is sent to a point and p is bijective otherwise.

The key: a homeomorphism $B^n \to S^n \setminus \{\text{point}\}$.

$$B^n \xrightarrow[\varphi_1]{\cong} \mathbb{R}^n$$
$$x \longmapsto f(\|x\|)\frac{x}{\|x\|}$$



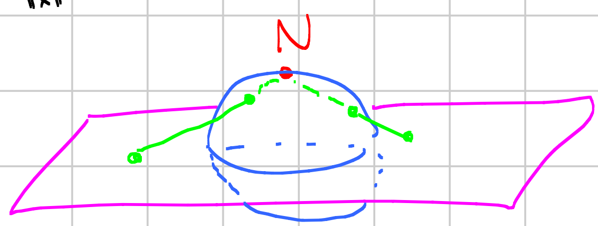
$$S^n \setminus \{\text{north pole}\} \xrightarrow[\varphi_2]{\cong} \mathbb{R}^n$$

N $(0,\dots,0,1)$

unit sphere in $\mathbb{R}^{n+1}$

$$(x_1,\dots,x_{n+1}) \longmapsto \frac{1}{1-x_{n+1}}(x_1,\dots,x_n)$$

$$\frac{2}{1+\|y\|^2}\left(y_1,\dots,y_n,\frac{\|y\|^2-1}{2}\right) \longleftarrow (y_1,\dots,y_n)$$



stereographic projection.

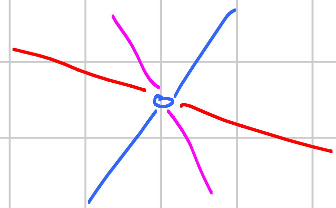
So: $B^n \xrightarrow{\varphi_1} \mathbb{R}^n \xrightarrow{\varphi_2^{-1}} S^n \setminus \{N\}$

Define p by $\quad p(x) = \begin{cases} \varphi_2^{-1}\varphi_1(x) & x\in B^n \\ N & x\in S^{n-1}. \end{cases}$

Then just check that p is continuous.

4. $\underline{\mathbb{RP}^n}$: real projective n-space. 3 constructions as identification spaces X^*

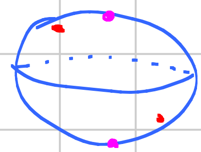
① $X_1 = \mathbb{R}^{n+1} \setminus \{0\}$. Say $\vec{x} \sim \vec{x}' \Leftrightarrow \vec{x}, \vec{x}'$ are collinear i.e. $\vec{x}' = \lambda \vec{x}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.



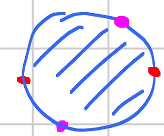
Equivalence classes are $\{\lambda \vec{x} \mid \lambda \in \mathbb{R} \setminus \{0\}\}$.

② $X_2 = S^n \ (\subset \mathbb{R}^{n+1})$. Say $\vec{x} \sim -\vec{x}$.

Equiv classes are $\{x, -x\}$.



③ $X_3 = \bar{B}^n \ (\subset \mathbb{R}^n)$. Say for $x \in S^{n-1} = \partial \bar{B}^n$, $x \sim -x$.

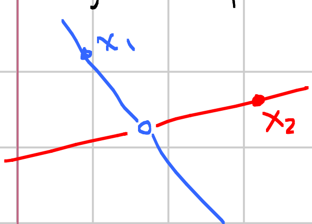


Equiv classes are $\{x\}$ $(x \in B^n)$ and $\{x, -x\}$ $(x \in S^{n-1})$.

$\underline{\text{Claim}}$ Write X_1^*, X_2^*, X_3^* for the quotient spaces. Then X_1^*, X_2^*, X_3^* are homeomorphic. Call these all $\mathbb{RP}^n$.

$\underline{\text{Lemma}}$ X_1^* is Hausdorff.

$\underline{\text{Pf}}$ Suppose $y_1, y_2 \in X_1^*$ and choose $x_1, x_2 \in X_1 = \mathbb{R}^{n+1} \setminus \{0\}$ such that $y_1, y_2 =$ equiv. classes of x_1, x_2. Then the angle between x_1 and x_2 is $\neq 0, \pi$.



