Math 621 — part 2

Note Title 2/14/2017

Riemannian Metrics

Def A Riemannian metric is a (0,2) tensor g, i.e. a smoothly varying

$$g : T_xM \otimes T_xM \longrightarrow \mathbb{R}$$
$$v \otimes w \mapsto g(v,w) \text{ or } \langle v, w \rangle$$

that is: • SYMMETRIC : $g(v,w) = g(w,v)$

• POSITIVE DEFINITE : $g(v,v) \geq 0$, equality $\Leftrightarrow v = 0$.

In coordinates, write $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$.

Def A Riemannian manifold is (M,g), M = smooth mfd, g = Riem. metric.

When are two such mfds the "same"?

Def An isometry between (M,g) and (N,h) is a diffeomorphism $\varphi : M \to N$ such that $g = \varphi^* h$, i.e.:

$$h(\varphi_* v, \varphi_* w) = g(v,w) \qquad \forall\ v,w \in T_pM$$

$\varphi : M \to N$ is a local isometry at $p \in M$ if $\exists$ nbd U of p with $\varphi|_U : U \to \varphi(U)$ isometry.

Ex 1. $\mathbb{R}^n$, g = standard inner product: $g_{ij} = \delta_{ij}$ Kronecker delta.
Euclidean space.

Ex 2. Lie groups.

Def A Riem metric $\langle , \rangle$ on G is left invariant if L_h is an isometry $\forall h \in G$.

$\forall h \in G$, $\forall v, w \in T_g G$, $\quad \underset{\text{at } g}{\langle v, w \rangle} = \underset{\text{at } hg}{\langle (L_h)_* v, (L_h)_* w \rangle}$

Similarly: right invariant. If both, then biinvariant.

Easy to construct left invt metric: given an inner product $\langle , \rangle$ on $\mathfrak{g} = T_e G$, define $\underset{\text{at } g}{\langle v, w \rangle} = \underset{\text{at } e}{\langle (L_{g^{-1}})_* v, (L_{g^{-1}})_* w \rangle}$.

Fact: any compact Lie group has a biinvariant metric
(see doCarmo p. 46 #7). (not true in general!)

Prop Let $\langle , \rangle$ be the left invt metric on G induced by $\langle , \rangle_e$ on $\mathfrak{g}$. Then $\langle , \rangle$ is biinvariant $\Leftrightarrow$
$$0 = \langle [X,Y], Z \rangle_e + \langle Y, [X,Z] \rangle_e \quad \forall X, Y, Z \in \mathfrak{g}.$$

Pf. $\Rightarrow$. Let $X, Y, Z \in \mathfrak{g}$. For $t \in \mathbb{R}$, recall $\exp(tX) = \varphi_t(e)$; time t flow of X
and $[X,Y] = \frac{d}{dt}\big|_{t=0} Ad(\exp(tX))\, Y$ where $Ad(h) = (R_{h^{-1}})_*$.

Then $\langle Y, Z \rangle = \langle (R_{h^{-1}})_* Y, (R_{h^{-1}})_* Z \rangle \qquad h = \exp(tX)$
$= \langle Ad(\exp(tX))\, Y, Ad(\exp(tX))\, Z \rangle$

and $\frac{d}{dt}$: $\quad 0 = \langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle$.

$\Leftarrow$: HW.

Ex 3. Given an immersion $f: M \to N$ (often use $N = \mathbb{R}^{n+k}$),
a Riem. metric $\langle\, ,\rangle_N$ on N induces one $\langle\, ,\rangle_M$ on M:

$$\langle v, w\rangle_M = \langle f_* v, f_* w\rangle_N .$$

Note immersion $\Rightarrow$ $df = f_* : T_pM \to T_{f(p)}N$ is injective.
Thus $\langle\, ,\rangle_M$ is pos def since if $\langle v, v\rangle_M = 0$ then $f_* v = 0 \Rightarrow v = 0$.

So eg. embedded submfds $M \hookrightarrow \mathbb{R}^{n+k}$ have metric induced from Euclidean metric on $\mathbb{R}^{n+k}$.

ex: $S^n \subset \mathbb{R}^{n+1}$ unit sphere inherits the "round metric".
$\{x_1^2 + \dots + x_{n+1}^2 = 1\}$ (for S^1, this agrees with flat metric on $\mathbb{R}/\mathbb{Z}$)

More general way to get submfds:
$h: N^{n+k} \to P^k$ smooth.

- $p \in N$ is a critical point if dh_p not surjective
- $q \in P$ is a critical value if $q = h(p)$, some critical p
- $q \in P$ is a regular value if not critical value.

Prop q = regular value $\Rightarrow M = h^{-1}(q) \subset N$ is a smooth n-manifold.
So a metric on N induces one on $M \hookrightarrow N$.

Pf $p \in M \Rightarrow dh_p : T_pN \to T_{h(p)}P$ is surjective: Can choose coords $x_1, \dots, x_n, y_1, \dots, y_k$ near p st. $h = (h_1, \dots, h_k)$, $\left(\frac{\partial(h_1, \dots, h_k)}{\partial(y_1, \dots, y_k)}\right)$ is nonsingular.
Implicit Function Thm $\Rightarrow$ $\exists\, g: \mathbb{R}^n \to \mathbb{R}^k$ with $h(\underline{x}, g(\underline{x})) = q$,
and this gives a chart $\begin{aligned} \mathbb{R}^n &\to M \\ x &\mapsto (x, g(x)) \end{aligned}$. $\square$

$$k\left\{\underbrace{\left[\quad \boxed{\frac{\partial h}{\partial y}}\right]}_{n+k}\right.$$

Ex 4 Product manifolds.

(M_1,g_1), (M_2,g_2) Riem mfds $\rightsquigarrow M_1 \times M_2$.

Recall $T_{(p_1,p_2)}(M_1\times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$.

If $(v_1,v_2), (w_1,w_2) \in T_{(p_1,p_2)}(M_1\times M_2)$ then define

$$g((v_1,v_2),(w_1,w_2)) = g_1(v_1,w_1) + g_2(v_2,w_2).$$

easy to check: this is a metric.

So e.g. $T^n = \underbrace{S^1\times\cdots\times S^1}_{n}$ has a metric induced from S^1 (the book calls it the "flat metric").

Prop Any smooth mfd has a Riem metric.

Def $\{V_\alpha \subset M\}$ is locally finite if $\forall p\in M$, $p\in V_\alpha$ for only finitely many α.

Def $f\in C^\infty(M)$. The support of f is $\operatorname{supp} f = \overline{\{p\in M \mid f(p)\neq 0\}}$.

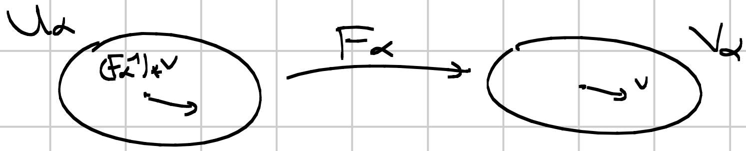
Def $\{V_\alpha\}$ locally finite open cover of M. A partition of unity subordinate to $\{V_\alpha\}$ is a collection $f_\alpha \in C^\infty(M)$ s.t.

1. $f_\alpha(p) \geq 0 \ \forall p$
2. $\operatorname{supp} f_\alpha \subset V_\alpha$
3. $\sum_\alpha f_\alpha(p) = 1 \ \forall p$ (note finite sum for each p).

Thm M smooth mfd, $\{V_\alpha\}$ open cover. Then $\exists$ locally finite open cover $\{V'_\beta\}$ subordinate to $\{V_\alpha\}$, i.e. $\forall \beta$ $V'_\beta \subset V_\alpha$ for some α, and a partition of unity subordinate to $\{V'_\beta\}$.

Pf of Prop. Let $\{(F_\alpha, U_\alpha, V_\alpha)\}$ be an atlas for M. By thm, assume $\{V_\alpha\}$ locally finite and f_α = partition of unity subordinate to $\{V_\alpha\}$.

Each α determines a metric $\langle\, ,\rangle_\alpha$ on V_α: $\langle v, w\rangle_\alpha = \langle F_\alpha^* v, F_\alpha^* w\rangle$



Then define $\langle\, ,\rangle$ on M by

$v, w \in T_pM \Rightarrow \quad \langle v, w\rangle = \sum_\alpha f_\alpha(p) \langle v, w\rangle_\alpha$. (note: finite sum)

Easy to check: symmetric, bilinear, pos def. $\square$

Uses for metrics.

- Length of curves. $\gamma : [a,b] \to M$ piecewise smooth.

  $\rightsquigarrow$ Length $\ell(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{1/2}\, dt$ $\quad (\gamma'(t) \in T_{\gamma(t)}M)$

  Usual pf: indep of parametrization.

- Isom between tangent + cotangent.

  $p \in M \Rightarrow \quad g(p) : T_pM \otimes T_pM \to \mathbb{R}$

  $\Rightarrow \quad \tilde{g}(p) : T_pM \to T_p^*M$.

  g Pos def $\Rightarrow$ isomorphism. So get $\tilde{g} : TM \xrightarrow{\cong} T^*M$ bundle isom.

  Note: works for any nondeg $(0,2)$-tensor.

- Volume form. Assume M oriented.

  $g \rightsquigarrow \omega \in \Omega^n(M)$ given in local coords by

  $\omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n$. $\quad \left(g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)$.

Check well-defined: change of coords $x_1 \cdots x_n \rightsquigarrow y_1 \cdots y_n$.

$$\frac{\partial}{\partial y_i} = \sum \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}.$$

Write Jacobian $\left(\frac{\partial x_j}{\partial y_i}\right) = M$: then if $\tilde{g}_{ij} = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$, then

$$(\tilde{g}_{ij}) = M(g_{ij})M^T \quad \therefore$$

$$\tilde{g}_{ij} = \sum \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j} g_{kl}.$$

Also $\begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = M^T \begin{pmatrix} dy_1 \\ \vdots \\ dy_n \end{pmatrix} \Rightarrow dx_1 \wedge \cdots \wedge dx_n = (\det M)\, dy_1 \wedge \cdots \wedge dy_n.$

$$\Rightarrow \sqrt{\det \tilde{g}_{ij}}\, dy_1 \wedge \cdots \wedge dy_n = |\det M| \sqrt{\det g_{ij}}\, dy_1 \wedge \cdots \wedge dy_n$$

(if $\det M > 0$) $\longrightarrow = \sqrt{\det g_{ij}}\, dx_1 \wedge \cdots \wedge dx_n$.

Affine Connections

Motivations:

① How to take directional derivatives of vector fields?

Would like: X tangent vector, $Y = \sum b_i \frac{\partial}{\partial x_i}$

$\Rightarrow$ "directional derivative" "$X(Y)$" $= \sum X(b_i) \frac{\partial}{\partial x_i}$.

Problem: not coord independent.

ex.- $M = \mathbb{R}$ $\qquad X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial x} \Rightarrow$ "$X(Y)$" $= 0$.

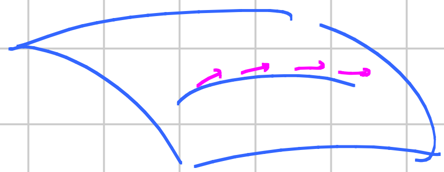
But: another coord system $y = x^3$.

$$\frac{\partial}{\partial x} = \frac{dy}{dx} \frac{\partial}{\partial y} = 3y^{2/3} \frac{\partial}{\partial y}$$

$$\rightarrow \text{"}X(Y)\text{"} = 3y^{2/3} \frac{\partial}{\partial y}(3y^{2/3}) \frac{\partial}{\partial y} = 6y^{1/3} \frac{\partial}{\partial y} \neq 0.$$

② What's the analogue of a straight line in a mfld M?

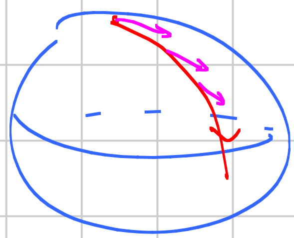
Need notion of "parallel" vector along a curve.

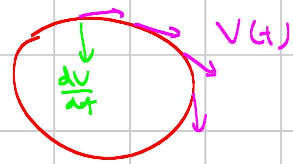


γ path in M, V = vector field along γ i.e. $V(t) \in T_{\gamma(t)}M$.

What does it mean for V to be parallel? "$\frac{dV}{dt} = 0$"

For $M \subset \mathbb{R}^3$ surface, $V(t) \in \mathbb{R}^3$ so $\frac{dV}{dt}$ is a vector in $\mathbb{R}^3$ but not nec. in $T_{\gamma(t)}M$.





Maybe define

$$\frac{dV}{dt} = \text{orthogonal projection of } V'(t) \text{ to } T_{\gamma(t)}M.$$

In general:

__Def__ An __affine connection__ D on M is a map

$$D : \mathrm{Vect}\,M \times \mathrm{Vect}\,M \to \mathrm{Vect}\,M$$
$$(X, Y) \mapsto D_X Y$$

Satisfying:

1. $D_{X_1+X_2} Y = D_{X_1} Y + D_{X_2} Y$; $D_X(Y_1+Y_2) = D_X Y_1 + D_X Y_2$
2. $D_{fX} Y = f D_X Y$
3. $D_X(fY) = f D_X Y + X(f) Y$.

__Note__: for fixed Y, the map $D_{\cdot}Y : \mathrm{Vect}\,M \to \mathrm{Vect}\,M$ is a tensor

Interlude: coords and indices

Convention: write $x^1, \dots, x^n$ for coords.

T_pM gen'd by $\frac{\partial}{\partial x^1} = \partial_1, \dots, \frac{\partial}{\partial x^n} = \partial_n$; T_p^*M gen'd by $dx^1, \dots, dx^n$.

Vector field $\sum_i a^i \partial_i = a^i \partial_i$; 1-form $\sum_i b_i dx^i = b_i dx^i$.

Einstein summation notation: sum over repeated indices (one upper, one lower)

$X = a^i \partial_i \Rightarrow \quad Xf = \sum a^i \frac{\partial f}{\partial x^i} = a^i \partial_i f$.

Metric $g_{ij} = \langle \partial_i, \partial_j \rangle$; $g = g_{ij}\, dx^i \otimes dx^j$.

for future use: write inverse matrix to (g_{ij}) as (g^{ij}); note $g_{ik} g^{kj} = \delta_i^j$.

Write $\nabla_i = \nabla_{\partial/\partial x^i}$.

Connection in coordinates.

Def $x^1, \dots, x^n$ coords. The Christoffel symbol $\Gamma_{ij}^k \in C^\infty(U)$, $1 \le i, j, k \le n$, is defined by

$$\nabla_{\partial/\partial x^i}\left(\frac{\partial}{\partial x^j}\right) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{i.e.} \quad \nabla_i \partial_j = \Gamma_{ij}^k \partial_k.$$

Connections are local operators (exc) and on a chart, Γ_{ij}^k determine the connection:

if ∇ = connection then

$$\nabla_{(a^i\partial_i)}(b^j\partial_j) = a^i \nabla_i(b^j \partial_j) = a^i\left(b^j \nabla_i \partial_j + (\partial_i b^j)\partial_j\right) = a^i b^j \Gamma_{ij}^k \partial_k + a^i (\partial_i b^j)\partial_j.$$

Conversely can define $\nabla_X Y$ $\forall X, Y$ by this formula $\rightarrow$ and check that this gives a connection.

Ex $M = \mathbb{R}^n$, $\Gamma_{ij}^k = 0 \quad \forall i,j,k$.

$X = a^i \partial_i$, $Y = b^j \partial_j \Rightarrow D_X Y = a^i (\partial_i b^j) \partial_j = X(b^j)\partial_j$

(note in a different coord system, Γ_{ij}^k might not be 0).

Now: given γ = curve in M, V = vector field along γ, use ∇ to define $\frac{dV}{dt}$. ("covariant der. along a curve")

Prop M smooth, ∇ = affine connection.

Then $\exists!$ way to associate, to a curve $\gamma(t)$ and a vector field V along γ, another vector field $\frac{DV}{dt}$ along γ, s.t.:

① $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$

② $\frac{D}{dt}(fV) = f\frac{DV}{dt} + \frac{df}{dt}V$

③ If V extends to a vector field Y on M (or a nbd of γ) then
$$\frac{DV}{dt} = \nabla_{\gamma'(t)} Y.$$

Pf Write $\gamma(t) = (x^1(t), \dots, x^n(t)) \Rightarrow \gamma'(t) = (x^i)' \partial_i$.

