Math 431 Homework 10—due November 14

Fall 2017

This problem set consists of one problem about the Contraction Mapping Principle. The problem is basically 5.7 #8 on p. 209, but I've filled in some detail. The problem set is due on November 14 but there won't be a quiz on that day.

1. Let $\mathcal{M}$ denote
$$\mathcal{M} = \{f \in C(\mathbb{R}) \, : \, |f(x)| \leq 1 \text{ for all } x\}.$$
Consider $\mathcal{M}$ as a metric space equipped with the sup norm ρ_∞.

   (a) Prove that $(\mathcal{M}, \rho_\infty)$ is a complete metric space.

   (b) Define the map T on $\mathcal{M}$ by
$$T(\psi)(x) = \frac{1}{2}\cos(x) + \frac{1}{2}\int_{x-\frac{1}{4}}^{x+\frac{1}{4}} (\sin(x-y))(\psi(y))^2 \, dy.$$
Show that T maps $\mathcal{M}$ to $\mathcal{M}$; that is, if $\psi \in \mathcal{M}$ then $T(\psi) \in \mathcal{M}$.

   Please note: for the sake of making the problem simpler, you may assume that if ψ is continuous then $T(\psi)$ is continuous. It's possible to prove this using the techniques that you know—essentially, the proof is a souped-up version of the proof of continuity of ψ_{n+1} on p. 184 in the proof of Theorem 5.4.1—but you don't have to do it here. (The reason that this setting is more complicated than the setting of Theorem 5.4.1 is that here, both the integrand *and* the limits of integration depend on x.)

   (c) Show that $T : \mathcal{M} \to \mathcal{M}$ is a contraction.

   (d) Show that there is a unique $\psi \in \mathcal{M}$ satisfying the following integral equation:
$$\psi(x) = \frac{1}{2}\cos(x) + \frac{1}{2}\int_{x-\frac{1}{4}}^{x+\frac{1}{4}} (\sin(x-y))(\psi(y))^2 \, dy.$$