Math 612 - Final Exam - Solutions Outline

Note Title 12/4/2014

1. UCT + naturality $\Rightarrow$ $f^* : \underset{\mathbb{Z}}{\underset{\wr\wr}{H^2(\mathbb{CP}^1;\mathbb{Z})}} \longrightarrow \underset{\mathbb{Z}}{\underset{\wr\wr}{H^2(\mathbb{CP}^1;\mathbb{Z})}}$ is mult. by d.

Commutative diagram

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "f"] \arrow[d, "i"'] & \mathbb{CP}^1 \arrow[d, "i"] \\ \mathbb{CP}^n \arrow[r, "f"] & \mathbb{CP}^n \end{tikzcd}$$

$i[z_0, z_1] = [z_0, z_1, 0, \dots, 0]$

$\Rightarrow$ if α generates $H^2(\mathbb{CP}^n;\mathbb{Z})$ then $f^*(\alpha) = d\cdot\alpha$.

$\Rightarrow$ $f^* : \underset{\mathbb{Z}[\alpha]/(\alpha^{n+1})}{\underset{\wr\wr}{H^*(\mathbb{CP}^n;\mathbb{Z})}} \circlearrowleft$ sends $\alpha^k \longmapsto d^k\cdot\alpha^k$.

2. (a) $a : S^n \to S^n$ antipodal map, n odd $\Rightarrow a^*\omega = \omega$.
Cover S^n by $\{U_\alpha\}$ with $U_\alpha \cap a(U_\alpha) = \emptyset$ $\forall\,\alpha$. Then $\mathbb{RP}^n$ is covered by $\{\pi(U_\alpha)\}$.

$\pi(U_\alpha) \to V_\alpha$ Coord charts in $\mathbb{RP}^n$

$U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha$, $\omega \rightsquigarrow \omega_\alpha \in \Omega^n(V_\alpha)$.

Define $\eta \in \Omega^n(\mathbb{RP}^n)$ by $\eta_\alpha = \omega_\alpha \in \Omega^n(V_\alpha)$.
The transition functions for $\mathbb{RP}^n$ are either $\varphi_\alpha \circ \varphi_\beta^{-1}$ or $\varphi_\alpha \circ a \circ \varphi_\beta^{-1}$; either way, η_α pulls back to η_β since ω_α pulls back to ω_β.

(b) η is nowhere 0 since $p^*\eta = \omega$ and ω is nowhere 0
$\Rightarrow$ η is a volume form.

(c) $H^n_{DR}(\mathbb{RP}^n) \cong \mathbb{R}$ by Poincaré duality, and
$\int_{\mathbb{RP}^n} \eta > 0$ $\Rightarrow$ $[\eta]$ generates.

3. (c) $H(K)=4K/16K$, $H(\mathcal{F}^1K)=4K/32K$, $H(\mathcal{F}^2K)=4K/0$,
$H(\mathcal{F}^3K)=8K/0$, $H(\mathcal{F}^4K)=16K/0$, $H(\mathcal{F}^5K)=32K/0$