Uniqueness: by ③,
$$\frac{D}{dt}(\partial_j) = \nabla_{(x^i)'\partial_i} \partial_j = (x^i)' \Gamma_{ij}^k \partial_k$$

Then for general vector field along γ, $V = V^j(t)\partial_j$,

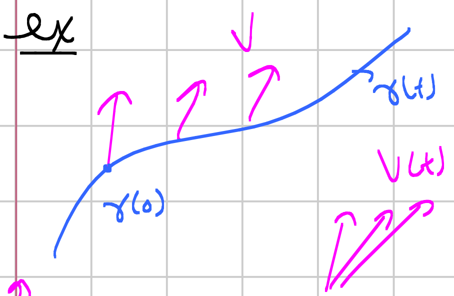
$$\frac{DV}{dt} = V^j(t)\frac{D}{dt}\partial_j + \frac{dV^j}{dt}\partial_j \qquad \text{(① and ②)}$$
$$= (x^i)' V^j(t) \Gamma_{ij}^k \partial_k + \frac{dV^j}{dt}\partial_j$$

$$(*) \quad \boxed{\frac{DV}{dt} = \left(\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma_{ij}^k\right)\partial_k}.$$

Existence: define $\frac{DV}{dt}$ by $(*)$; check ② (①, ③ obvious).

Note: on overlapping charts, values must agree by uniqueness. □

ex



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$$\frac{DV}{dt}\Big|_{t=0} = \nabla_{\gamma'(0)} V = a^i(\partial_i b^j)\,\partial_j = a^i \frac{\partial b^j}{\partial x^i}\frac{\partial}{\partial x^j}$$

$V = b^j \partial_j$

$\gamma'(0) = a^i \partial_i$

$V(\gamma(t)) = b^j(\gamma(t))\,\partial_j$

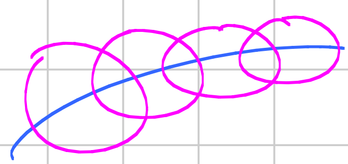
$$= \frac{d}{dt}\Big|_{t=0} V(\gamma(t)).$$

Parallel transport: $\gamma(t)$ curve, V = vector field along γ.

Def V is parallel if $\frac{DV}{dt} = 0$.

Prop $V_0 \in T_{\gamma(0)}M$. $\exists!$ parallel vector field $V(t)$ with $V(0) = V_0$: this is called "parallel transport" of V_0 along γ.

Pf Suffices to prove in a coord chart: then cover γ by overlapping charts.



We want to find

$V(t) = V^j(t)\,\partial_j$ satisfying:

$$\underbrace{\left(\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma_{ij}^k\right)}_{=0\ \forall k} \partial_k = 0$$

$$\Leftrightarrow \quad \frac{dV^k}{dt} = -V^j(t)\,\frac{dx^i}{dt}\,\Gamma_{ij}^k .$$

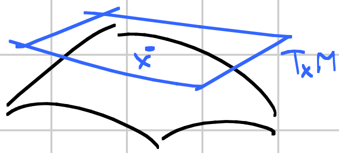
This is a system of n 1st order diff eq's in n variables $\Rightarrow \exists!$ solution given initial conditions. $\square$

Important example: $M \subset \mathbb{R}^{n+k}$.

Define affine connection $\bar\nabla$ on $\mathbb{R}^{n+k}$ as before.

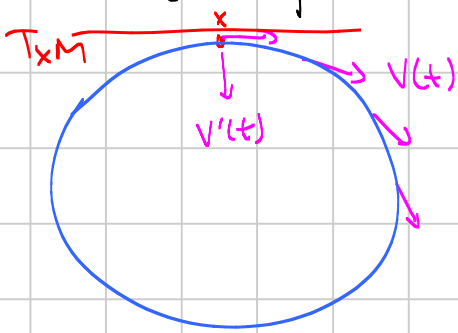
This induces a affine connection ∇ on M:

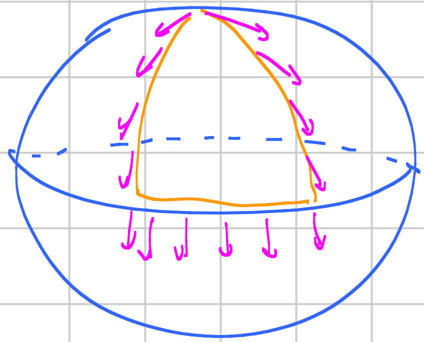
$X, Y \in Vect(M)$, expanded to $\bar X, \bar Y \in Vect(\mathbb{R}^{n+k}) \Rightarrow \nabla_X Y = (\bar\nabla_{\bar X} \bar Y)^T$: orthogonal projection $T_x\mathbb{R}^{n+k} \to T_xM$.



check: this is an affine connection.

Parallel transport: $\frac{DV}{dt} = 0 \Rightarrow$ if $V(t) \in \mathbb{R}^{n+k}$ then $(V'(t))^T = 0$.





Levi-Civita Connection.

g = Riem metric on M

Def. A connection ∇ on M is <u>compatible</u> with g if $\forall X, Y, Z \in Vect(M)$,

$$X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Prop γ = curve on M, ∇ compatible with $g = \langle\,,\rangle$.

1. If V, W are vector fields along γ, then
$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle.$$
2. If V = parallel vector field along γ, then $|V(t)| = \langle V(t), V(t)\rangle^{1/2}$ is constant.

Pf 1. Extend V, W to vector fields near γ, and use $\frac{DV}{dt} = \nabla_{\gamma'(t)} V$ etc.

2. Clear. $\square$

Def A connection ∇ on M is torsion-free ("symmetric") if
$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \forall\ X, Y \in \mathrm{Vect}\, M.$$

Note: in coords $(\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k = \nabla_i \partial_j - \nabla_j \partial_i = [\partial_i, \partial_j] = 0 \Rightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$.

Thm M Riem. There exists a unique connection that is torsion-free and compatible with g, "Levi-Civita connection".

Pf Uniqueness:

$$\begin{aligned}
\oplus\quad & X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
\oplus\quad & Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\
\ominus\quad & Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle
\end{aligned}$$

$$\Big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle\Big) = 2\langle \nabla_X Y, Z\rangle + \Big(\langle [X, Z], Y\rangle + \langle X, [Y, Z]\rangle - \langle Z, [X, Y]\rangle\Big)$$

So $\langle \nabla_X Y, Z\rangle$ is determined $\forall X, Y, Z \Rightarrow \nabla_X Y$ is determined.

Existence: HW. $\square$

In coord: choose $X = \partial_i$, $Y = \partial_j$, $Z = \partial_\ell$.

$$\langle \nabla_X Y, Z \rangle = \Gamma_{ij}^m \langle \partial_m, \partial_\ell \rangle = \Gamma_{ij}^m g_{m\ell}$$

$$\Rightarrow \Gamma_{ij}^m g_{m\ell} = \tfrac{1}{2}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

$$\Rightarrow \Gamma_{ij}^k = \Gamma_{ij}^m g_{m\ell} g^{\ell k}$$

$$\boxed{\Gamma_{ij}^k = \tfrac{1}{2}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) g^{\ell k}}$$

<u>Ex</u> $\mathbb{R}^n$, Euclidean metric. Then $g_{ij} = \delta_{ij}$ so $\Gamma_{ij}^k = 0 \ \forall i, j, k$.

<u>Geodesics.</u>

<u>Def</u> (M, g) Riem mfd, ∇ = Levi-Civita connection.

A curve γ on M is a <u>geodesic</u> if $\frac{D}{dt}(\gamma'(t)) = 0$

i.e. $\gamma'(t)$ is a parallel vector field along γ, "γ has zero acceleration".

<u>Observation</u>: γ = geodesic $\Rightarrow$

$$\frac{d}{dt}|\gamma'(t)|^2 = \frac{d}{dt}\langle \gamma'(t), \gamma'(t) \rangle = 2\left\langle \frac{D}{dt}\gamma'(t), \gamma'(t) \right\rangle = 0$$

So $|\gamma'(t)|$ is constant. (usually assume not constant map: $|\gamma'(t)| > 0$).

<u>Ex</u>
- Straight lines in $\mathbb{R}^n$
- for $M \subset \mathbb{R}^{n+k}$: γ = geodesic if tangential component of acceleration $\gamma''(t) \in \mathbb{R}^{n+k}$ is 0.

In coordinates: $\gamma(t) = (x^1(t), \dots, x^n(t))$, $\quad \gamma'(t) = \frac{dx^i}{dt}\partial_i$.

recall $\frac{DV}{dt} = \left(\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma_{ij}^k\right)\partial_k$

$\Rightarrow \frac{D}{dt}(\gamma'(t)) = \left(\frac{d^2}{dt^2}x^k + \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^k\right)\partial_k$

so a geodesic satisfies a second order system

$$\boxed{\frac{d^2}{dt^2}x^k + \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^k = 0} \qquad k = 1, \dots, n.$$

Notes 1. This is homogeneous: if $\gamma(t) : (a,b) \to M$ is a geodesic then so is the reparametrization $\tilde{\gamma}(t) = \gamma(ct) : (\frac{a}{c}, \frac{b}{c}) \to M$.

(note $\tilde{\gamma}'(0) = c\gamma'(0)$).

2 Can reformulate in terms of the tangent bundle.

U = coord chart on M, coords $x^1, \dots, x^n$

$\Rightarrow U \times \mathbb{R}^n$ = coord chart on TM, coords $x^1, \dots, x^n, y^1, \dots, y^n$.

γ = curve in M $\rightsquigarrow$ $\tilde{\gamma}$ = curve in TM given by

$\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$.

Then if γ = geodesic, $\tilde{\gamma} = (x^1, \dots, x^n, y^1, \dots, y^n)$ is given by:

first order system of diff eq's
$$\left\{\begin{array}{ll} y^1(t) = \frac{d}{dt}x^1(t) & \frac{d}{dt}y^1(t) = -y^iy^j\Gamma_{ij}^1 \\ \vdots & \vdots \\ y^n(t) = \frac{d}{dt}x^n(t) & \frac{d}{dt}y^n(t) = -y^iy^j\Gamma_{ij}^n. \end{array}\right.$$

Def The geodesic vector field on TM is the vector field given in coordinates by

$$\left(\underset{\uparrow \atop \frac{\partial}{\partial x^1}}{y^1}, \dots, \underset{\uparrow \atop \frac{\partial}{\partial x^n}}{y^n}, \underset{\uparrow \atop \frac{\partial}{\partial y^1}}{-y^iy^j\Gamma_{ij}^1}, \dots, \underset{\uparrow \atop \frac{\partial}{\partial y^n}}{-y^iy^j\Gamma_{ij}^n}\right).$$

(from the following discussion: coord independent).

The geodesic vector field is constructed so that $\tilde{\gamma}$ is a flow of it.
This implies short time existence of geodesics:
for $(x,v)\in TM$, let $\tilde{\gamma}(t)$ be the time t flow of the geodesic vector field in TM;
then $\gamma(t)=\pi\circ\tilde{\gamma}(t)$ is a geodesic in M with $\gamma(0)=x$, $\gamma'(0)=v$,
and conversely. So $\exists!$ geodesic $\gamma:(-\epsilon,\epsilon)\to M$ with
$\gamma(0)=x$, $\gamma'(0)=v$, small ϵ.

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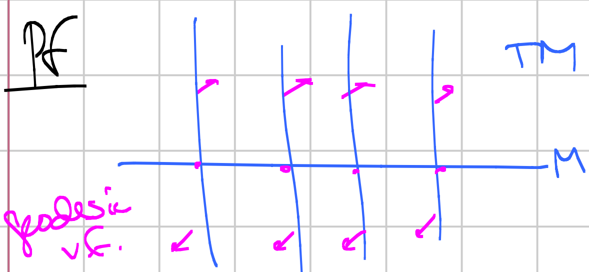
Prop $\forall\, p\in M$ $\exists\, \epsilon>0$, nbd of p, and smooth map
$$\gamma:(-2,2)\times\Omega\to M \qquad \text{where } \Omega=\{(x,v)\mid x\in U, |v|<\epsilon\}\subset TM$$
s.t. $t\mapsto\gamma(t,x,v)$ is the unique geodesic γ
with $\gamma(0)=x$, $\gamma'(0)=v$.

Pf



Short time existence:
Can find $\bar{\Omega}$, nbd of $(p,0)\in TM$,
such that for $(x,v)\in\bar{\Omega}$, the time t
flow of the geodesic v.f. starting at (x,v)
is defined for $|t|<\delta$: write this flow as
$$t\mapsto\tilde{\gamma}(t,x,v).$$

Might as well assume $\bar{\Omega}$ is of the form
$$\bar{\Omega}=\{(x,v)\mid x\in U, |v|<\epsilon_0\}$$ for some ϵ_0, some nbd U of p.

Write $\gamma=\pi\circ\tilde{\gamma}$. Then:

$t\mapsto\gamma(t,x,v)$ is a geodesic for $|t|<\delta$, $|v|<\epsilon_0$

$\shortparallel$
$\gamma(\frac{t}{c},x,cv)$ (reparametrization — homogeneity)

$\Rightarrow$ $t\mapsto\gamma(t,x,cv)$ is a geodesic for $|t|<\frac{\delta}{c}$, $|v|<\epsilon_0$.

Choose $c=\frac{\delta}{2}\Rightarrow$

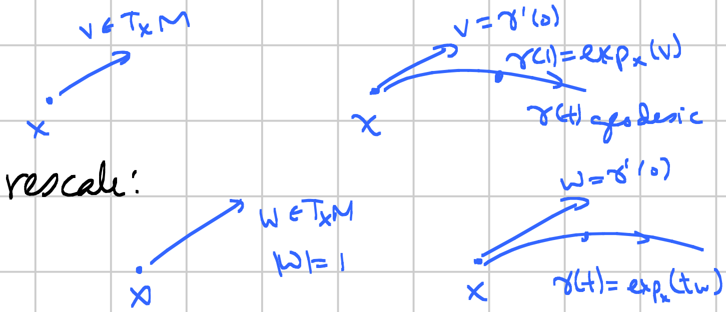
$t\mapsto\gamma(t,x,v)$ is a geodesic for $|t|<2$, $|v|<\frac{\epsilon_0\delta}{2}$. □

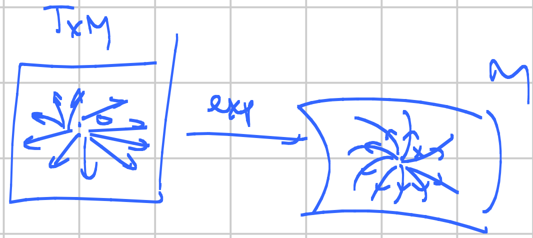
Def Ω as above: $\Omega = \{(x,v) \mid x \in U, |v| < \epsilon\}$

The *exponential map* $\exp : \Omega \to M$ is defined by

$$\exp(x,v) = \gamma(1,x,v) = \gamma\left(|v|, x, \frac{v}{|v|}\right). \qquad x \in U, |v| < \epsilon.$$

For fixed x, get $\exp_x : B_\epsilon(0) \to M$, $\exp(0) = x$.





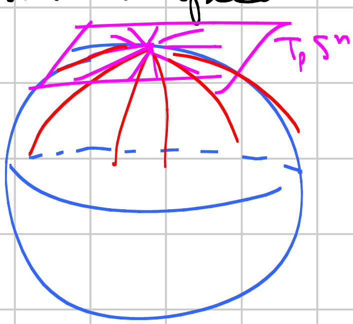
Prop Fix $x \in M$. $\exists\, \epsilon > 0$ s.t. $\exp_x : B_\epsilon(0) \to M$ is a diffeomorphism onto an open subset of M.

Pf
$$\begin{aligned} d(\exp_x)(0)(v) &= \frac{d}{dt}\Big|_{t=0} \exp_x(tv) \\ &= \frac{d}{dt}\Big|_{t=0} \gamma(1,x,tv) \\ &= \frac{d}{dt}\Big|_{t=0} \gamma(t,x,v) \\ &= v \end{aligned}$$

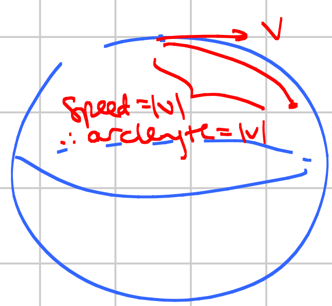
So $d(\exp_x)(0)$ = isomorphism. Now use Inverse Function Thm. $\square$

Ex (S^n, round metric).

Geodesics are (arcs of) great circles, parametrized with constant speed.



$\exp_p : \mathbb{R}^n \to S^{n+1}$. $\exp_p(v)$:



$\exp_p : B_r(0) \to S^n$ is a diffeo onto its image for $r < \pi$.

$\exp_p(\{|v| = \pi\})$ = antipodal pt

$\exp_p(\{|v| = 2\pi\}) = p$

etc.