Let $U_1 = \{$points $\in X_1$ at angle $< \epsilon$ or $> \pi - \epsilon$ from $x_1\}$

$U_2 = \{ \qquad \qquad \qquad x_2\}$

For small $\epsilon > 0$, $U_1 \cap U_2 = \emptyset$. Also U_1, U_2 are $\pi^{-1}(V_1, V_2)$ for sets V_1, V_2 in X_1^* (if $x \in U_1$ then the entire equivalence class of x is in U_1), and so V_1, V_2 are open sets containing y_1, y_2 and $V_1 \cap V_2 = \emptyset$. $\square$

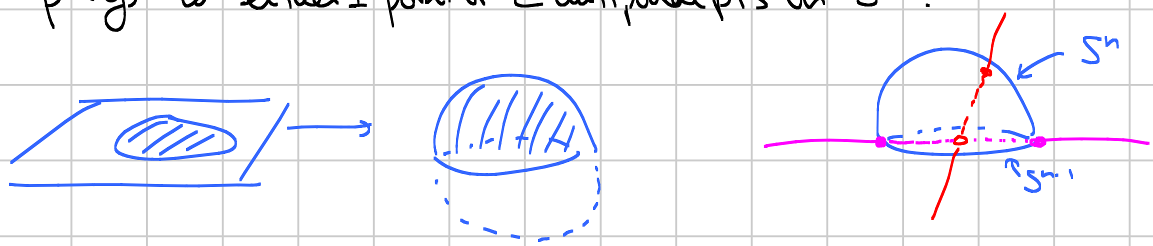
$\underline{\text{Pf of claim}}$ $X_1^* \cong X_2^*$: define $p: X_2 \to X_1^*$ by $\underset{S^n}{\overset{}{X_2}} \to \underset{\mathbb{R}^{n+1} \setminus 0}{X_1} \xrightarrow{\pi} X_1^*$.

X_2 cpt, p surjective, X_1^* Hausdorff $\Rightarrow p$ is a quotient map.

$p^{-1}(y)$ is 2 points $\pm x$ in S^n so the partition of X_2 given by p is the same as the one giving X_2^* $\Rightarrow$ X_2^* is homeo to X_1^* by Prop.

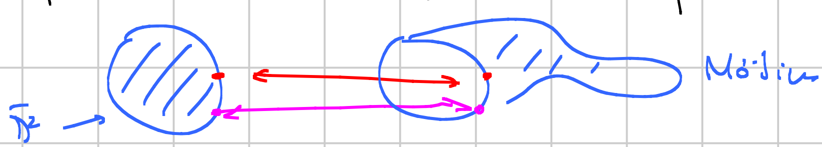
$X_1^* \cong X_3^*$: define $p: X_3 \to X_1^*$ by $X_3 = \bar{B}^n \to S^n \to \mathbb{R}^{n+1}\setminus\{0\} \to X_1^*$
$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,\sqrt{1-x_1^2-\dots-x_n^2})$

X_3 cpt, X_1^* Hausdorff, p surjective
and $p^{-1}(y)$ is either 1 point or 2 antipodal pts on S^{n-1}.



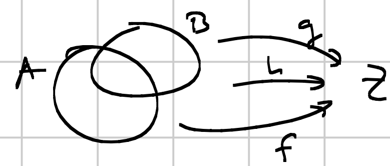
End of material for 2nd midterm ↑

<u>Fun fact</u>: $\mathbb{RP}^2 = \bar{D}^2 \cup$ Möbius strip
where we view $\bar{D}^2 \cup$ Möbius as an identification space: both $\bar{D}^2$ and Möbius have boundary S^1 and we identify these circles pointwise.



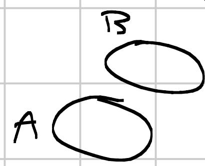
<u>Gluing maps</u> - application of quotient topology.

Suppose $X = A \cup B$ and suppose we have continuous maps $f: A \to Z$, $g: B \to Z$. If $f(x) = g(x)$ for all $x \in A \cap B$, then we can define the glued map $h: X \to Z$: $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$.



Question: is h necessarily continuous? (no)

Define a space $A \amalg B$



(topology induced from A, B; alternatively,
$$A \amalg B = \{(x,0) \mid x \in A\} \cup \{(x,1) \mid x \in B\} \subset X \times [0,1].)$$

and define $j: A \amalg B \to X$ by $j(x) = \begin{cases} x, & x \in A \\ x, & x \in B \end{cases}$.

This is continuous : $j^{-1}(U) = (U \cap A) \amalg (U \cap B)$ ← open ↑

<u>Prop</u> Suppose $f: A \to Z$, $g: B \to Z$ are continuous with $f(x) = g(x)$ $\forall x \in A \cap B$. If j is a quotient map then $h: X \to Z$ is continuous.