$$\begin{tikzcd}
 & & H(K) \arrow[d, equal] & H(\mathcal{F}^1K) \arrow[d, equal] & H(\mathcal{F}^2K) \arrow[d, equal] & H(\mathcal{F}^3K) \arrow[d, equal] & H(\mathcal{F}^4K) \arrow[d, equal] & H(\mathcal{F}^5K) \arrow[d, equal] & \\
 & & 4K/16K \arrow[d, equal] & 4K/32K \arrow[l, "\times1"'] \arrow[d, equal] & 4K \arrow[l, "\times1"'] \arrow[d, equal] & 8K \arrow[l, "\times1"'] \arrow[d, equal] & 16K \arrow[l, "\times1"'] \arrow[d, equal] & 32K \arrow[l, "\times1"'] \arrow[d, equal] & \\
A_1: & \cdots & \mathbb{Z}/4 \arrow[l, "\cong"'] \arrow[d, "0"'] & \mathbb{Z}/8 \arrow[l, "\times1"'] \arrow[d, "0"'] & \mathbb{Z}/16 \arrow[l, "\times1"'] \arrow[d, "\times1"'] & \mathbb{Z}/8 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/4 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & 0 \arrow[l] \\
E_1: & & \mathbb{Z}/2 \arrow[ur, "\times4"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[ur, "\times8"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[ur, "0"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[ur, "0"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[ur, "0"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 & \\
 & & K/2K \arrow[u, equal] & 2K/4K \arrow[u, equal] & 4K/8K \arrow[u, equal] & 8K/16K \arrow[u, equal] & 16K/32K \arrow[u, equal] & 32K \arrow[u, equal] & \\
 & & H(K/\mathcal{F}^1) \arrow[u, equal] & H(\mathcal{F}^1/\mathcal{F}^2) \arrow[u, equal] & H(\mathcal{F}^2/\mathcal{F}^3) \arrow[u, equal] & H(\mathcal{F}^3/\mathcal{F}^4) \arrow[u, equal] & H(\mathcal{F}^4/\mathcal{F}^5) \arrow[u, equal] & H(\mathcal{F}^5) \arrow[u, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & H(K) & iH(\mathcal{F}^1) & iH(\mathcal{F}^2) & iH(\mathcal{F}^3) & iH(\mathcal{F}^4) & iH(\mathcal{F}^5) & \\
A_2: & \cdots & \mathbb{Z}/4 \arrow[l] \arrow[d, "0"'] & \mathbb{Z}/4 \arrow[l, "\times1"'] \arrow[d, "0"'] & \mathbb{Z}/8 \arrow[l, "\times1"'] \arrow[d, "\times1"'] & \mathbb{Z}/8 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/4 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & 0 \arrow[l] \\
E_2: & & \mathbb{Z}/2 \arrow[urr, "\times4"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[urr, "\times4"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[urr, "0"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[urr, "0"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & H(K) & iH(\mathcal{F}^1) & i^2H(\mathcal{F}^2) & i^2H(\mathcal{F}^3) & i^2H(\mathcal{F}^4) & i^2H(\mathcal{F}^5) & \\
A_3: & \cdots & \mathbb{Z}/4 \arrow[l] \arrow[d, "0"'] & \mathbb{Z}/4 \arrow[l, "\times1"'] \arrow[d, "0"'] & \mathbb{Z}/4 \arrow[l, "\times1"'] \arrow[d, "\times1"'] & \mathbb{Z}/4 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/4 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & 0 \arrow[l] \\
E_3: & & \mathbb{Z}/2 \arrow[urrr, "\times2"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[urrr, "\times2"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[urrr, "0"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & H(K) & iH(\mathcal{F}^1) & i^2H(\mathcal{F}^2) & i^3H(\mathcal{F}^3) & i^3H(\mathcal{F}^4) & i^3H(\mathcal{F}^5) & \\
A_4: & \cdots & \mathbb{Z}/4 \arrow[l] \arrow[d, "0"'] & \mathbb{Z}/4 \arrow[l, "\times1"'] \arrow[d, "0"'] & \mathbb{Z}/4 \arrow[l, "\times1"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times1"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times0"'] \arrow[d, "\times1"'] & 0 \arrow[l] \\
E_4: & & \mathbb{Z}/2 \arrow[urrrr, "\times1"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[urrrr, "\times1"] \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & H(K) & iH(\mathcal{F}^1) & i^2H(\mathcal{F}^2) & i^3H(\mathcal{F}^3) & i^4H(\mathcal{F}^4) & i^4H(\mathcal{F}^5) & \\
A_5: & \cdots & \mathbb{Z}/4 \arrow[l] & \mathbb{Z}/4 \arrow[l, "\times1"'] & \mathbb{Z}/4 \arrow[l, "\times1"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times2"'] \arrow[d, "\times1"'] & \mathbb{Z}/2 \arrow[l, "\times1"'] & 0 \arrow[l] & 0 \arrow[l] \\
E_5: & & & & \mathbb{Z}/2 \arrow[r, phantom, "\oplus"] & \mathbb{Z}/2 & & & = E_\infty
\end{tikzcd}$$

3. (a) Cont'd.

$$\boxed{E_1 \cong E_2 \cong E_3 \cong E_4 \cong (\mathbb{Z}/2)^6, \quad E_5 \cong \cdots \cong E_\infty \cong (\mathbb{Z}/2)^2}$$

(b) $E_\infty \cong (\mathbb{Z}/2)^2$, $H(K) \cong \mathbb{Z}/4 \Rightarrow E_\infty \not\cong H(K)$.