Ex $SO(n)$ with biinvariant metric. By HW, geodesics are 1-parameter subgroups. If $M \in so(n)$, define

$$\exp(M) = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots \in SO(n).$$

Then $\exp(tM)$ is a 1-parameter subgroup $\Rightarrow$ geodesic, and

$$\frac{d}{dt}\Big|_{t=0} \exp(tM) = M \ ;$$

so $\quad \exp_e(tM) = \exp(tM)$.

$e \in SO(n)$, $tM \in T_e SO(n)$

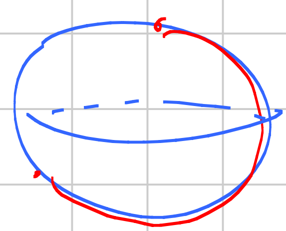
If $\gamma : [a,b] \to M$ is a piecewise smooth (in particular, continuous) path in M, define length

$$\ell(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2} dt.$$

Claim: if the endpoints p, q of γ are sufficiently close, and γ is a geodesic, then $\forall$ piecewise smooth $\tilde{\gamma}$ with same endpts,

$$\ell(\tilde{\gamma}) \geq \ell(\gamma).$$

Note: not true in general:



$\leftarrow$ geodesic

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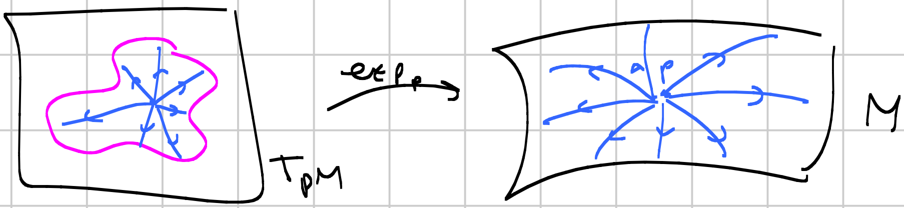
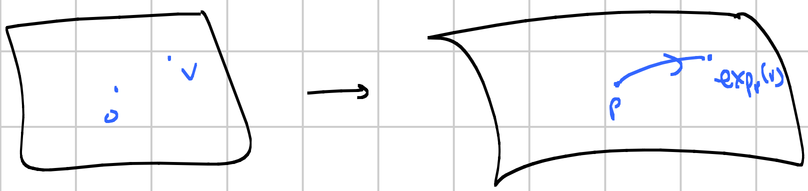
Review Geodesic $= \gamma(t)$: $\nabla_{\gamma'}\gamma' = 0$.

$\exp_p$: (subset of T_pM) $\longrightarrow M$

$v \longmapsto \gamma(1,p,v)$ where $\gamma(t,p,v)=$ geodesic with $\gamma(0)=p$, $\gamma'(0)=v$.





geodesic $\gamma(t,p,v)$, $|\gamma'(t)|=|v|$

length $=\int_0^1 \langle\gamma',\gamma'\rangle\,dt = |v|$.

Note if we fix v, then $\exp(tv)=\gamma(1,p,tv)=\gamma(t,p,v)$

So $\underset{\gamma(t)}{\exp(tv)}$ is the geodesic with initial conditions $\gamma(0)=p$, $\gamma'(0)=v$.

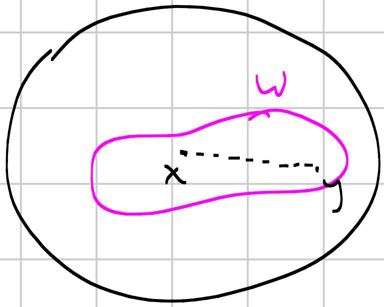
Normal Neighborhoods

eventual goal: geodesics are length minimizers.

Def $V=$ nbd of $p\in M$ is a <u>normal neighborhood of p</u> if $\exists$ $U=$ nbd of $0\in T_pM$ such that $\exp_p: U\to V$ is a diffeomorphism.

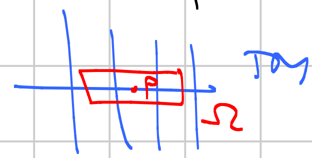
Prop $p\in M$. $\exists$ nbd W of p and $\epsilon>0$ such that:

1. $\forall x\in W$, $\exp_x: B_\epsilon(0)\to M$ is a diffeo onto its image and $\exp_x(B_\epsilon(0))\supset W$ (so $W=$ normal nbd of each $x\in W$)
2. $\forall x,y\in W$, $\exists!$ $v\in T_xM$ with $|v|<\epsilon$ st. $y=\exp_x v$; i.e. $\exists!$ geodesic of length $<\epsilon$ joining x and y.



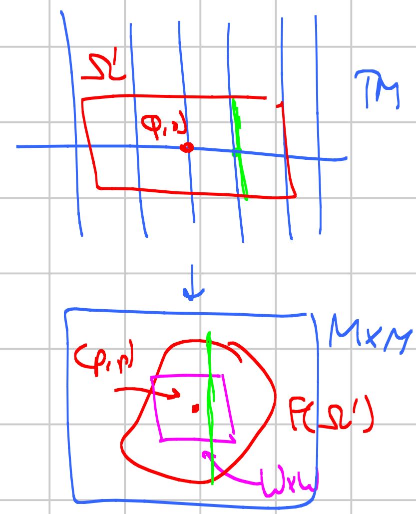
W is called a <u>totally normal neighborhood</u> of p.

<u>Pf</u> $\Omega = \{(x,v) \mid x \in U, |v| < \epsilon\} \subset TM$ as before
(U = nbd of x).



$\exp : \Omega \to M$, $(x,v) \mapsto \exp_x v$. $\rightsquigarrow$ write $F : \Omega \to M \times M$, $(x,v) \mapsto (x, \exp_x v)$.

Then $dF(p,0) = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ so F is a local diffeo near $(p,0)$.
Thus $\exists$ nbd $U' \subset U$ of p, $\epsilon' < \epsilon$, s.t. F is a diffeo on
$\Omega' = \{(x,v) \mid x \in U', |v| < \epsilon'\}$.



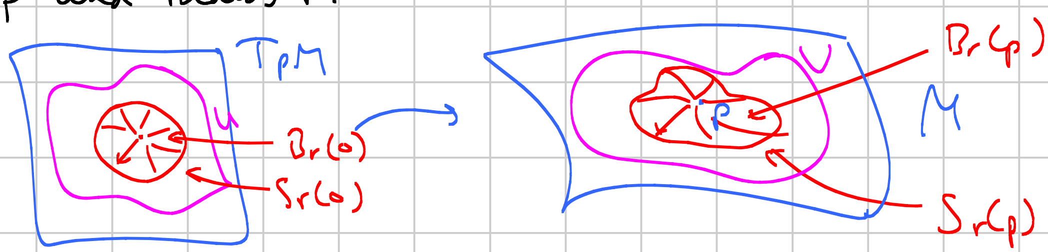
There is a nbd W of p with $W \times W \subset F(\Omega')$.
If $x \in W$ then $F(\{x\} \times B_{\epsilon'}(0)) \supset \{x\} \times W$
So $\exp_x(B_{\epsilon'}(0)) \supset W$. This proves #1.

#2 follows directly from #1. $\square$

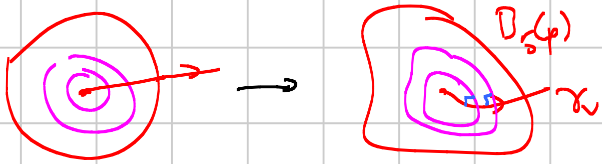
<u>Notation</u>: a <u>normal ball</u> with center p and radius r is
$\exp_p(B_r(0))$ whose closure lies in a normal nbd V of p:
i.e. $\overline{B}_r(0) \subset U$ where $\exp_p : \underset{0}{\overset{\cup}{U}} \to \underset{p}{\overset{\cup}{V}}$ is a diffeo.

Then $\exp_p(\underbrace{S_r(0)}_{= \partial B_r(0)})$ is called the <u>normal sphere</u> with
center p and radius r.

Write
$B_r(p) = \exp_p(B_r(0))$
$S_r(p) = \exp_p(S_r(0))$



Now: suppose $B_{r_0}(p)$ is a normal ball.
For $v \in T_pM$ with $|v|=1$, let $\gamma_v : [0, r_0) \to M$ be the geodesic
$\gamma_v(r) = \exp_p(rv)$. ("radial geodesic")

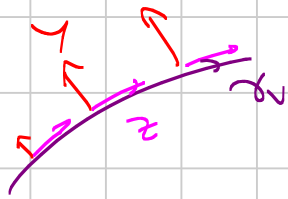


<u>Gauss Lemma</u> γ_v is normal to $S_r(p)$ $\forall$ $r \in (0, r_0)$.

<u>Pf</u> Let $\frac{\partial}{\partial r}$ be the radial vector field on $B_1(0) - \{0\} \subset T_pM$.
Define $Z := (\exp_p)_* \frac{\partial}{\partial r}$. Then
$$Z_{\gamma_v(r)} = \gamma_v'(r).$$
(note $|Z| = 1$)

Want: any tangent vector to $S_r(p)$ at $\gamma_v(r)$ is $\perp$ to $Z_{\gamma_v(r)}$.

Let X be any vector field defined on $S_1(0)$; extend this to vector fields:
X on $B_1(0) - \{0\}$ by $X_{rw} = X_w$, $w \in S_1(0)$
$\tilde{X}$ on $B_1(0)$ by $\tilde{X}_{rw} = r X_w$.
It suffices to show that
$$Y := (\exp_p)_* \tilde{X} \quad \text{and} \quad \gamma_v' \text{ are orthogonal along } \gamma_v.$$



Fix v, and write $f(t)=\langle Y_{\gamma_v(t)}, Z_{\gamma_v(t)}\rangle$. Want $f(t)\equiv 0$.

$$\frac{d}{dt} f(t) = Z\langle Y, Z\rangle = \langle D_Z Y, Z\rangle + \langle Y, \underbrace{D_Z Z}\rangle$$

$$= \underbrace{\langle D_Y Z, Z\rangle} + \langle [Z,Y], Z\rangle$$

=0 since $Z=\gamma_v'$ and γ_v is a geodesic

=0 since $2\langle D_Y Z, Z\rangle = Y\langle Z, Z\rangle = 0$

$$[Z,Y] = \left[(\exp_p)_*\frac{\partial}{\partial r}, (\exp_p)_*\tilde{X}\right] = (\exp_p)_*\left[\frac{\partial}{\partial r}, \tilde{X}\right] \longleftarrow \left[\frac{\partial}{\partial r}, rX\right] = X$$

$\left[\frac{\partial}{\partial r}, X\right]=0$ (exercise) $\Downarrow$

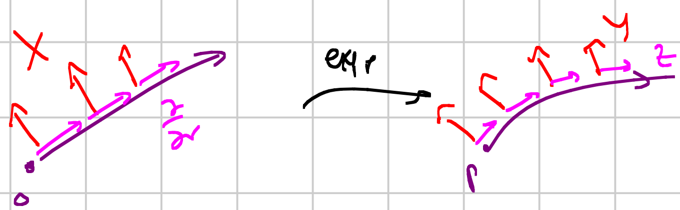
$$= \frac{1}{r}(\exp_p)_*\tilde{X}$$

$$= \frac{1}{r} Y$$

$\Rightarrow \frac{d}{dt} f(t) = \frac{1}{t} f(t)$

$\Rightarrow f(t) = ct$ for some constant c.

$\Rightarrow c = \frac{1}{t}\langle Y, Z\rangle_{\gamma_v(t)} = \langle (\exp_p)_* X, (\exp_p)_*\frac{\partial}{\partial r}\rangle_{\gamma_v(t)}$



Now along the arc $\{tv\}$, we can extend X and $\frac{\partial}{\partial r}$ to 0, and the metric varies continuously. So

$$c = \lim_{t\to 0}\langle (\exp_p)_* X, (\exp_p)_*\frac{\partial}{\partial r}\rangle_{\gamma_v(t)} = \langle (\exp_p)_* X, (\exp_p)_*\frac{\partial}{\partial r}\rangle_p$$

$$= \langle X, \frac{\partial}{\partial r}\rangle_0$$

$$= 0$$

Since (as we saw last time) $(\exp_p)_*(0): T_0(T_pM) \to T_pM$ is the identity.
$T_0(T_pM) = T_pM$

So $f(t)\equiv 0$. $\square$

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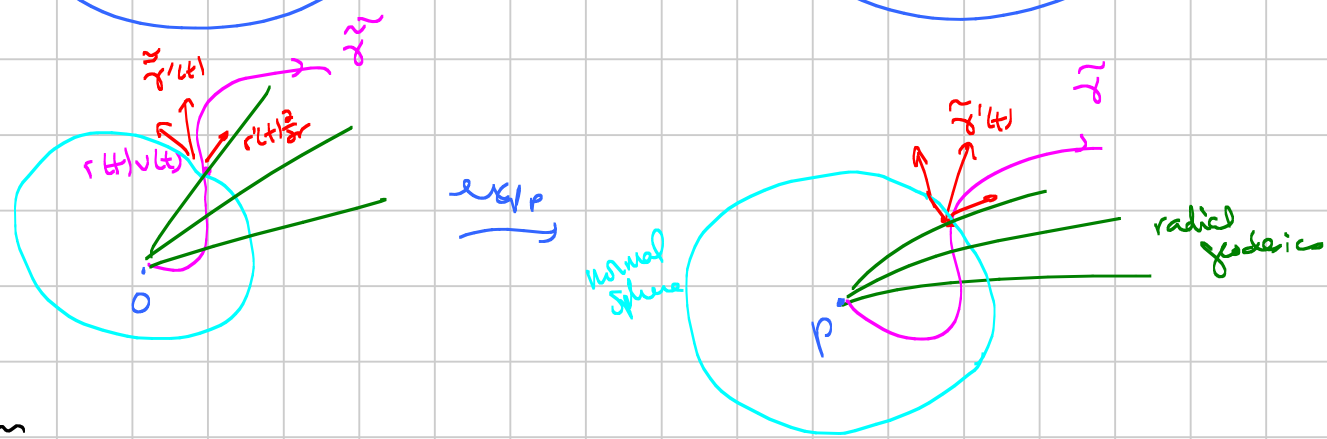
Prop $p \in M$, $B = B_{r_0}(p)$ normal ball centered at p.

$q \in B$, $\gamma : [0,1] \to B$ is the geodesic with $\gamma(0) = p$, $\gamma(1) = q$.

$(\gamma(t) = \exp_p(t \exp_p^{-1}(q)))$.

If $\tilde{\gamma} : [0,1] \to M$ is piecewise smooth with $\tilde{\gamma}(0) = p$, $\tilde{\gamma}(1) = q$, then $\ell(\tilde{\gamma}) \geq \ell(\gamma)$, with equality $\Leftrightarrow$ $\tilde{\gamma}$ = reparametrization of γ.

Pf First suppose $\tilde{\gamma}$ lies in B, and define $\tilde{\tilde{\gamma}} : [0,1] \to B_\epsilon(0)$ by $\exp_p \circ \tilde{\tilde{\gamma}} = \tilde{\gamma}$. Write $\tilde{\tilde{\gamma}}(t) = r(t) v(t)$, $|v(t)| = 1$.



Then

$$\tilde{\tilde{\gamma}}'(t) = \left(r'(t)\frac{\partial}{\partial r}\right) + (\text{tangent to sphere})$$

$$\Rightarrow \tilde{\gamma}'(t) = r'(t) Z + (\text{tangent to normal sphere})$$

and $Z \perp$ normal sphere by Gauss Lemma

$$\Rightarrow |\tilde{\gamma}'(t)| \geq |r'(t)|\,|Z| = |r'(t)|,$$

so

$$\ell(\tilde{\gamma}) = \int_0^1 |\tilde{\gamma}'(t)|\,dt \geq \int_0^1 |r'(t)|\,dt \geq \int_0^1 r'(t)\,dt = r(1) = \ell(\gamma)$$

with equality $\Leftrightarrow$ no normal component and $r' > 0$ $\Leftrightarrow$ $\tilde{\gamma} = \gamma$.

Now: if $\tilde{\gamma}$ doesn't lie in B, let t_0 be first time with $\tilde{\gamma}(t_0) \notin B$. Then
$$\ell(\tilde{\gamma}) = \int_0^1 |\gamma'(t)|\,dt \geq \int_0^{t_0} |\gamma'(t)|\,dt \geq r_0 > \ell(\gamma). \quad \square$$

Converse? Are length minimizers geodesics?

Prop $\gamma : [a,b] \to M$ piecewise smooth, constant speed. If $\ell(\gamma) \leq \ell(\tilde{\gamma})$ for any $\tilde{\gamma}$ with same endpts, then γ is a geodesic.