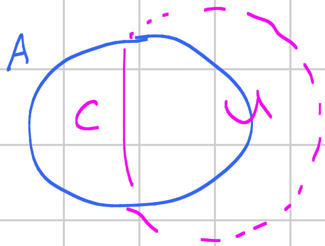
PF f, g continuous $\Rightarrow$ $k: A \sqcup B \to Z$ defined by $k(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$ is continuous. But $k = h \circ j$:

$$\begin{tikzcd} A \sqcup B \arrow[r, "j"] \arrow[rr, bend right, "k"'] & X \arrow[r, "h"] & Z \end{tikzcd}$$

and k is continuous $\Leftrightarrow$ h is continuous. $\square$

Cor ("Pasting Lemma" Thm 18.3) If A, B are both closed (or both open) in X, $f: A \to Z$, $g: B \to Z$ continuous, and $f(x) = g(x)$ $\forall\, x \in A \cap B$, then $h: X \to Z$ is continuous.

PF A, B closed $\Rightarrow$ j sends closed sets to closed sets: if $C \subset A$ is closed then $A \setminus C$ open in A $\Rightarrow$ $A \setminus C = A \cap U$ for open $U \subset X \Rightarrow$



$C = (X \setminus U) \cap A$ is closed $\subset X$.

Thus j is a quotient map. $\square$

Ex Gluing paths. Suppose $\gamma_1, \gamma_2: [0,1] \to X$ are paths with $\gamma_1(1) = \gamma_2(0)$. Then $\gamma: [0,1] \to X$ $\quad \gamma(t) = \begin{cases} \gamma_1(t) & t \le \frac{1}{2} \\ \gamma_2(2t-1) & t \ge \frac{1}{2} \end{cases}$ is continuous:

$[0,1] \xrightarrow{\gamma_1} X \qquad [1,2] \xrightarrow{t \mapsto \gamma_2(t-1)} X \rightsquigarrow [0,1] \longrightarrow [0,2] \longrightarrow X$.

$t \longmapsto 2t \qquad t \longmapsto \begin{cases} \gamma_1(t) & t \le 1 \\ \gamma_2(t-1) & t \ge 1 \end{cases}$.

Space-Filling Curve

$I=[0,1]$

<u>Thm</u> There is a <u>continuous</u> map $f: I \to I \times I$ that is <u>surjective</u>. "Peano space-filling curve"

<u>Def</u> (X,d) metric space. A <u>Cauchy sequence</u> is $\{x_n\}$ such that for any $\epsilon > 0$ there is N such that whenever $m,n \geq N$, $d(x_m, x_n) < \epsilon$.

<u>Def</u> A metric space (X,d) is <u>complete</u> if any Cauchy sequence $\{x_n\}$ converges to some $x \in X$: $\lim_{n\to\infty} d(x_n, x) = 0$.

<u>Thm</u> $\mathbb{R}$ is complete. (essentially equivalent to least upper bound property)

<u>Cor</u> $I \times I$ is complete $\qquad d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Pf: If $\{(x_n,y_n)\}$ is Cauchy then for any ϵ there is N s.t.

$m,n \geq N \Rightarrow \underbrace{d((x_m,y_m),(x_n,y_n))}_{\geq |x_m - x_n|} < \epsilon$ so $\{x_n\}$ is Cauchy $\Rightarrow x_n \to x \Rightarrow$

and similarly $\{y_n\}$ is Cauchy $\Rightarrow y_n \to y$.

$x_n \in I \Rightarrow x \in I$ since I is closed, and similarly $y \in I$. $\square$

<u>Prop</u> X, Y metric spaces, Y complete. Define $C(X,Y) = \{\text{continuous } X \to Y\}$.

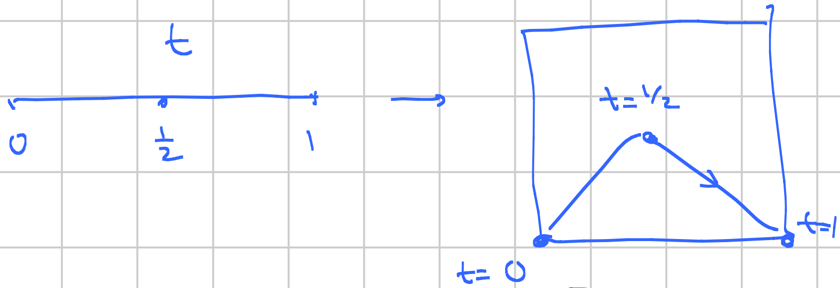
metric ρ_∞: $\quad f,g \in C(X,Y) \to \rho_\infty(f,g) = \sup_{x\in X} d(f(x),g(x))$.