One possibility for another filtration:

$\mathcal{F}^0 = K$, $\mathcal{F}^1 = 16K$, $\mathcal{F}^2 = 0$.

$$\begin{tikzcd}[row sep=small]
 & H(K) \arrow[d, equal] & H(\mathcal{F}^1) \arrow[d, equal] & H(\mathcal{F}^2) \\
 & 4K/16K \arrow[d, equal] & 16K \arrow[l, "\times 1"'] \arrow[d, equal] & \\
A_1: \quad \cdots & \mathbb{Z}/4 \arrow[l] & \mathbb{Z}/4 \arrow[l, "0"'] & 0 \arrow[l]
\end{tikzcd}$$

$\Rightarrow E_\infty \cong \mathrm{Gr}\, H(K) \cong \mathbb{Z}/4 \oplus 0 = \mathbb{Z}/4 \cong H(K)$.

4. Leray: E_2:
$$\begin{tikzcd}
\mathbb{Z} \arrow[drr, "d_2"] & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z}
\end{tikzcd}$$
$d_2 = n$ for some $n \in \mathbb{Z}$.

$\Rightarrow E_\infty = E_3 = \boxed{\begin{matrix} \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{matrix}}$ if $n=0$, $\boxed{\begin{matrix} 0 & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z}/n \end{matrix}}$ if $n \neq 0$.

In all cases $H^*(M;\mathbb{Z}) \cong \mathrm{Gr}\, H^*(M;\mathbb{Z})$ since $\forall k$, $\mathrm{Gr}\, H^k(M;\mathbb{Z})$ has at most one nonzero summand.

Also get $H_*(M)$ from $H^*(M;\mathbb{Z})$ by UCT: $H_k(M) = F_k \oplus T_k$, $H^k(M;\mathbb{Z}) = F_k \oplus T_{k-1}$.

(Note UCT applies because $H_*(M)$ is finitely generated, since M is compact: not hard to show any sequence in M has a convergent subsequence.)

- $H^* = \overset{0}{\mathbb{Z}} \oplus \overset{1}{\mathbb{Z}} \oplus \overset{2}{\mathbb{Z}} \oplus \overset{3}{\mathbb{Z}}$, $\quad H_* = \overset{0}{\mathbb{Z}} \oplus \overset{1}{\mathbb{Z}} \oplus \overset{2}{\mathbb{Z}} \oplus \overset{3}{\mathbb{Z}}$: example is $M = S^1 \times S^2$
- $H^* = \overset{0}{\mathbb{Z}} \oplus \overset{1}{0} \oplus \overset{2}{0} \oplus \overset{3}{\mathbb{Z}}$, $\quad H_* = \overset{0}{\mathbb{Z}} \oplus \overset{1}{0} \oplus \overset{2}{0} \oplus \overset{3}{\mathbb{Z}}$: example is S^3 (Hopf)

(more on next page!)

4. Cont'd.

- $H^* = \overset{0}{\mathbb{Z}} \oplus \overset{1}{0} \oplus \overset{2}{\mathbb{Z}/n} \oplus \overset{3}{\mathbb{Z}}$, $H_* = \overset{0}{\mathbb{Z}} \oplus \overset{1}{\mathbb{Z}/n} \oplus \overset{2}{0} \oplus \overset{3}{\mathbb{Z}}$: example is $L(n,1)$:

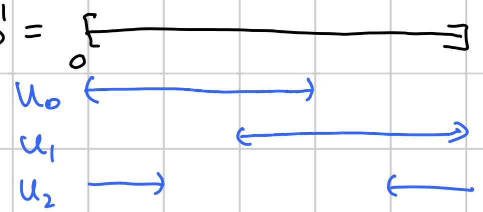
  $(n \geq 2)$

  $$L(n,1) = \underbrace{\{|z|^2+|w|^2=1\}}_{S^3} \Big/ \big((z,w)\sim(\zeta z, \zeta w)\big), \quad \zeta = e^{2\pi i/n}.$$

  Hopf map $S^3 \to \mathbb{CP}^1$ descends to $L(n,1) \to \mathbb{CP}^1$ and the fibers are $\{|\lambda|=1\}/(\lambda \sim \zeta^k \lambda \ \forall k) = S^1$.

  (In fact any $L(p,q)$ has an S^1 fibration over S^2!)

5. (a) Usual good cover $S^1 = [0,1]/\sim$

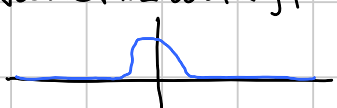


For each $U = U_\alpha$ or $U_{\alpha\beta}$, $\pi^{-1}(U) \cong U \times \mathbb{R}$ and Poincaré says $H^*_{cv}(\pi^{-1}(U)) \cong H^{*-1}(U) \cong \begin{cases} \mathbb{R} & *=1 \\ 0 & \text{else} \end{cases}$.