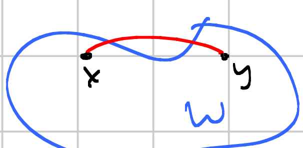
Pf $t \in [a,b]$, W = totally normal nbd of $\gamma(t)$; so γ maps $[a_1, b_1]$ to W for some $a_1 < t < b_1$. Then $\gamma(a_1) \to \gamma(b_1)$ is a curve in a normal ball. If $\tilde{\gamma}$ = geodesic joining $\gamma(a_1)$ to $\gamma(b_1)$ then $\ell(\tilde{\gamma}) \leq \ell(\gamma) \Rightarrow \ell(\tilde{\gamma}) = \ell(\gamma) \Rightarrow \tilde{\gamma} = \gamma$ up to reparametrization $\Rightarrow \gamma|_{[a_1,b_1]}$ is a geodesic.

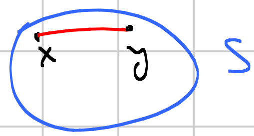
This is true for all t. $\square$

Geodesic Convexity

See do Carmo, ch3 sec4

Recall: any pt has a totally normal neighborhood: $W \ni p$ and $\epsilon > 0$ such that any $x, y \in W$ can be connected by a geodesic of length $< \epsilon$: but this could go outside W.

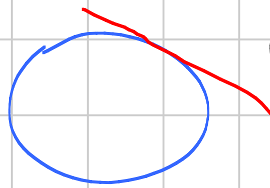




Def A subset $S \subset M$ is (geodesically) convex if $\forall\, x, y \in \bar{S}$, $\exists!$ length minimizing geodesic γ between p and q such that the interior of $\gamma \subset S$.

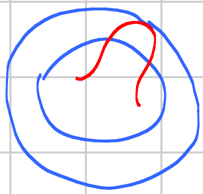
Prop $p \in M$. $\exists\, \epsilon > 0$ st. any normal ball $B_r(p)$ with $r < \epsilon$ is geodesically convex.

Idea: Lemma For any suff. small ϵ, any geodesic tangent to $S_\epsilon(p)$ lies outside $B_\epsilon(p)$.



(derivative estimate for distance to p)

Then: Connect x, y by a geodesic.



If this strays outside $B_\epsilon(p)$, then it's tangent to $B_r(p)$ for some r but lies inside.

Geodesics and Topology

Def M Riem. The distance between two points $p, q \in M$ is

$$d(p,q) = \inf \ell(\gamma) \qquad \text{over all piecewise smooth } \gamma \text{ from } p \text{ to } q.$$

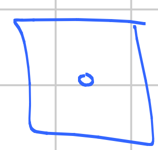
(note: if γ achieves inf then γ is a geodesic).

Prop (M,d) is a metric space, and the topology on M agrees with the metric topology.

Pf Recall if $q \in B_r(p)$ then $d(p,q) = \ell(\gamma) < r$ where γ = geodesic from p to q.
Thus the normal ball $B_r(p)$ is $\{q \mid d(p,q) < r\}$ = metric ball. $\square$

When is there always a minimal geodesic between two points?

not here →

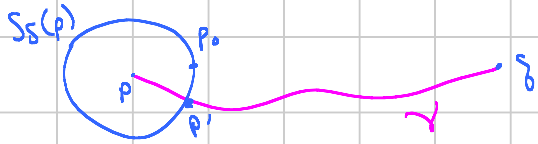


Def M is geodesically complete if all geodesics can be extended to have domain $\mathbb{R}$: i.e., $\forall p \in M$, $\exp_p$ is defined on all of T_pM.

Thm M connected, $p \in M$. If $\exp_p$ is defined on all of T_pM, then $\forall q \in M$ $\exists$ geodesic γ joining p to q with $\ell(\gamma) = d(p,q)$.

Lemma $p, q \in M$. For suff small δ, $\exists\ p_0 \in S_\delta(p)$ with
$$d(p,p_0) + d(p_0,q) = d(p,q).$$

Pf Choose δ such that $B_\delta(p)$ = normal ball, and choose $p_0 \in S_\delta(p)$ minimizing $d(p_0,q)$ ($\exists$ since $d(\cdot,q) : S_\delta(p) \to \mathbb{R}$ is continuous).



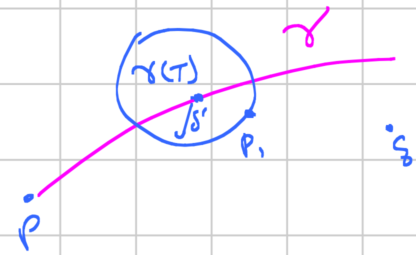
If γ joins p to q then if $p' \in \gamma \cap S_\delta(p)$,   both $=\delta$
$$\ell(\gamma) \geq d(p,p') + d(p',q) \geq d(p,p_0) + d(p_0,q)$$
So $$d(p,q) \geq d(p,p_0) + d(p_0,q) \geq d(p,q). \quad \square$$

Pf of Thm Suppose $d(p,q) = r$. Choose δ, p_0 as in lemma and write $p_0 = \exp_p(\delta v)$, $|v| = 1$.
Let γ be the geodesic $\gamma(t) = \exp_p(tv)$. **Claim**: $\gamma(r) = q$.

Define $I = \{ t \in [0,r] \mid d(\gamma(t), q) = r - t \}$. Note $\delta \in I$:
$$d(\gamma(\delta), q) = d(p_0, q) = d(p,q) - \delta = r - \delta.$$
Also I is closed $\Rightarrow$ Write $T = \max I$. $T \in I$, $\delta \leq T \leq r$.

If $T=r$ then $d(\gamma(r), q)=0 \Rightarrow$ done.

If $T<r$ then apply lemma to $\gamma(T), q$.

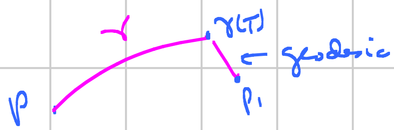


$\Rightarrow \exists\, \delta', \ p_1 \in S_{\delta'}(\gamma(T))$ with

$$d(\gamma(T), p_1)=\delta', \quad d(p_1, q)= r-T-\delta' \qquad (d(\gamma(T), q) = r-T)$$

$$\Rightarrow \quad r = d(p,q) \le d(p,p_1) + d(p_1, q)$$

$$\Rightarrow \quad d(p,p_1) \ge r-(r-T-\delta') = T+\delta'.$$

Now the path  has length $T+\delta' \Rightarrow$ it's a geodesic

$\Rightarrow T$ isn't maximal. $\square$

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Thm (Hopf-Rinow) (M,g) connected Riem. TFAE:

1. (M,g) is geodesically complete, i.e. $\forall p$, $\exp_p$ is defined on all of T_pM
2. for some p, $\exp_p$ is defined on all of T_pM
3. all closed bounded (w.r.t. d) subsets of M are compact
4. (M,d) is complete as a metric space.

Moreover (by previous thm), any of these imply that $\forall p,q \in M$, $\exists$ geodesic γ between p and q s.t. $\ell(\gamma) = d(p,q)$.

Pf. $1 \Rightarrow 2$ obvious.

$2 \Rightarrow 3$: $K \subset M$ closed, bounded. Then $\exists\, R$ with $K \subset B^d_R(p)$ = ball centered at p with radius R in d metric.

$\Rightarrow K \subset \exp_p(B_R(0))$: $\forall q \in K$, $d(p,q)<R$, so by previous thm $\exists$ geodesic of length $<R$ between p, q

$\Rightarrow K \subset \exp_p(\overline{B_R(0)})$ = compact since it's the continuous image of compact

$\Rightarrow K$ = cpt since closed.

$3 \Rightarrow 4$: $\{p_n\}$ Cauchy $\Rightarrow$ bounded $\Rightarrow$ sits in some compact $\overline{B^d_R(p)}$

$\Rightarrow$ has a convergent subsequence $\Rightarrow$ converges.

$4 \Rightarrow 1$: Let γ be a geodesic, assume speed $= 1$. Say its maximal domain is $I \subset \mathbb{R}$. Local existence $\Rightarrow$ I is open.

Claim I is closed. Let $\{t_n\} \subset I$ satisfy $t_n \to t$; want $t \in I$.

Note $d(\gamma(t_m), \gamma(t_n)) \leq$ length of γ between t_m and t_n $= |t_m - t_n|$.

$\Rightarrow \{\gamma(t_n)\}$ is Cauchy $\Rightarrow$ has a limit point $p \in M$.

Let $W \ni p$ be a totally normal nbd, and ϵ st. $W \subset B_\epsilon(x)$ $\forall x \in W$.

Any geodesic of speed 1 starting at any pt $x \in W$ is defined at least on $(-\epsilon, \epsilon)$. Choose n st. $\gamma(t_n) \in W$ and $|t - t_n| < \epsilon/2$.

Then γ is defined at least on $(t_n - \epsilon, t_n + \epsilon)$ and thus at t. $\square$

<u>Cor</u> The following are geodesically complete:

- any compact M
- any closed submanifold of a geodesically complete mfd (eg. Euclidean $\mathbb{R}^n$).

Curvature

Gauss: defined "Gaussian curvature" for surfaces.

Then can extend to 2D slices of a mfd: "Sectional Curvature."

Modern formulation: <u>Curvature tensor</u>, measures deviation from being <u>flat</u> (isometric to Euclidean space).

Def The <u>curvature tensor</u> of (M,g) is the $(1,3)$ tensor
$$X,Y,Z \in \mathrm{Vect}(M) \longrightarrow R(X,Y)Z \in \mathrm{Vect}(M)$$
defined by
$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$
where ∇ = Levi-Civita connection.

⚠ <u>CAUTION</u>: many people use the opposite sign convention.

<u>Note on name</u>: fixing X, Y, the map $Z \mapsto R(X,Y)Z$ is a $(1,1)$ tensor: think of $R(X,Y)$ as being an endomorphism of TM, $R(X,Y): T_pM \circlearrowleft$

<u>Check tensor</u>: $R(fX,Y)Z = \underbrace{\nabla_Y(f\nabla_X Z)}_{f\nabla_Y\nabla_X Z + \cancel{(Yf)\nabla_X Z}} - f\,\nabla_X\nabla_Y Z - \nabla_{\underbrace{[fX,Y]}_{f[X,Y] - \cancel{(Yf)X}}} Z$
$$= f R(X,Y,Z);$$
$$R(X,Y)(fZ) = \nabla_Y \nabla_X(fZ) - \nabla_X\nabla_Y(fZ) + \nabla_{[X,Y]}(fZ)$$
$$= fR(X,Y)Z + (Yf)\nabla_X Z - (Xf)\nabla_Y Z$$
$$+ \underbrace{\nabla_Y((Xf)Z)}_{(YXf)Z + (Xf)\nabla_Y Z} - \underbrace{\nabla_X((Yf)Z)}_{(XYf)Z + (Yf)\nabla_X Z} + ([X,Y]f)Z$$
$$= fR(X,Y)Z.$$

Prop (First) Bianchi identity) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$

Pf $\mathrm{LHS} = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$ $\quad\to \nabla_{[X,Y]}Z - \nabla_Z[X,Y]$
$\quad + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X$ $\quad = [[X,Y],Z]$
$\quad + \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y$
$= [[X,Y],Z]$ + cyclic permutations $= 0$. $\square$

We can turn the (1,3) tensor $R(X,Y)Z$ into a (0,4) tensor by using the metric:

define $\quad R(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle.$

$\quad X,Y,Z,W \in T_pM \Rightarrow \mathbb{R}$

Prop

1. $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$
2. $R(X,Y,Z,W) = -R(Y,X,Z,W)$
3. $R(X,Y,Z,W) = -R(X,Y,W,Z)$
4. $R(X,Y,Z,W) = R(Z,W,X,Y)$

Pf. 1. Bianchi; 2. obvious

3. equivalently: $R(X,Y,Z,Z) = 0$.

$$R(X,Y,Z,Z) = \langle \nabla_Y\nabla_X Z, Z\rangle - \langle \nabla_X\nabla_Y Z, Z\rangle + \langle \nabla_{[X,Y]}Z, Z\rangle$$

$$Y\langle \nabla_X Z, Z\rangle - \cancel{\langle \nabla_X Z, \nabla_Y Z\rangle} \qquad X\langle \nabla_Y Z, Z\rangle - \cancel{\langle \nabla_Y Z, \nabla_X Z\rangle}$$

$$= \tfrac{1}{2} YX\langle Z,Z\rangle - \tfrac{1}{2} XY\langle Z,Z\rangle + \tfrac{1}{2}[X,Y]\langle Z,Z\rangle$$

$$= 0.$$

4.

$$\begin{array}{rl}
 & R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0 \\
 & R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) = 0 \\
 & R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) = 0 \\
+ & R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) = 0 \\
\hline
 & 2R(Z,X,Y,W) + 2R(W,Y,Z,X) = 0
\end{array}$$

$$\Rightarrow \quad R(Z,X,Y,W) = -R(W,Y,Z,X) = R(Y,W,Z,X). \qquad \square$$

In coordinates: Write $R_{ijkl} = \langle R(\partial_i,\partial_j)\partial_k, \partial_l\rangle$, $R(\partial_i,\partial_j)\partial_k = R_{ijk}^{\ \ \ l}\partial_l$.

Then $\qquad R_{ijkl} = R_{ijk}^{\ \ \ m} g_{ml}$, $\quad R_{ijk}^{\ \ \ l} = R_{ijkm} g^{ml}$

$1 \Rightarrow \quad R_{ijkl} + R_{jkil} + R_{kijl} = 0$

$2,3,4 \Rightarrow \quad R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$

Formula for R_{ijk}^{ℓ} :

$$R(\partial_i,\partial_j)\partial_k = \nabla_j\nabla_i\partial_k - \nabla_i\nabla_j\partial_k$$
$$= \nabla_j(\Gamma_{ik}^{\ell}\partial_\ell) - \nabla_i(\Gamma_{jk}^{\ell}\partial_\ell)$$
$$= \Gamma_{ik}^{\ell}\Gamma_{j\ell}^{m}\partial_m + (\partial_j\Gamma_{ik}^{\ell})\partial_\ell - \Gamma_{jk}^{\ell}\Gamma_{i\ell}^{m}\partial_m - (\partial_i\Gamma_{jk}^{\ell})\partial_\ell$$

$$\Rightarrow \boxed{R_{ijk}^{m} = \Gamma_{ik}^{\ell}\Gamma_{j\ell}^{m} - \Gamma_{jk}^{\ell}\Gamma_{i\ell}^{m} + \partial_j\Gamma_{ik}^{m} - \partial_i\Gamma_{jk}^{m}}$$

ex. $\mathbb{R}^n$, Euclidean metric: $R_{ijk}^{m} = 0 \ \forall\, i,j,k,m$.

Sectional Curvature

Idea: if $n=2$ then curvature tensor is determined by one number, R_{1212}. In general, slice by a plane to get a 2-d mfd.

<u>Def</u> $\sigma \subset T_pM$ 2-dimensional subspace. Then the <u>sectional curvature</u> of σ at p is, for any basis X,Y of σ,

$$K(\sigma) := \frac{R(X,Y,X,Y)}{|X\wedge Y|^2} \quad \text{where } |X\wedge Y|^2 := \det\begin{bmatrix} g(X,X) & g(X,Y) \\ g(Y,X) & g(Y,Y)\end{bmatrix}.$$

<u>Check</u> if we replace X,Y by another basis, this doesn't change.

Just need to check 3 elementary changes of basis:

1. $(X,Y)\to(Y,X)$: $R(X,Y,X,Y) = R(Y,X,Y,X)$ and $|X\wedge Y|^2 = |Y\wedge X|^2$
2. $(X,Y)\to(\lambda X,Y)$: $R(\lambda X,Y,\lambda X,Y) = \lambda^2 R(X,Y,X,Y)$ and $|(\lambda X)\wedge Y|^2 = \lambda^2|X\wedge Y|^2$
   $\det\begin{bmatrix}\lambda^2 g(X,X) & \lambda g(X,Y)\\ \lambda g(Y,X) & g(Y,Y)\end{bmatrix}$
3. $(X,Y)\to(X+Y,Y)$: $|(X+Y)\wedge Y|^2 = |X\wedge Y|^2$ and
   $$R(X+Y,Y,X+Y,Y) = R(X,Y,X+Y,Y) + R(Y,Y,X+Y,Y)$$
   $$= R(X,Y,X,Y) + R(X,Y,Y,Y) = R(X,Y,X,Y).$$

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Prop The sectional curvature at a point <u>determines</u> the curvature tensor.

