Math 621 HW 1 – Outline of Solutions

Note Title 1/24/2018

1. For $(z_1,\dots,z_{n+1}) \in \mathbb{C}^{n+1} - 0$, write $[z_1,\dots,z_{n+1}]$ for the equivalence class of $(z_1,\dots,z_{n+1})$ in $\mathbb{CP}^n$. $\mathbb{CP}^n$ is covered by

$$V_j = \{[z_1,\dots,z_{n+1}] \mid z_j \neq 0\} \subset \mathbb{CP}^n, \quad 1 \le j \le n+1.$$

We have coord charts

$$f_j : U_j = \mathbb{C}^n \cong \mathbb{R}^{2n} \longrightarrow V_j \subset \mathbb{CP}^n$$

$$(x_1,y_1,\dots,x_n,y_n) \longmapsto [x_1+iy_1,\dots,x_{j-1}+iy_{j-1},1,x_j+iy_j,\dots,x_n+iy_n]$$

(think of this as $(z_1,\dots,z_n)$ where $z_k = x_k+iy_k$)

We'll calculate transition function $f_j^{-1}\circ f_i$ for $i>j$ (note not $\sqrt{-1}$) ($i<j$ is similar).

$$f_j^{-1}\circ f_i : \underset{\{(x_1,y_1,\dots,x_n,y_n) \mid (x_j,y_j)\neq(0,0)\}}{\underset{\shortparallel}{f_i^{-1}(V_i\cap V_j)}} \longrightarrow \underset{\{(x_1,y_1,\dots,x_n,y_n) \mid (x_{i-1},y_{i-1})\neq(0,0)\}}{\underset{\shortparallel}{f_j^{-1}(V_i\cap V_j)}}$$

$$\begin{aligned}
f_j^{-1}\circ f_i(x_1,y_1,\dots,x_n,y_n) &= f_j^{-1}\big([x_1+iy_1,\dots,x_{i-1}+iy_{i-1},1,x_i+iy_i,\dots,x_n+iy_n]\big)\\
&= \frac{1}{x_j+iy_j}\big(x_1+iy_1,\dots,x_{j-1}+iy_{j-1},x_{j+1}+iy_{j+1},\dots,x_{i-1}+iy_{i-1},1,\\
&\qquad x_i+iy_i,\dots,x_n+iy_n\big) \quad (\in \mathbb{C}^n)\\
&= \frac{1}{x_j^2+y_j^2}\big(x_1x_j+y_1y_j,\ x_jy_1-x_1y_j,\ \dots,\ x_{j-1}x_j+y_{j-1}y_j,\ x_jy_{j-1}-x_{j-1}y_j,\\
&\qquad x_{j+1}x_j+y_{j+1}y_j,\ x_jy_{j+1}-x_{j+1}y_j,\ \dots,\ x_{i-1}x_j+y_{i-1}y_j,\ x_jy_{i-1}-x_{i-1}y_j,\\
&\qquad x_j,\ -y_j,\ x_ix_j+y_iy_j,\ x_jy_i-x_iy_j,\ \dots,\\
&\qquad x_nx_j+y_ny_j,\ x_jy_n-x_ny_j\big).
\end{aligned}$$

Note this is smooth for $(x_j,y_j)\neq(0,0)$.

2. (a) Cover $\mathbb{RP}^2$ by three coord. charts $\mathbb{R}^2 \to \mathbb{RP}^2$

$(x,y) \mapsto [x,y,1]$
$[x,1,y]$
$[1,x,y]$

The first chart f_1 gives

$$F \circ f_1(x,y) = \frac{1}{1+x^2+y^2}(x^2-y^2, xy, x, y).$$

The derivative matrix for this is

$$\frac{1}{(1+x^2+y^2)^2}\begin{pmatrix} 2x(1+2y^2) & y(1-x^2+y^2) & 1-x^2+y^2 & -2xy \\ -2y(1+2x^2) & x(1+x^2-y^2) & -2xy & 1+x^2-y^2 \end{pmatrix}$$

and it's straightforward to check this has rank 2 $\forall (x,y)$; similarly for the other two charts.

(b) Show directly that $F(x,y,z) = F(x',y',z') \Rightarrow (x,y,z) = \pm(x',y',z')$ case-by-case, depending on which of x,y,z,x',y',z' are zero and which are nonzero.

3. (a) $\Leftarrow$: Given an oriented atlas for M such that all φ_g's preserve orientation, construct an atlas for M/G as in class (or Ex 0.4.8). Tracing through the construction, we see that the transition functions on M/G are of the form

$h_q^{-1} \circ h_p = f_q^{-1} \circ \varphi_g \circ f_p$, and since φ_g preserves orientation, by definition $\det d(h_q^{-1} \circ h_p) > 0$. Thus this atlas for M/G is oriented.

$\Rightarrow$: Suppose M/G has an oriented atlas $\{(f_\alpha, U_\alpha, V_\alpha)\}$. For any $p \in M$, let $V_p \subset M$ be a nbd. of p as in the def. of properly discontinuous ($V_p \cap \varphi_g(V_p) = \emptyset\ \forall g \neq id$). Choose α such that $\pi(p) \in V_\alpha$. Then π is a bijection

$$V_p \cap \pi^{-1}(V_\alpha) \longrightarrow \pi(V_p) \cap V_\alpha$$

3.(a) Cont'd. and we can define a chart

$$\pi^{-1}\circ f_\alpha : \underset{\substack{\cap \\ U_\alpha \subset \mathbb{R}^n}}{f_\alpha^{-1}(\pi(V_p))} \longrightarrow \underset{\substack{\cap \\ M}}{V_p \cap \pi^{-1}(V_\alpha)} .$$

The collection of all such charts over all $p \in M$ gives an atlas for M, and this is oriented since the atlas for M/G is oriented. Furthermore, straightforward arrow chasing shows that φ_g preserves orientation $\forall\ g \in G$.