So $\mathcal{H}^*_{cv} = 0$ for $* \neq 1$.

For $*=1$, let $f \in C^\infty_c(\mathbb{R})$ be a symmetric bump function with $\int f = 1$.

Then $H^1_{cv}(\pi^{-1}(U))$ is generated by $[f(y)\,dy]$ for all U except U_2, while $H^1_{cv}(\pi^{-1}(U_2))$ is generated by $[\tilde{f}(y)\,dy]$, $\tilde{f}(y) = \begin{cases} f(y) & y < \frac{1}{2} \\ -f(y) & y > \frac{1}{2} \end{cases}$.



So we get

$$\begin{tikzcd} & \mathcal{H}^1_{cv}(U_0) \arrow[dl] \arrow[dr] & \\ \mathcal{H}^1_{cv}(U_{01}) & & \mathcal{H}^1_{cv}(U_{02}) \\ \mathcal{H}^1_{cv}(U_1) \arrow[u] \arrow[r] & \mathcal{H}^1_{cv}(U_{12}) & \mathcal{H}^1_{cv}(U_2) \arrow[l] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} & \mathbb{R} \arrow[dl, "1"'] \arrow[dr, "1"] & \\ \mathbb{R} & & \mathbb{R} \\ \mathbb{R} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{R} & \mathbb{R} \arrow[l, "-1"'] \arrow[u, "1"'] \end{tikzcd}.$$

5. (b) $H^*_{cv}(E) \cong H^{*-1}(\mathcal{U}, \mathcal{H}^1_{cv})$.

Cont'd

Easy to check: $H^*(\mathcal{U}, \mathcal{H}^1_{cv}) = 0 \quad \forall *$

$\Rightarrow H^*_{cv}(E) = 0$.

Not orientable since $H^*_{cv}(E) \not\cong H^{*-1}(S^1)$.

(c) Consider the map $\varphi: [0,1] \times \mathbb{R} \to \mathbb{R}^3$ defined by $\varphi(\theta, t) = (\cos \pi\theta, \sin \pi\theta, t)$. Note the image of this map avoids the line $x=y=0$, and $\varphi(0,t) = (1,0,t) = -(-1,0,-t) = -\varphi(1,-t)$.

If we define $\tilde{p}: \mathbb{R}^3 - 0 \to \mathbb{RP}^2$ (restricting to $\tilde{p}: \mathbb{R}^3 - \{x=y=0\} \to U$) in the obvious way, then $\tilde{p} \circ \varphi$ descends to a map $E \to U$.

It's straightforward to check this is a diffeomorphism and sends the zero section to the equator.

(d) M-V:

$$\begin{array}{ccccccc}
\to H^2_c(U \cap V) & \xrightarrow{\cong} & H^2_c(U) \oplus H^2_c(V) & \to & H^2_c(\mathbb{RP}^2) & \to & 0 \\
\mathbb{R} & & 0 \oplus \mathbb{R} & & & & \\
\to H^1_c(U \cap V) & \to & H^1_c(U) \oplus H^1_c(V) & \to & H^1_c(\mathbb{RP}^2) & & \\
\mathbb{R} & & 0 \oplus 0 & & & & \\
0 \to H^0_c(U \cap V) & \to & H^0_c(U) \oplus H^0_c(V) & \to & H^0_c(\mathbb{RP}^2) & & \\
0 & & 0 \oplus 0 & & & &
\end{array}$$

- $U \cap V \cong S^1 \times \mathbb{R} \Rightarrow H^*_c(U \cap V) \cong H^{*-1}(S^1)$
- $H^*_c(U) \cong H^*_c(E) \cong H^*_{cv}(E)$
- $V \cong \mathbb{R}^2 \Rightarrow H^*_c(V) \cong H^{*-2}_c(pt)$

The map $\underset{\mathbb{R}}{H^2_c(U \cap V)} \xrightarrow{i_*} \underset{\mathbb{R}}{H^2_c(V)}$ is onto: sends a bump form to a bump form: note $U \cap V \cong S^1 \times \mathbb{R}$, $V \cong \mathbb{R}^2$ are orientable.

$\Rightarrow$
$$\boxed{\begin{aligned} H^0(\mathbb{RP}^2) &= H^0_c(\mathbb{RP}^2) \cong \mathbb{R} \\ H^1(\mathbb{RP}^2) &= H^1_c(\mathbb{RP}^2) = 0 \\ H^2(\mathbb{RP}^2) &= H^2_c(\mathbb{RP}^2) = 0 \end{aligned}}$$