Pf

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\Big|_{\alpha=\beta=0}\Big[\overbrace{R(X+\alpha Z, Y+\beta W, X+\alpha Z, Y+\beta W)}^{\text{determined by } K(\text{plane thru } X+\alpha Z,\, Y+\beta W)} - \underbrace{R(X+\alpha W, Y+\beta Z, X+\alpha W, Y+\beta Z)}_{\text{determined by } K(\text{plane thru } X+\alpha W,\, Y+\beta Z)}\Big]$$

$$= R(Z,W,X,Y) + R(Z,Y,X,W) + R(X,W,Z,Y) + R(X,Y,Z,W)$$
$$- R(W,Z,X,Y) - R(W,Y,X,Z) - R(X,Z,W,Y) - R(X,Y,W,Z)$$

$R(Z,Y,X,W) + R(X,Z,Y,W)$ $\qquad$ $-R(X,W,Y,Z) - R(W,Y,X,Z)$

(Bianchi)

$$= 4\,R(X,Y,Z,W) - R(Y,X,Z,W) + R(Y,X,W,Z)$$
$$= 6\,R(X,Y,Z,W). \quad \square$$

Def (M,g) has <u>constant sectional curvature</u> if

$$K(\sigma) = \text{Constant } K_0 \quad \forall\, p \in M \text{ and } \forall\, \sigma = \text{2-plane at } p.$$

Ex. $(\mathbb{R}^n, \text{flat})$ $\qquad$ $(\mathbb{R}^n/\mathbb{Z}^n = T^n, \text{flat})$

(S^n, round) $\qquad$ more generally M/G, M = const sectional curvature, G = gp of isometries

$(\mathbb{H}^n, \text{hyperbolic})$

in fact this give all const sectional curvature.

Prop Constant sectional curvature K_0

$$\Longleftrightarrow \quad R(X,Y,Z,W) = K_0\big(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\big).$$

Pf Call RHS $= R'(X,Y,Z,W)$. Easy to check: this satisfies all same properties as R. Also "sectional curvature" $K'(X,Y) = \frac{R'(X,Y,X,Y)}{|X\wedge Y|^2} = \frac{K_0(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle\langle Y,X\rangle)}{|X\wedge Y|^2} = K_0$

So since sectional curvature determines Riem curv, $R = R'$. $\square$

Two more curvatures

So far: $X\,Y\,Z\,W \to R(X\,Y\,Z\,W)$

$\sigma \to K(\sigma)$ sectional curv.

Several approaches.

First, in terms of an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM:

Def $X, Y \in T_pM$. The

{ other books may have different normalizations

Ricci tensor $\quad Ric_p(X,Y) = \frac{1}{n-1}\sum_{i=1}^{n} R(X, e_i, Y, e_i)$ $\quad$ (0,2) tensor

Ricci curvature $\quad Ric_p(X) = Ric_p(X,X)$ $\quad$ quadratic form

[note: $Ric_p(X,Y)$ is a symmetric bilinear form so it's determined by $Ric_p(X)$]

Scalar curvature $\quad S(p) = \frac{1}{n}\sum_{i=1}^{n} Ric_p(e_i) \quad \left(= \frac{1}{n(n-1)}\sum_{i,j} R(e_i, e_j, e_i, e_j)\right).$

These are independent of the choice of $e_1, \dots, e_n$:

Define $\varphi_{X,Y} : T_pM \to T_pM$ by $\varphi_{X,Y}(Z) = R(X,Z)Y$.

Then

$$Ric_p(X,Y) = \frac{1}{n-1}\operatorname{Tr}\varphi_{X,Y} \quad \text{indep. of basis.}$$

Now $Ric_p(X,Y)$ is a bilinear form so $\exists\, T : T_pM \to T_pM$ linear s.t.

$$Ric_p(X,Y) = \langle T(X), Y\rangle.$$

Then

$$S(p) = \frac{1}{n}\operatorname{Tr} T \; : \quad \text{note } Ric_p(e_i, e_i) = \langle T(e_i), e_i\rangle = (i,i) \text{ entry of } T.$$

Local coordinates: note $\{\partial_i\}$ isn't usually orthonormal.

$$R_{ij} := Ric_p(\partial_i, \partial_j).$$

The map $\varphi_{\partial_i,\partial_j} : Z \mapsto R(\partial_i, Z)\partial_j$ sends $\partial_k \mapsto R_{ikj}{}^{l}\partial_l$

so the (k,l) entry of $\varphi_{\partial_i,\partial_j}$ is $R_{ikj}{}^{l}$

$$\to \quad R_{ij} = \frac{1}{n-1}\operatorname{tr}\varphi_{\partial_i,\partial_j} = \boxed{\frac{1}{n-1}R_{ikj}{}^{k} = \frac{1}{n-1}R_{ikjl}\,g^{lk} = R_{ij}}$$

$$R_{ij} = \langle T(\partial_i), \partial_j \rangle = \langle T_i^k \partial_k, \partial_j \rangle = T_i^k g_{kj} \Rightarrow T_i^k = R_{ij} g^{jk}$$

$$\Rightarrow S = \frac{1}{n} \operatorname{Tr} T = \frac{1}{n} T_i^i = \frac{1}{n} R_{ij} g^{ji} = \boxed{\frac{1}{n(n-1)} R_{ikj}^{\ \ \ k} g^{ji} = S \\ = \frac{1}{n(n-1)} R_{ikjl} g^{ij} g^{kl}}$$

What are these curvatures?

Weyl: View the curvature tensor algebraically.

Let $V = T_pM$. Recall a $(0,2)$ tensor is in $V^* \otimes V^*$;

antisymmetric is in $\Lambda^2 V^*$. The curvature tensor R is in:

$$V^* \otimes V^* \otimes V^* \otimes V^* \supset (\Lambda^2 V^*) \otimes (\Lambda^2 V^*) \supset \operatorname{Sym}^2 \Lambda^2 V^*.$$

Define the Bianchi map $\quad b: \operatorname{Sym}^2 \Lambda^2 V^* \circlearrowleft$

$$b(T)(X\,Y\,Z\,W) = T(X\,Y\,Z\,W) + T(Y\,Z\,X\,W) + T(Z\,X\,Y\,W).$$

Then $\quad R \in \ker b =: \underbrace{C(V)}_{\text{space of curvature tensors on } V} \subset \operatorname{Sym}^2 \Lambda^2 V^*$

Note $O(n)$ acts on V, V^*, $\operatorname{Sym}^2 \Lambda^2 V^*$, $C(V)$.

Fact: we can orthogonally decompose $C(V)$ as an $O(n)$-module:

$$C(V) = \underbrace{\mathbb{R} \oplus \operatorname{Sym}_0^2 V^*}_{\text{all symm bilinear forms} = \operatorname{Sym}^2 V^*} \oplus W(V)$$

$\operatorname{Sym}_0^2 V^*$: traceless symm bilinear forms

where the map $C(V) \to \operatorname{Sym}^2 V^*$ is $(X, Y) \mapsto \operatorname{tr} T(X, \cdot, Y, \cdot)$

and the map $\operatorname{Sym}^2 V^* \to \mathbb{R}$ is $T \mapsto \operatorname{tr} T$.

$$\text{So} \quad \underset{C(V)}{\overset{R}{\cap}} \longrightarrow \underset{\operatorname{Sym}^2 V^*}{\overset{Ric(X,Y)}{\cap}} \longrightarrow \underset{\mathbb{R}}{\overset{S(p)}{\cap}}$$

What about $W(V)$? The component of R in $W(V)$ is the <u>Weyl tensor</u> of R. Facts: · if $n \geq 5$, $W(V)$ is irreducible

· if $n \leq 3$, $W(V) = 0$. (R is determined by Ric!)

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<u>Review:</u>

<u>(1,3) curvature tensor</u> $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$

<u>(0,4) curvature tensor</u> $R(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle$

<u>Sectional curvature</u> $\sigma \subset T_pM$ 2-plane, e_1, e_2 = ONB for σ

$\Rightarrow$ $K(\sigma) = R(e_1, e_2, e_1, e_2)$

<u>Ricci tensor</u> $\mathrm{Ric}_p(X,Y) = \frac{1}{n-1} \sum_{i=1}^{n} R(X, e_i, Y, e_i)$ $\{e_1, \ldots, e_n\}$ ONB for T_pM

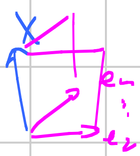
<u>Ricci curvature</u> $\mathrm{Ric}_p(X) = \mathrm{Ric}_p(X,X)$

<u>Scalar curvature</u> $S(p) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{Ric}_p(e_i)$.

If X = unit vector in T_pM, then

$\mathrm{Ric}_p(X)$ = average sectional curvature of planes through X.

Complete X to an ONB $X, e_2, \ldots, e_n$, and let $\sigma_i = \langle X, e_i \rangle$.



$K(\sigma_i) = R(X, e_i, X, e_i)$,

$\mathrm{Ric}_p(X) = \frac{1}{n-1} \sum_{i=2}^{n} K(\sigma_i)$ (note $R(X,X,X,X) = 0$).

Nice computation: parametrize planes through X by

$v \in \{\text{unit vectors} \perp \text{to } X\} = S^{n-2}$. With the usual measure on S^{n-2},

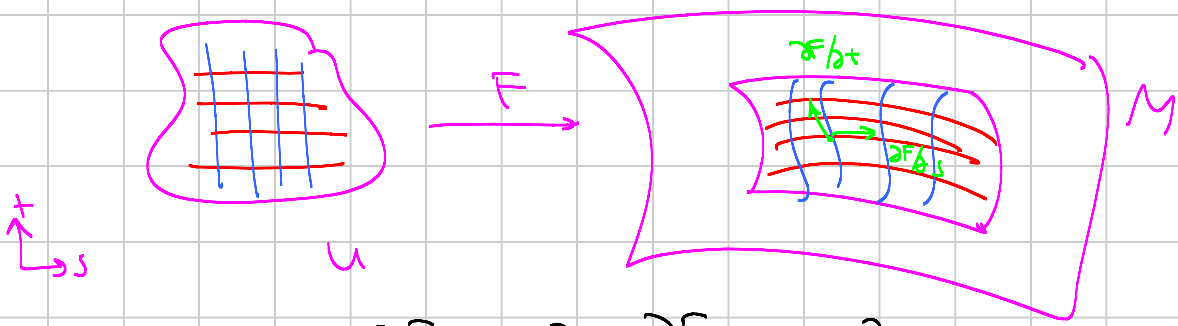
the average $K(\sigma_{X,v})$ over all v is $\mathrm{Ric}_p X$.

<u>In particular</u>: for a space of constant sectional curvature K_0,

$\mathrm{Ric}_p(X) = K_0 |X|^2$, $\mathrm{Ric}_p(X,Y) = K_0 \langle X, Y \rangle$, $S = K_0$.

Derivatives on Surfaces

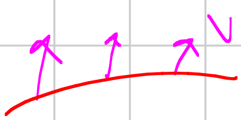
Def A <u>parametrized surface</u> is a smooth map $F: U \to M$. ($U \subseteq \mathbb{R}^2$)



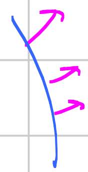
Write $\frac{\partial F}{\partial s} = F_*\left(\frac{\partial}{\partial s}\right)$, $\frac{\partial F}{\partial t} = F_*\left(\frac{\partial}{\partial t}\right)$.

Note $F(s_0, \cdot)$, $F(\cdot, t_0)$ are curves in M for fixed s_0, t_0.

A vector field V along the surface is in particular a vector field along each of these curves: can define "$\frac{D}{dt}$" for V along these curves.



$F(\cdot, t_0) \to \frac{D}{\partial s} V$



$F(s_0, \cdot) \to \frac{D}{\partial t} V$.

$\Rightarrow$ get vector fields $\frac{D}{\partial s} V$, $\frac{D}{\partial t} V$ on the surface.

Prop $\frac{D}{\partial s} \frac{\partial F}{\partial t} = \frac{D}{\partial t} \frac{\partial F}{\partial s}$.

Pf. $\nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t} - \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial s} = \left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right] = F_*\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0$.

Prop $\frac{D}{\partial t} \frac{D}{\partial s} V - \frac{D}{\partial s} \frac{D}{\partial t} V = R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) V$.

Pf More or less by definition: $\frac{D}{\partial s} V = \nabla_{\frac{\partial F}{\partial s}} V$, $\frac{D}{\partial t} V = \nabla_{\frac{\partial F}{\partial t}} V$

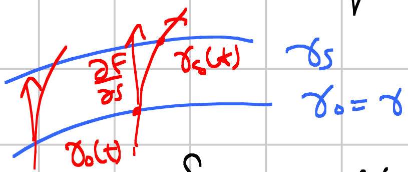
and $\left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right] = F_*\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0$. $\square$

Jacobi vector fields

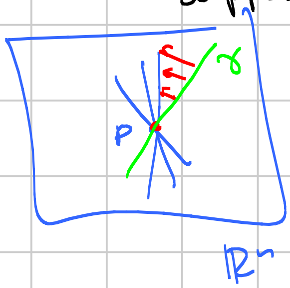
Let γ = geodesic.

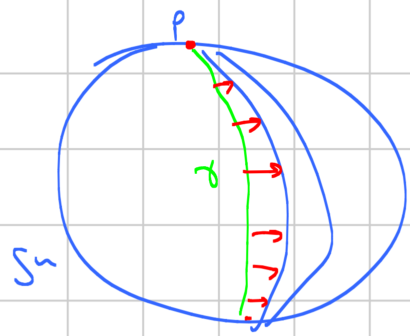
Say we have a 1-parameter family of geodesics $\gamma_s(t)$, $s \in (-\epsilon, \epsilon)$, smoothly varying, $\gamma_0 = \gamma$. Define $F(s,t) = \gamma_s(t)$.

The infinitesimal change in geodesic is $\frac{\partial}{\partial s} F(s,t)$: at $s=0$ this is a vector field along γ_0.



Suppose $\gamma_s(0) = p$ $\forall s$. Then $\frac{\partial F}{\partial s}$ measures the "spread" of the geodesics.



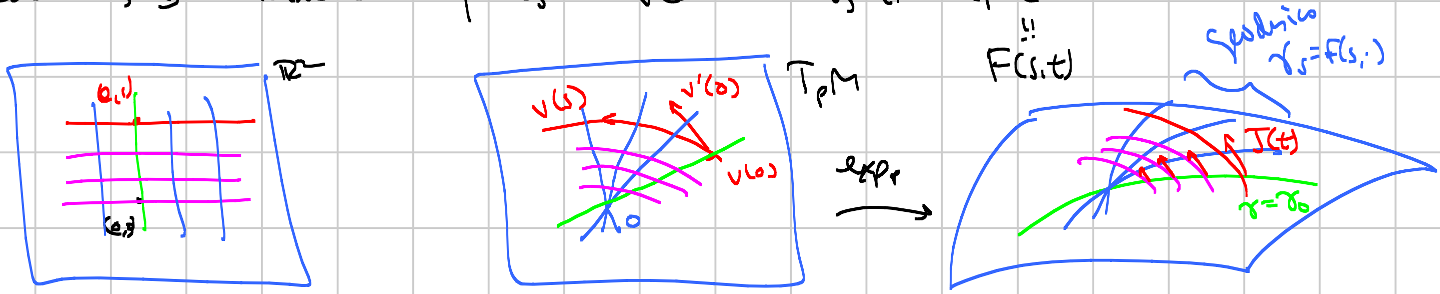


The difference between these is curvature.

Special case

Let $\gamma_s(t)$ be a family of geodesics with $\gamma_s(0) = p$ $\forall s$.

Each γ_s is determined by $\gamma_s'(0) = v(s)$: $\gamma_s(t) = \exp(tv(s))$.

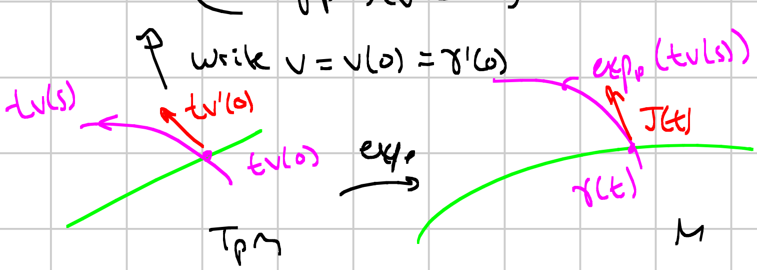


$(s,t) \longmapsto tv(s)$

Then the infinitesimal change at $\gamma = \gamma_0$ is

$$J(t) := \frac{\partial F}{\partial s}(0,t) = (d(\exp_p))_{tv}(tw) \quad \text{where} \quad v = v(0) = \gamma'(0)$$

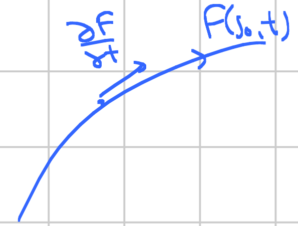
write $v = v(0) = \gamma'(0)$ $\qquad w = v'(0) \in T_pM$.