(b), (c). $\underline{\mathbb{RP}^n}$: The antipodal map $a: S^n \to S^n$ satisfies

$$\pi_S \circ a \circ \pi_N^{-1}(x) = -x \qquad \forall\ x \in \mathbb{R}^n$$

stereographic projection

Since $\{\pi_N^{-1}, \pi_S^{-1}\circ h\}$ is an oriented atlas for S^n, and

$\det d\big((\pi_S^{-1}\circ h)^{-1}\circ a \circ \pi_N^{-1}\big) = \det d\big(h \circ (\pi_S \circ a \circ \pi_N^{-1})\big) = (-1)^{n+1}$,

a is orientation preserving if and only if n is odd.

Thus by (a), $\mathbb{RP}^n$ is orientable $\Leftrightarrow$ n is odd.

$\underline{\text{Möbius band}}$: $\text{Möbius} = (\mathbb{R}\times(0,1))/(\text{group generated by } \varphi)$

where $\varphi(x,y) = (x+1, 1-y)$.

Since $\det(d\varphi) = -1$, φ is orientation reversing, so by (a), the Möbius band is nonorientable.

4. (a) For the usual atlas $\{f_i : \mathbb{R}^n \to \mathbb{RP}^n\}$ with $f_i(y_1,\dots,y_n) = [y_1,\dots,y_{i-1},1,y_i,\dots,y_n]$ and $i > j$, we have :

4. (a) Cont'd.

$y_1 \cdots y_n \quad f_1 \cdots f_m$

$f: \mathbb{R}^n \to \mathbb{R}^m \quad \begin{bmatrix} df_1 \\ df_m \end{bmatrix}$

$$d(f_j^{-1}\circ f_i) = \begin{array}{r} \\ \\ \\ j-1\to \\ j\to \\ \\ i-2\to \\ i-1\to \\ i\to \\ \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccccccccc} & & j-1 & j & j+1 & & i-1 & i & & \\ & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \end{array} \\
\left[\begin{array}{cccccccccc}
\frac{1}{y_j} & \cdots & 0 & -\frac{y_1}{y_j^2} & & & & & & \\
\vdots & \ddots & \vdots & \vdots & & 0 & & & 0 & \\
0 & \cdots & \frac{1}{y_j} & -\frac{y_{j-1}}{y_j^2} & & & & & & \\
 & & & -\frac{y_{j+1}}{y_j^2} & \frac{1}{y_j} & \cdots & 0 & & & \\
 & 0 & & \vdots & \vdots & \ddots & \vdots & & 0 & \\
 & & & -\frac{y_{i-1}}{y_j^2} & 0 & \cdots & \frac{1}{y_j} & & & \\
0 & \cdots & 0 & -\frac{1}{y_j^2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & & & -\frac{y_{i+1}}{y_j^2} & & & & \frac{1}{y_j} & \cdots & 0 \\
 & 0 & & \vdots & & 0 & & \vdots & \ddots & \vdots \\
 & & & -\frac{y_n}{y_j^2} & & & & 0 & \cdots & \frac{1}{y_j}
\end{array}\right] \end{array}$$

And linear algebra gives $\det d(f_j^{-1}\circ f_i) = \frac{(-1)^{i+j}}{y_j^{n+1}}$

(the only entries contributing to this are circled in green).

So define $\tilde{f}_i : \mathbb{R}^n \to \mathbb{RP}^n$ by $\tilde{f}_i = \begin{cases} f_i & ; \ i \text{ even} \\ f_i\circ h & ; \ i \text{ odd} \end{cases}$

with $h(x_1,\dots,x_n) = (x_1,\dots,x_{n-1},-x_n)$; then

$\det d(\tilde{f}_j^{-1}\circ\tilde{f}_i) = \frac{1}{y_j^{n+1}} > 0$ and $\tilde{f}_i$ is an oriented atlas.

(b) Use the atlas from #1. The Jacobian $d(f_j^{-1}\circ f_i)$ is a $2n\times 2n$ matrix, of the same form as in (a) but with each entry in the matrix from (a) replaced by a 2×2 matrix; and this 2×2 matrix is 0 if the corresponding entry from (a) is 0. Furthermore, each of the circled entries is replaced by a matrix as follows:

$$\frac{1}{y_j} \longrightarrow \frac{1}{x_j^2+y_j^2}\begin{bmatrix} x_j & y_j \\ -y_j & x_j \end{bmatrix} \qquad \left(\det = \frac{1}{x_j^2+y_j^2}\right)$$

$$-\frac{1}{y_j^2} \longrightarrow \frac{1}{(x_j^2+y_j^2)^2}\begin{bmatrix} y_j^2-x_j^2 & -2x_jy_j \\ 2x_jy_j & y_j^2-x_j^2 \end{bmatrix} \qquad \left(\det = \frac{1}{(x_j^2+y_j^2)^2}\right).$$

Because of the form of the matrix, a bit of linear algebra gives

$$\det d(f_j^{-1}\circ f_i) = \frac{}{(x_j^2+y_j^2)^{n+1}} > 0$$

So the atlas from #1 is oriented.