Then $(C(X,Y), \rho_\infty)$ is complete.

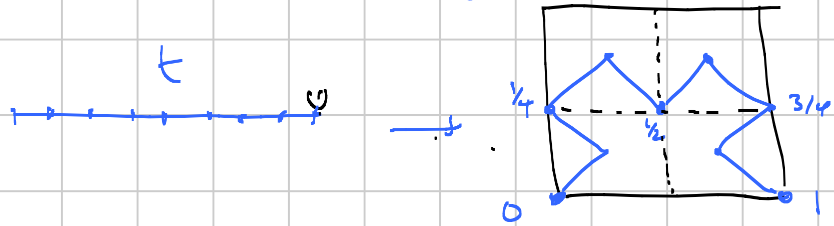
<u>Cor</u> $C([0,1],[0,1]^2)$ is complete.

Now we'll construct a sequence of functions $f_n : [0,1] \to [0,1]^2$ such that f_n is continuous and $\{f_n\}$ is Cauchy $\Rightarrow$ $f_n \to f$ for some continuous $f: [0,1] \to [0,1]^2$.

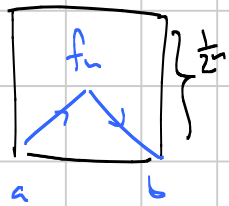
f_0:



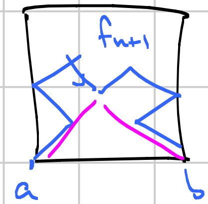
f_1:



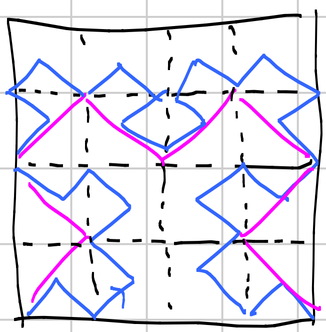
Each step replaces



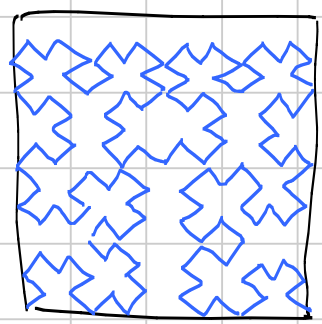
by



So: f_2:



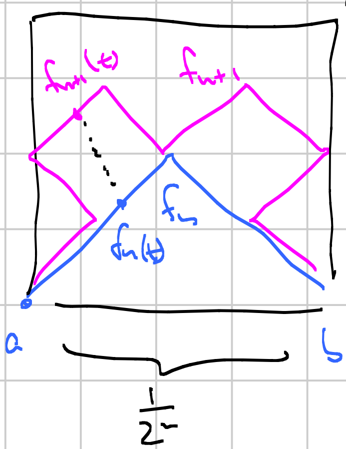
f_3:



etc.

Claim 1: $\{f_n\}$ is Cauchy in $C([0,1],[0,1]^2)$.

Note: $f_n(t)$, $f_{n+1}(t)$ are both in a $\frac{1}{2^n}\times\frac{1}{2^n}$ square



So

$$\rho(f_n(t), f_{n+1}(t)) \le \frac{\sqrt{2}}{2^n} \text{ for all } t.$$

Thus $\boxed{\rho_\infty(f_n, f_{n+1}) \le \frac{\sqrt{2}}{2^n}}$.

Consequence: $\rho_\infty(f_m, f_n) \le \frac{\sqrt{2}}{2^{m-1}}$ if $m<n$

$\Rightarrow$ $\{f_n\}$ is Cauchy $\rightarrow$ $f_n \to f$ continuous.

Claim 2 f is onto.

Choose any $(x,y) \in [0,1] \times [0,1]$.

Note that f_n maps into each $\frac{1}{2^n} \times \frac{1}{2^n}$ square.

For each n, (x,y) sits in one of these squares.

So there is some t_n such that $\rho\left(f_n(t_n), (x,y)\right) \le \frac{\sqrt{2}}{2^n}$.

The sequence $\{t_n\}$ is bounded, so by Bolzano-Weierstrass it has a convergent subsequence $\{t_{n_k}\} \to t$.

Then $f(t) = \lim f(t_{n_k}) = \lim f_{n_k}(t_{n_k}) = (x,y)$. $\square$