Differential equation for $J(t)$:

Fixing $s=s_0 \Rightarrow F(s_0,\cdot)$ is a geodesic $\Rightarrow \frac{D}{\partial t}\frac{\partial F}{\partial t}=0$

$$\Rightarrow 0 = \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial F}{\partial t}$$
$$= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial F}{\partial t} - R\left(\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t}$$
$$= \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial F}{\partial s} + R\left(\frac{\partial F}{\partial t},\frac{\partial F}{\partial s}\right)\frac{\partial F}{\partial t}.$$



Plug in $s=0$:

$$\boxed{0 = \frac{D^2}{dt^2}J(t) + R(\gamma',J)\gamma'} \qquad (*)$$

<u>Def</u> A vector field J along a geodesic $\gamma:[0,a]\to M$ is a <u>Jacobi field</u> if $(*)$ holds.

So. if $F(s,t)=\gamma_s(t)$ is a family of geodesics (no assumption on $\gamma_s(0)$) then $\frac{\partial F}{\partial s}(0,t)=J(t)$ is a Jacobi field along $\gamma_0=\gamma$.

<u>Prop</u> $\exists!$ Jacobi field for sufficiently small $\underline{a}$ and specified initial conditions

$$J(0),\ J'(0) \in T_pM.$$
$$J'(0) = \frac{D}{dt}J(0)$$

Pf Choose ONB $e_1,\dots,e_n$ of $T_pM \rightsquigarrow$ by parallel transport, get $e_1(t),\dots,e_n(t)$ ONB along γ. Want to find

$$J(t) = (\Sigma)\, f^i(t)\, e_i(t)$$
$$\Rightarrow \frac{DJ}{dt} = \frac{df^i}{dt}e_i(t) \qquad \text{since } e_i(t) \text{ parallel}$$
$$\Rightarrow \frac{D^2J}{dt} = \frac{d^2f^i}{dt}e_i(t).$$

Write $r_{ij}(t) = \langle R(\gamma'(t), e_i(t))\gamma'(t), e_j(t)\rangle$. Then

(*) holds $\iff \frac{d^2 f^i}{dt} e_i(t) + f^j R(\gamma', e_j)\gamma' = 0$

$\iff \frac{d^2 f^i}{dt} + f^j(t) r_{ij}(t) = 0$.

This is a 2nd order linear system of ODEs. □

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Cor The vector space of Jacobi v.f.s along a geodesic is 2n-dimensional.

Two "trivial" Jacobi v.f.s:

- $J_1(t) = \gamma'(t)$ $\quad \frac{D}{dt}\gamma' = 0 \quad J_1(0) = \gamma'(0),\ J_1'(0) = 0$

  this corresponds to the variation $F(s,t) = \gamma_s(t) = \gamma(t+s)$
- $J_2(t) = t\gamma'(t)$ $\quad \frac{D^2}{dt^2}(t\gamma'(t)) = 0 \quad J_2(0) = 0,\ J_2'(0) = \gamma'(0)$

  this corresponds to $F(s,t) = \gamma_s(t) = \gamma((s+1)t)$.

Note: $\{\text{Jacobi along } \gamma(t)\} \longrightarrow \mathbb{R}^2$

$J(t) \longmapsto (\langle J(0), \gamma'(0)\rangle, \langle J'(0), \gamma'(0)\rangle)$

this is a surjective linear map because of J_1, J_2. The kernel is $\{\text{Jacobi with } J(0), J'(0) \perp \gamma'(0)\} = (2n-2)$-dim'l vector space.

Prop $\langle J(t), \gamma'(t)\rangle = \langle J'(0), \gamma'(0)\rangle t + \langle J(0), \gamma'(0)\rangle$

so $J(0), J'(0) \perp \gamma'(0) \iff J(t) \perp \gamma'(t)\ \forall t$.

Pf Write $|\gamma'(t)| = a$. Then $e_1(t) = \frac{\gamma'(t)}{a}$.

$\langle J(t), \gamma'(t)\rangle = a f^1(t)$, $\quad r_{1j}(t) = \langle R(\gamma', e_j)\gamma', e_1\rangle = 0$

$\Rightarrow \frac{d^2 f^1}{dt^2} = 0 \Rightarrow \langle J(t), \gamma'(t)\rangle = At + B$.

$t=0 \Rightarrow B = \langle J(0), \gamma'(0)\rangle$; $A = \frac{d}{dt}\big|_{t=0}\langle J, \gamma'\rangle = \langle J', \gamma'\rangle(0)$. □

Break up {Jacobi v.f.} into subspace:

dim	no restriction	J normal to γ	
$J(0)$ arbitrary	$2n$	$2n-2$	$\leftarrow$ cut out J_1, J_2
$J(0)=0$	n	$n-1$	$\leftarrow$ cut out J_2

These are achieved by the special case $F(s,t) = \exp_p(tv(s))$ (infinitesimal variation γ_s with $\gamma_s(0)=p\ \forall s$). In this case:

$J(0)=0$ since $F(s,0)=p$. Also:

$J(t) = (d\exp_p)_{tv}(tw)$ where $w = v'(0)$

$$\Rightarrow \frac{DJ}{dt}(0) = \frac{D}{dt}\Big|_{t=0}\left(t(d\exp_p)_{tv}(w)\right) = (d\exp_p)_0(w) = w.$$

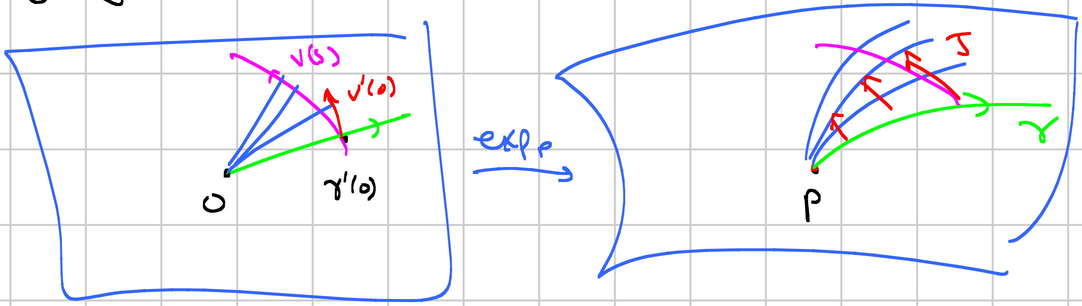
So: $J(0)=0$, $J'(0) = \frac{DJ}{dt}(0) = w$.

Conversely:

<u>Prop</u> $\gamma:[0,a]\to M$ geodesic, J = Jacobi vector field along γ with $J(0)=0$.
Then $\exists$ family of geodesics $F(s,t) = \gamma_s(t)$ with $\gamma_0 = \gamma$
s.t. $J(t) = \frac{\partial F}{\partial s}(0,t)$.

actually holds without this too.

<u>Pf</u> Try $F(s,t) = \exp_p(tv(s))$. Choose path $\{v(s)\}\subset T_pM$ with $v(0)=\gamma'(0)$, $v'(0) = J'(0)$. Then $\frac{\partial F}{\partial s}(0,t) =: \bar{J}(t)$ is a Jacobi vector field, and $J, \bar{J}$ have same initial conditions so $J = \bar{J}$. $\square$



Now: $F(s,t) = \exp_p(t v(s))$, $J(t) = \frac{\partial F}{\partial s}(0,t)$, $v(0) = v = \gamma'(0)$

$J(0) = 0$, $J'(0) = w = v'(0)$. <u>Assume $|w| = 1$</u>.

<u>Prop</u> With these assumptions:

$$|J(t)|^2 = t^2 - \frac{1}{3} R(v,w,v,w)\, t^4 + \underbrace{o(t^4)}_{\lim_{t\to 0} \frac{o(t^4)}{t^4} = 0}.$$

<u>Pf</u> $J(0)=0$, $J'(0)=w$, $J''(0) = -(R(\gamma',J)\gamma')(0) = 0$.

$\langle J,J\rangle(0) = 0$

$\langle J,J\rangle'(0) = 2\langle J(0), J'(0)\rangle = 0$

$\langle J,J\rangle''(0) = 2\langle J, J''\rangle(0) + 2\langle J',J'\rangle(0) = 2$

$\langle J,J\rangle'''(0) = 2\langle J, J'''\rangle(0) + 6\langle J', J''\rangle(0) = 0$

$\langle J,J\rangle^{(4)}(0) = \underset{0}{\cancel{2\langle J, J^{(4)}\rangle(0)}} + 8\langle J', J'''\rangle(0) + \underset{0}{\cancel{6\langle J'', J''\rangle(0)}}$

To calculate $J'''(0)$, note for any vector field W along γ:

$$\langle R(\gamma',J)\gamma', W\rangle = \langle R(\gamma',W)\gamma', J\rangle$$

$\left.\frac{d}{dt}\right|_{t=0} \Rightarrow \quad \langle \frac{D}{dt} \underbrace{R(\gamma',J)\gamma'}_{-J''}, W\rangle = \langle R(\gamma',W)\gamma', J'\rangle(0)$

$\qquad \| \qquad\qquad\qquad\qquad = \langle R(\gamma',J')\gamma', W\rangle(0)$

$-\langle J'''(0), W(0)\rangle$

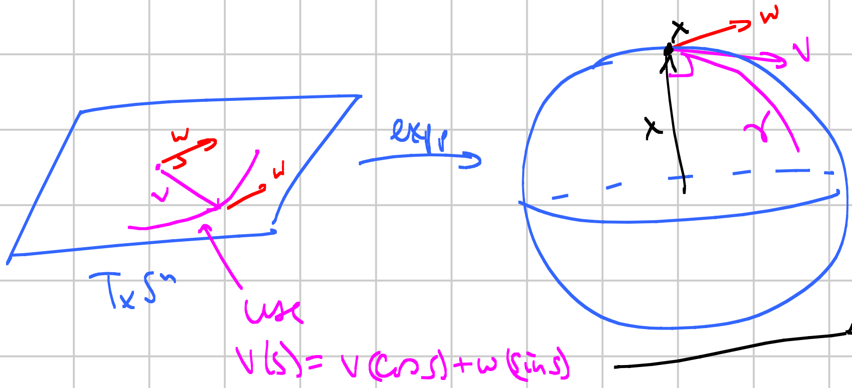
$\Rightarrow \quad J'''(0) = -R(\gamma',J')\gamma'(0) = -R(v,w)v$

$\Rightarrow \quad \langle J,J\rangle^{(4)}(0) = 8\langle J', J'''\rangle(0) = -8R(v,w,v,w)$. $\square$

<u>Cor</u> If $|v|=1$ (γ par. by arclength), $|w|=1$, $\langle v,w\rangle = 0$, and σ = plane gen'd by v,w, then

$$|J(t)|^2 = t^2 - \frac{1}{3}K(\sigma)\, t^4 + o(t^4).$$

Can use this to calculate sectional curvature for (S^n, round metric).



$x \in S^n$, $v, w \in T_xS^n$ ONB for $\sigma(v,w)$.

$\gamma(t) = x\cos t + v\sin t$

$F(s,t) =$
$\gamma_s(t) = x\cos t + (v\cos s + w\sin s)\sin t$

$\Rightarrow J(t) = \frac{\partial F}{\partial s}(0,t) = w\sin t$

$\Rightarrow |J(t)|^2 = \sin^2 t = \left(t - \frac{t^3}{6} + \dots\right)^2 = t^2 - \frac{t^4}{3} + \dots$

$\Rightarrow \quad K(\sigma(v,w)) = 1.$

(Similarly for sphere of radius R: $K(\sigma(v,w)) = \frac{1}{R^2}$.)

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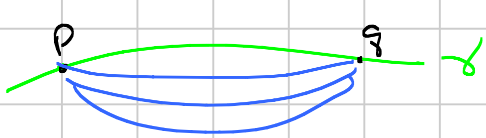
Conjugate Points

Def $\gamma : [a,b] \to M$ geodesic. Two points $p = \gamma(t_0)$, $q = \gamma(t_1)$ along γ are *conjugate* if $\exists$ nonzero Jacobi field J along γ with $J(t_0) = J(t_1) = 0$.

One way to get this: γ = part of a family of geodesics between p and q.



The dimension of the vector space $\{J(t_0) = J(t_1) = 0,\ J \text{ Jacobi}\}$ is the *multiplicity* of the conjugate point (think: k-dim'l family of geodesics).

Ex. (S^n, round): antipodal pts are conjugate along any geodesic, with multiplicity $(n-1)$.

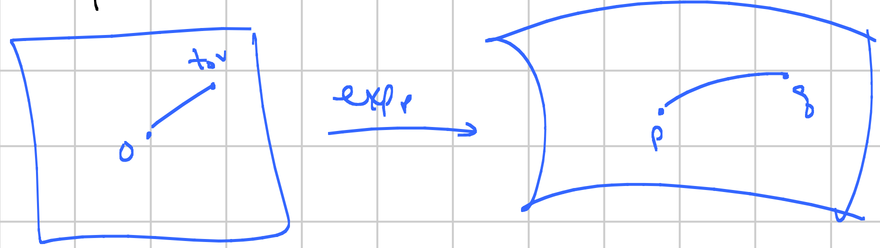
Def q is the first <u>conjugate</u> to p along γ if no other conjugate points before q. The <u>conjugate locus</u> of p is $\{$first conjugate$\}$ over all geodesics.

Ex: S^n: conjugate locus of p is $\{-p\}$

2-d ellipsoid: (more generic)

$\mathbb{R}^n$: no conjugate locus (Jacobi fields satisfy $J''(t)=0$)

Prop $\gamma: [0,a] \to M$ geodesic, $\gamma(0)=p$, $\gamma(t)=\exp_p(tv)$, $v=\gamma'(0)$.
Then $q=\gamma(t_0)=\exp_p(t_0 v)$ is conjugate to p along $\gamma \iff$ $t_0 v$ is a <u>critical point</u> of $\exp_p$, i.e. $d(\exp_p)_{t_0 v}: T_pM \to T_qM$ is not surjective.



Pf q conjugate $\iff \exists$ nonzero Jacobi field $J(t)$ with $J(0)=J(t_0)=0$.
Recall if $J'(0)=w(\neq 0)$ then $J(t)=d(\exp_p)_{tv}(tw)$, so

$\exists J^{\neq 0}$ with $J(t_0)=0 \iff \exists w^{\neq 0}$ with $d(\exp_p)_{t_0 v}(w)=0 \iff \ker d(\exp_p)_{t_0 v} \neq 0$. $\square$

Hadamard (Hadamard–Cartan) Thm

by Hopf–Rinow, $\exp_p$ is defined on all of T_pM

M complete Riemannian with nonpositive sectional curvature: $K(\sigma) \le 0 \ \forall \sigma \subset T_pM \ \forall p$.

Then $\forall p \in M$, $\exp_p : \underset{\cong \mathbb{R}^n}{T_pM} \to M$ is a covering map.

If M is simply connected, then $\exp_p : \mathbb{R}^n \to M$ is a diffeomorphism.

Ex: $\mathbb{R}^n$, $\mathbb{H}^n$; not S^n (Cor: S^n can't have a metric of ≤ 0 sect. curv.)

Lemma 1 M (geod.) complete Riem, nonpositive sectional curvature.

(1) Conjugate locus $(p) = \emptyset \ \forall p \in M$: no conjugate pts to p along any geodesic.

(2) $\exp_p$ is a local diffeo.

Pf (1) J = Jacobi field along γ, $\gamma(0)=p$, $\gamma(t_0)=q$, $J(0)=J(t_0)=0$.

$$\frac{d}{dt}\langle J(t), J(t)\rangle = 2\langle J', J\rangle$$

$$\begin{aligned}\frac{d^2}{dt^2}\langle J, J\rangle &= 2\langle J', J'\rangle + 2\langle J'', J\rangle \\ &= 2|J'|^2 - 2\langle R(\gamma', J)\gamma', J\rangle \\ &= 2|J'|^2 - 2\underset{\text{gen'd by } \gamma', J}{K(\sigma)}\ \underbrace{|\gamma' \wedge J|^2}_{\ge 0} \\ &\ge 0.\end{aligned}$$

But if J has two zeros then $|J|^2$ has a max, where $\frac{d^2}{dt^2}|J|^2 < 0$.

(2) Follows from previous prop. □

Lemma 2 M, N Riem, M geod. complete, $f: M \to N$ surjective local isometry (in particular, local diffeo). Then f is a covering map.

Pf $p\in N$, $f^{-1}(p)=\{p_i\}$. Let $B_r(p)$ = normal ball in N, $\exp_p: B_r(0)\to B_r(p)$ diffeo.
Write $U=B_r(p)$, $U_i=\exp_{p_i}(B_r(0))\subset M$ ($\exists$ since complete).

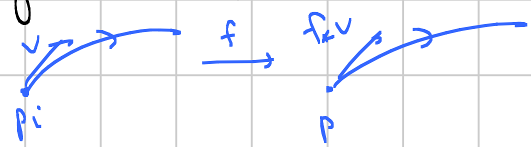
Claim: $f^{-1}(U)=\coprod U_i$, $f: U_i\to U$ diffeo.

① $f(U_i)\subset U$: $q\in U_i \Rightarrow \exists$ geodesic γ from p_i to q, $\ell(\gamma)<r$.
Local isometry $\Rightarrow$ $f\circ\gamma$ = geodesic from p to $f(q)$,
$\ell(f\circ\gamma)<r \Rightarrow f(q)\in U$.

② $f: U_i\to U$ diffeo.

$$\begin{tikzcd} T_{p_i}M \supset B_r(0) \arrow[r, "\exp_{p_i}"] \arrow[d, "f_*"', "\cong"] & U_i \arrow[d, "f"] \\ T_pN \supset B_r(0) \arrow[r, "\exp_p", "\cong"'] & U \end{tikzcd}$$

Diagram commutes:



Since f_*, $\exp_p$ are diffeos, $\exp_{p_i}$ is injective $\Rightarrow$ bijective,
so f is bijective $\Rightarrow$ $\exp_{p_i}$, f are diffeos.

③ Pf of claim: $f^{-1}(U)\overset{?}{=}\coprod U_i$. Suppose $\bar q\in f^{-1}(U)$, $q=f(\bar q)$.

Reverse geod. $p\to q$ to get geod. γ from q to p.
Write $v=\gamma'(0) \Rightarrow \exists$ geodesic $\bar\gamma$ with $\bar\gamma(0)=\bar q$,
$\bar\gamma'(0)=(f_*)^{-1}(v)$. Then $f\circ\bar\gamma=\gamma$ so
the endpt of $\bar\gamma$ is some $p_i \Rightarrow \bar q\in U_i$.
If $\exists$ 2 geods $\bar\gamma_1,\bar\gamma_2$ from $\bar q$ to p_1,p_2 then they must
project to the same geodesic from q to p by uniqueness. $\Rightarrow\Leftarrow$ $\square$

Pf of Hadamard: $\exp_p : T_pM \to M$ well-defined, surjective. Lemma 1 $\Rightarrow$ local diffeo. So can pull back metric on M to get metric on T_pM; then $\exp_p$ is a local isometry. Now straight lines $\{tv\}$ through $0 \in T_pM$ are geodesics since they map to geodesics, so by Hopf–Rinow, T_pM is geod. complete. Lemma 2 $\Rightarrow$ covering map. $\square$

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Variations of Energy

Idea: $p, q \in M \to$ let $\mathcal{P}(p,q) = \{$piecewise diff'ble paths from p to $q\}$.
Length gives a map $\ell : \mathcal{P}(p,q) \to \mathbb{R}_{\geq 0}$:

$$\ell(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2} dt \qquad \gamma : [a,b] \to M$$

A minimizing geodesic is a global minimum for ℓ.
More generally, a geodesic is a critical pt for ℓ:
"$d\ell_\gamma : T_\gamma \mathcal{P}(p,q) \to \mathbb{R}$" satisfies $d\ell_\gamma = 0$.
We'll make this precise.

First note: if γ is a length-minimizing curve from p to q,

$$\ell(\gamma) \leq \ell(\tilde{\gamma}) \quad \forall\ \tilde{\gamma} \in \mathcal{P}(p,q),$$

then γ is a reparametrization of a geodesic.
Can get rid of reparam. by considering instead the energy function

$$E(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle \, dt$$

which, unlike $\ell(\gamma)$, changes under reparam.

Prop $\gamma:[a,b]\to M$ length-min geodesic between p and q. Then
$$E(\gamma)\le E(\tilde\gamma)$$
for any $\tilde\gamma:[a,b]\to M$ between p and q, equality $\Leftrightarrow$ $\tilde\gamma$ = length-min. geodesic.

Lemma $\ell(\gamma)^2 \le (b-a)E(\gamma)$ for any path γ, equality $\Leftrightarrow$ constant speed.

Pf $\ell(\gamma)^2 = \left(\int_a^b |\gamma'(t)|\,dt\right)^2 \underset{\text{Cauchy-Schwarz}}{\le} \left(\int_a^b 1\,dt\right)\left(\int_a^b |\gamma'(t)|^2 dt\right) = (b-a)E(\gamma)$,

equality $\Leftrightarrow$ $|\gamma'(t)|, 1$ are proportional. $\square$

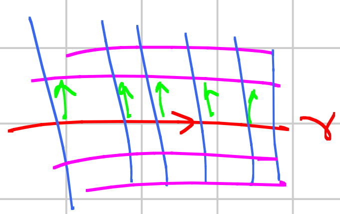
Pf of Prop $(b-a)E(\gamma) = \ell(\gamma)^2 \le \ell(\tilde\gamma)^2 \le (b-a)E(\tilde\gamma)$,

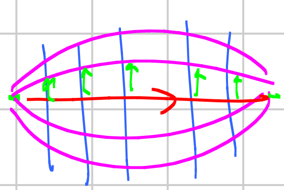
equality $\Leftrightarrow$ $\ell(\gamma)=\ell(\tilde\gamma)$ and $\tilde\gamma$ has constant speed. $\square$

Next: define "$T_\gamma \mathcal{P}(p,q)$" = infinitesimal deformation of a path.

Def A variation of $\gamma:[a,b]\to M$ is a smooth map
$$F:(-\epsilon,\epsilon)\times[a,b]\to M \quad \text{for some } \epsilon>0$$
with $F(0,t)=\gamma(t)$.

A proper variation is a variation with $F(s,a)=\gamma(a)$, $F(s,b)=\gamma(b)$.

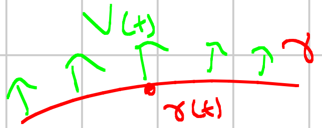




Variation $\to$ variational field $V(t)=\frac{\partial F}{\partial s}(0,t)$ vector field along γ.

Proper variation $\to$ $V(t)$ with $V(a)=V(b)=0$.

Prop If $V(t)$ = smooth vector field along γ then $\exists$ variation $F(s,t)$ (for some ϵ) s.t. V = its variational field. If $V(a)=V(b)=0 \Rightarrow \exists$ proper variation.



Pf $t\in[a,b] \Rightarrow \exists\, \epsilon(t)$ s.t. $s \mapsto \exp_{\gamma(t)}(sV(t))$ is defined for $|s|<\epsilon(t)$. Compactness $\Rightarrow \exists\, \epsilon$ s.t.

$$F(s,t) = \exp_{\gamma(t)}(sV(t)) \quad \text{is defined on } (-\epsilon,\epsilon)\times[a,b].$$

- Smooth since geodesic flow is smooth
- $\frac{\partial F}{\partial s}(0,t) = V(t)$ since $F(\cdot,t)$ = geod. through $\gamma(t)$, initial velocity $V(t)$
- $V(a)=0 \Rightarrow F(s,a)=\gamma(a)$. $\square$

Now: Consider $E: \mathcal{P}(p,q) \to \mathbb{R}$: want to find minima.
Want to calculate 1st and 2nd derivatives.

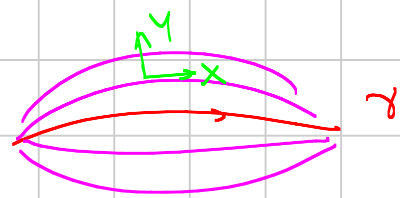
1st: $dE_\gamma : T_\gamma \mathcal{P}(p,q) \to \mathbb{R}$.

$$\updownarrow$$

$V(t)$ vector field along γ, $V(a)=V(b)=0$.

Prop (First Variation formula)
$F: (-\epsilon,\epsilon)\times[a,b] \to M$ variation of $\gamma:[a,b]\to M$, variational field $V(t)$. Then

$$\frac{1}{2}\frac{d}{ds}\Big|_{s=0} E(\gamma_s) = \langle V(t), \gamma'(t)\rangle \Big|_{t=a}^{t=b} - \int_a^b \langle V(t), \tfrac{D}{dt}\gamma'(t)\rangle\, dt$$



Notation: write $Y(s,t)=\frac{\partial F}{\partial s}$, $X(s,t)=\frac{\partial F}{\partial t}$
$Y(0,t)=V(t)$, $X(0,t)=\gamma'(t)$.
$X(s,t)=\gamma_s'(t)$

Pf $E(s) = \int_a^b \langle X, X\rangle\, dt$

$$\Rightarrow \tfrac{1}{2}E'(s) = \tfrac{1}{2}\int_a^b \frac{\partial}{\partial s}\langle X, X\rangle\, dt$$
$$= \int_a^b \left\langle \frac{D}{\partial s}X, X\right\rangle dt$$
$$= \int_a^b \left\langle \frac{D}{\partial t}Y, X\right\rangle dt$$
$$= \int_a^b \left(\frac{\partial}{\partial t}\langle Y, X\rangle - \left\langle Y, \frac{D}{\partial t}X\right\rangle\right) dt$$
$$= \langle Y, X\rangle\Big|_{t=a}^{t=b} - \int_a^b \left\langle Y, \frac{D}{dt}X\right\rangle dt.$$

$$\frac{DX}{\partial s} = \frac{D}{\partial s}\frac{\partial F}{\partial t} = \frac{D}{\partial t}\frac{\partial F}{\partial s} = \frac{DY}{\partial t}$$

Now plug in $s=0$. $\square$

Cor γ is a geodesic $\Leftrightarrow$ $\forall$ proper variations of γ, $E'(0)=0$.

Pf. $\Rightarrow$ follows from Prop.

$\Leftarrow$: by Prop, $\forall$ vector field $V(t)$ along γ with $V(a)=V(b)=0$,
$$\int_a^b \langle V(t), \nabla_{\gamma'}\gamma'\rangle dt = 0.$$

Choose $V(t) = f(t)\gamma'(t)$ where $f(a)=f(b)=0$, $f>0$ on (a,b).

$$\int_a^b f(t)\,|\nabla_{\gamma'}\gamma'|^2\, dt = 0 \Rightarrow \nabla_{\gamma'}\gamma' = 0. \quad \square$$

So geodesics = critical pts of E. To determine if they're minimizing, need 2nd derivative.

Prop (Second Variation Formula)

$\gamma: [a,b] \to M$ geodesic, $F: (-\epsilon, \epsilon)\times[a,b] \to M$ variation. Then

$$\tfrac{1}{2}E''(0) = \left\langle \frac{D}{\partial s}\frac{\partial F}{\partial s}(0,t), \gamma'\right\rangle\Big|_{t=a}^{t=b} + \int_a^b \left(|V'|^2 - R(V,\gamma',V,\gamma')\right) dt.$$

$$V' = \frac{DV}{dt}$$

Pf As before: $\frac{1}{2}\frac{\partial}{\partial s}\langle X,X\rangle = \langle \frac{DY}{\partial t}, X\rangle$

$$\Rightarrow \quad \frac{1}{2}\frac{\partial^2}{\partial s^2}\langle X,X\rangle = \underbrace{\langle \frac{D}{\partial s}\frac{DY}{\partial t}, X\rangle}_{\frac{D}{\partial t}\frac{DY}{\partial s} - R(\underbrace{\frac{\partial F}{\partial s}}_{Y}, \underbrace{\frac{\partial F}{\partial t}}_{X})Y} + \langle \frac{DY}{\partial t}, \underbrace{\frac{DX}{\partial s}}_{\frac{DY}{\partial t}}\rangle$$

$$= \underbrace{\langle \frac{D}{\partial t}\frac{DY}{\partial s}, X\rangle}_{\frac{\partial}{\partial t}\langle \frac{DY}{\partial s}, X\rangle - \underbrace{\langle \frac{DY}{\partial s}, \frac{DX}{\partial t}\rangle}_{=0 \text{ at } s=0 \text{ since } \gamma = \text{geodesic}}} - R(Y,X,Y,X) + \left|\frac{DY}{\partial t}\right|^2$$

$$\Rightarrow \quad \frac{1}{2}\frac{\partial^2}{\partial s^2}\langle X,X\rangle\Big|_{s=0} = \frac{\partial}{\partial t}\langle \frac{DY}{\partial s}, \gamma'\rangle - R(V,\gamma',V,\gamma') + |V'|^2;$$

now integrate from $t=a$ to $t=b$. $\square$

Cor If F = proper variation then

$$\frac{1}{2}E''(0) = \int_a^b \left(|V'|^2 - R(V,\gamma',V,\gamma')\right)dt$$

depends only on V.

Rmk: $E''(0)$ = "Hessian" $d^2E(V,V)$.

If γ is a local min for E then $E''(0) \geq 0 \ \forall \ V$.

4/7 I

Two applications: Myers' Thm and Synge's Thm.

Myers' (Bonnet-Myers) Thm

M complete Riem. Suppose there is $r>0$ such that

$$Ric_p(v) \geq \frac{1}{r^2}$$

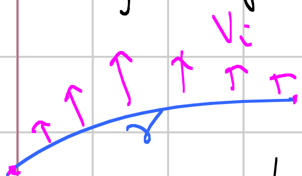
for all $p \in M$ and all $v \in T_pM$ with $|v|=1$.

Then M is compact and $\underbrace{\mathrm{diam}(M)}_{\sup\{d(p,q)\,|\,p,q\in M\}} \leq \pi r = \mathrm{diam}(S^n(r))$.

Pf Suffices to show $\forall\, p,q \in M$, γ = min. geodesic between p and $q \Rightarrow \ell(\gamma) \le \pi r$.
Then $\operatorname{diam}(M) \le \pi r \Rightarrow M$ bounded, complete $\Rightarrow$ compact.

Suppose $\ell(\gamma) = \ell$ and assume $|\gamma'| = 1$. Choose ONB $\{e_1 = \gamma'(0), e_2, \dots, e_n\}$ of $T_pM \rightarrow$ extend by parallel transport to ONB $\{e_1(t) = \gamma'(t), e_2(t), \dots\}$ along γ.

Along γ, define vector field $V_i(t) = \sin\left(\frac{\pi t}{\ell}\right) e_i(t)$, $2 \le i \le n$,



F_i = proper variation of γ with variational field V_i.

γ minimizes energy $\Rightarrow$ by 2nd variation

$$0 \le \frac{1}{2} E''(0) = \int_0^\ell \left(|V_i'|^2 - R(V_i, \gamma', V_i, \gamma') \right) dt$$

$$= \int_0^\ell \left(\left(\frac{\pi}{\ell} \cos \frac{\pi t}{\ell} \right)^2 - \sin^2\left(\frac{\pi t}{\ell}\right) R(e_i, \gamma', e_i, \gamma') \right) dt$$

Average over i:

$$0 \le \frac{\pi^2}{2\ell} - \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) \operatorname{Ric}_{\gamma(t)}(e_1(t))\, dt$$

$$\le \frac{\pi^2}{2\ell} - \int_0^\ell \frac{1}{r^2} \sin^2\left(\frac{\pi t}{\ell}\right) dt$$

$$= \frac{\pi^2}{2\ell} - \frac{\ell}{2r^2}$$

$\Rightarrow \ell \le \pi r$. $\square$

Cor M complete Riem, $\operatorname{Ric}_p(v) \ge \frac{1}{r^2}$.
Then the universal cover of M is compact and $\pi_1(M)$ is finite.

Pf Let $\tilde{M} \xrightarrow{\pi} M$ be the universal cover. Pull back metric on M to $\tilde{M}$; π = local isometry. Then $\tilde{M}$ is complete; apply Myers to $\tilde{M}$
$\Rightarrow \tilde{M}$ cpt, # of sheets = finite. $\square$

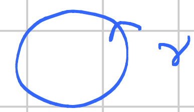
Note Complete, $\operatorname{Ric} > 0 \not\Rightarrow$ cpt, finite diameter. Ex: $\{z = x^2 + y^2\} \subset \mathbb{R}^3$.

Synge's Thm M cpt, even-dimensional, orientable, strictly positive sectional curvature. Then M = simply connected.

Rmks 1. Statement in odd dim: M cpt, odd-dim, $K > 0$. Then M is orientable. (see book)

2. Can't remove even-dim or orientable assumption: $\mathbb{RP}^n$ has $K > 0$ with metric induced from S^n.

Can't weaken to $K \geq 0$: $T^n = \mathbb{R}^n / \mathbb{Z}^n$.



Key to Synge: closed geodesics.

A geodesic $\gamma : [0,a] \to M$ is closed if $\gamma(0) = \gamma(a)$ and $\gamma'(0) = \gamma'(a)$: think of this as a smooth map $S^1 \to M$.

Def A homotopy class of free loops in M is a map $S^1 \to M$ up to homotopy.

Prop M compact. In any htpy class of free loops in M, there is a closed geodesic.

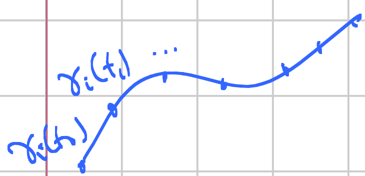
Pf Cpt $\Rightarrow \exists\, \epsilon > 0$ with $B_\epsilon(p)$ normal $\forall\, p \in M$.

Consider $l = \inf \{ l(\gamma) \mid \gamma \in \text{htpy class} \}$.

If $\exists\, \gamma$ with $l(\gamma) = l$ then γ locally minimizes length $\Rightarrow$ geodesic.

Otherwise, assume $\exists\, \gamma_i$ with $l(\gamma_i) \to l$. Reparametrize so $\gamma_i : [0,1] \to M$, $|\gamma_i'| = l(\gamma_i)$.

Choose $0=t_0<t_1<\dots<t_N=1$ with $t_{k+1}-t_k < \frac{\epsilon}{\max|\gamma_i'|}$.



Then $\gamma_i(t_k), \gamma_i(t_{k+1})$ lie in a normal ϵ-ball so we can replace γ_i by piecewise smooth curve s.t. $\gamma_i|[t_k,t_{k+1}]$ is a geodesic.

By passing to a subsequence, can assume $\forall k$, $\gamma_i(t_k) \xrightarrow{\text{converge}} p_k$, $\gamma_i'(t_k) \to v_k$

Now define $\gamma:[0,1]\to M$ by $\gamma|[t_k,t_{k+1}] =$ geodesic starting at p_k with deriv $= v_k$.
Then $\gamma_i|[t_k,t_{k+1}] \to \gamma|[\ ,\ ]$ so $\ell(\gamma)=\lim \ell(\gamma_i)=\ell$. $\square$

<u>Pf of Synge</u> Suppose M not simply connected. Then $\exists$ closed geodesic $\gamma:[0,a]\to M$ in a nontrivial htpy class of minimum length.
Say $\gamma(0)=\gamma(a)=p$.
Parallel transport along γ gives $P: T_pM \to T_pM$, orientation preserving, and $P(\gamma'(0))=\gamma'(0)$.
Let $T_p^\perp M =$ orthogonal complement to $\gamma'(0)$ in T_pM.
Then $P: T_p^\perp M \circlearrowleft$ so $P\in SO(n-1) \Rightarrow \exists\, v$ with $P(v)=v$.

Let $V=$ parallel vector field along γ with $V(0)=V(a)=v$, and let $\gamma_s=$ corresponding variation.
$\gamma=\gamma_0$ minimizes energy $\Rightarrow$ $E(s)=E(\gamma_s)$ has local min at $s=0$.
But

$$\frac{1}{2}E''(0) = \underbrace{\left\langle \frac{D}{\partial s}\frac{\partial f}{\partial s}(0,t),\gamma'\right\rangle\Big|_{t=0}^{t=a}}_{0 \text{ since } \gamma_s \text{ periodic}} + \int_0^a \left(|V'|^2 - R(V,\gamma',V,\gamma')\right)dt$$

$$= -\int_0^a \underbrace{R(V,\gamma',V,\gamma')}_{>0}\,dt$$

$$<0.$$

4/12 ↑ $\Rightarrow\Leftarrow$ $\square$

Constant Sectional Curvature

Three manifolds of constant sectional curvature

- $\mathbb{R}^n$, flat: $K \equiv 0$
- S^n, round: $K \equiv 1$
- H^n, hyperbolic: $K \equiv -1$ (see HW).

all complete, simply connected

If (M,g) has constant K, we can rescale to get $K \in \{1,0,-1\}$:
if $\lambda > 0$ then $\tilde{g} = \lambda g$ is a metric with $\tilde{\nabla} = \nabla$, $\underset{(0,4)\text{ tensor}}{\tilde{R} = \lambda R}$, $\tilde{K} = \lambda^{-1} K$.

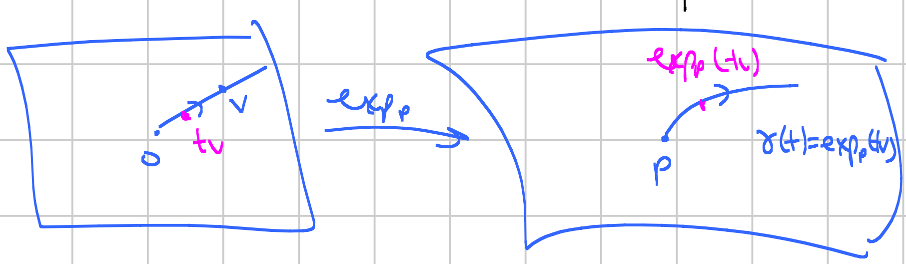
Can get more by quotienting by a group of isometries.

Thm M^n complete Riem mfd, constant sectional curvature $K \in \{1, 0, -1\}$. Then its universal cover is:

1. $\mathbb{R}^n$, flat if $K = 0$
2. S^n, round if $K = 1$
3. H^n, hyperbolic if $K = -1$,

i.e. any complete mfd with constant K is a quotient of one of them by isometries.

Pf Idea: $p \in M \rightsquigarrow \exp_p : T_pM \to M$ is a local isometry if $K = 0$, almost a local isometry if $K = \pm 1$. Choose $v \in T_pM$ unit vector.



Look at $\quad (d\exp_p)_{tv} : T_{tv}(T_pM) \longrightarrow T_{\exp_p(tv)}M.$

$$T_{tv}(T_pM) = T_pM$$

Note this maps $\quad v \longmapsto \gamma'(t)$ (these have the same length)

$\quad v^\perp \longrightarrow (\gamma'(t))^\perp$ by Gauss Lemma.

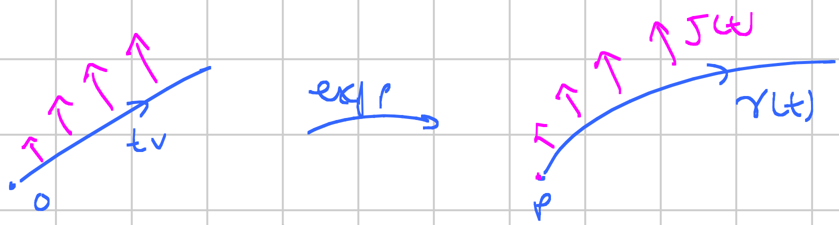
So it suffices to see what this does on $v^\perp \subset T_pM$.

Choose $u \in T_pM$ with $u \perp v$.

$\rightsquigarrow$ $U(t)$ = parallel vector field along γ with $U(0) = u$;

$J(t)$ = Jacobi field with $J(0) = 0$, $J'(0) = u$:

$$J(t) = (d\exp_p)_{tv}(tu).$$



How are these related? Jacobi equation $J'' = -R(\gamma', J)\gamma'$,

Constant sectional curvature $\Rightarrow \quad R(X\,Y\,Z\,W) = K\left(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\right)$

$\Rightarrow \quad R(\gamma', J)\gamma' = K\left(\langle \gamma',\gamma'\rangle J - \langle J,\gamma'\rangle\gamma'\right) = KJ$

$\Rightarrow \quad \boxed{J'' = -KJ}.$

Case 1. $K=0$: already done in HW.

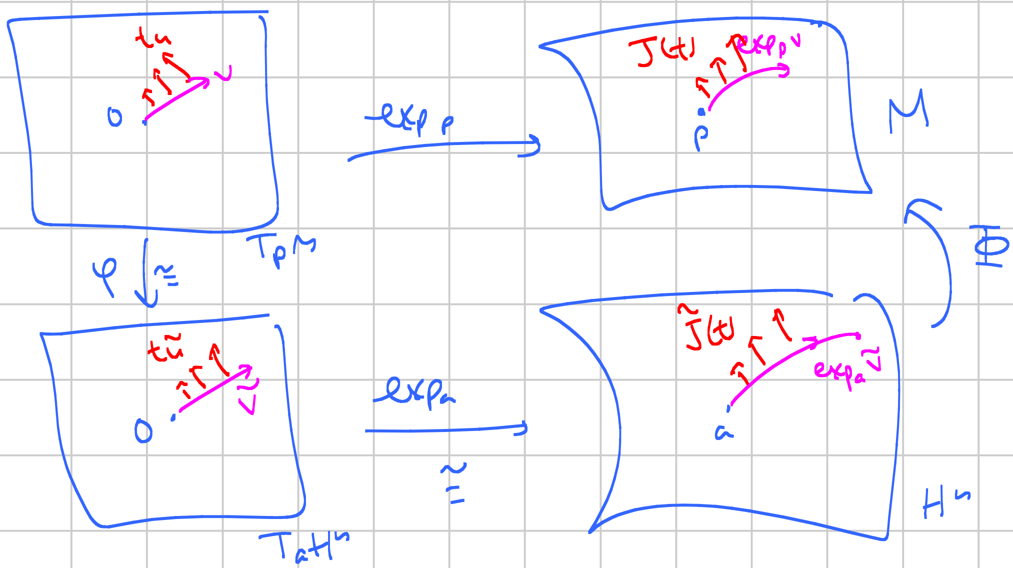
$J'' = 0 \Rightarrow J(t) = tU(t)$

$\Rightarrow (d\exp_p)_{tv}(u) = U(t)$

and parallel transport preserves inner product, so $(d\exp_p)_{tv}$ is an isometry on $v^\perp \Rightarrow$ on all of $T_{tv}(T_pM)$.

Thus $\exp_p$ is a local isometry $\underset{\text{"}\mathbb{R}^n\text{"}}{T_pM} \to M$, so by the proof of Hadamard, it's a covering map.

<u>Case 2</u> $K=-1$. $J''=+J \Rightarrow J(t)=(\sinh t)\,U(t)$.



Choose $a\in H^n$ and pick a linear isometry $\varphi: T_pM\to T_aH^n$.

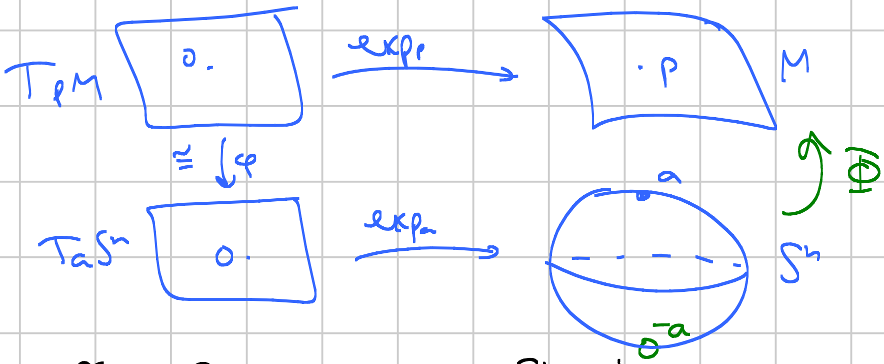
<u>Claim</u>: $\Phi := \exp_p\circ\varphi^{-1}\circ(\exp_a)^{-1}$ is a local isometry
$H^n\to M$ $\Rightarrow$ Covering map.

<u>Pf</u> Let $\tilde v\in T_aH^n$ be any unit vector, $v=\varphi^{-1}\tilde v$ = unit vector in T_pM.
Want to show: isometry at $\exp_a(t\tilde v)$. Suffices to show on $\perp$.

Choose $u\perp v$, $\tilde u=\varphi(u)\perp\tilde v \Rightarrow$ Jacobi fields $J(t)=d(\exp_p)_{tv}(tu)$,
$\tilde J(t)=d(\exp_a)_{t\tilde v}(t\tilde u)$. Note $J(t)=\Phi_*\tilde J(t)$.
Now if we have $u_1,u_2\perp v \rightsquigarrow J_1(t),J_2(t)$ on M, $\tilde J_1(t),\tilde J_2(t)$ on H^n,
then suffices to show $\langle J_1(t),J_2(t)\rangle=\langle\tilde J_1(t),\tilde J_2(t)\rangle$.

$J_i(t)=(\sinh t)\,U_i(t) \Rightarrow \langle J_1(t),J_2(t)\rangle=(\sinh t)^2\langle U_1(t),U_2(t)\rangle=\underline{\underline{(\sinh t)^2\langle u_1,u_2\rangle}}$
and similarly $\langle\tilde J_1(t),\tilde J_2(t)\rangle=(\sinh t)^2\langle\tilde u_1,\tilde u_2\rangle$
as desired.

Case 3 $K=1$. $J'' = -J \Rightarrow J(t) = (\sin t)\, U(t)$.

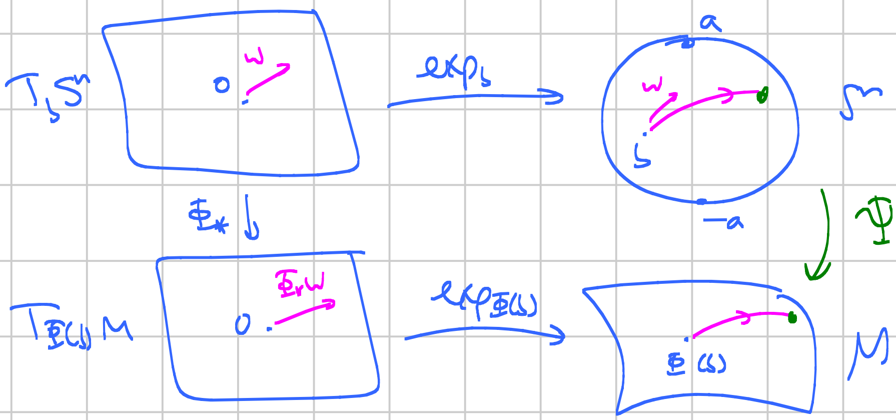


$\exp_a$ = diffeo from $B_\pi(0)$ to $S^n - \{-a\}$.

Same argument as before: $\Phi := \exp_p \circ \varphi^{-1} \circ (\exp_a)^{-1} : S^n - \{-a\} \to M$ is a local isometry.

Now choose $b \neq \pm a$ in S^n. Define

$$\Psi := \exp_{\Phi(b)} \circ \Phi_* \circ (\exp_b)^{-1} : S^n - \{-b\} \to M.$$



Claim: $\Phi = \Psi$. Why? A geodesic from b to c with tangent vector w maps under both Φ and Ψ to a geodesic from $\Phi(b)$ with tangent vector $\Phi_* w$, so Φ, Ψ map c to the same thing.

So Φ, Ψ together give $\Phi : S^n \to M$ local isometry
$\Rightarrow$ covering map, as before. $\square$

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Cut Locus

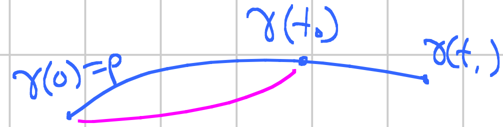
M complete $\Rightarrow \exp_p : T_pM \to M$ surjective. Can we always model M by part of T_pM? e.g. for S^n, $\exp_p : B_\pi(0) \to S^n - \{-p\}$ diffeo.

For a geodesic $\gamma(t)$, $0 \le t < \infty$, starting at p, we know that $\gamma|_{[0,t]}$ is a minimizing geodesic for small t. Define

$$I = \{ t_0 \ge 0 \mid \gamma|_{[0,t_0]} \text{ is a minimizing geodesic} \}.$$

i.e. $t_0 = d(p, \gamma(t_0))$

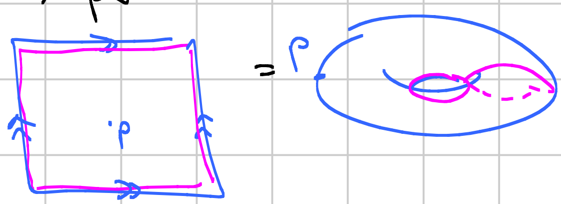
- I is closed
- if $t_0 \notin I$ and $t_1 > t_0$ then $t_1 \notin I$.



So either $I = [0,T]$ for some $T > 0$ or $I = [0,\infty)$.

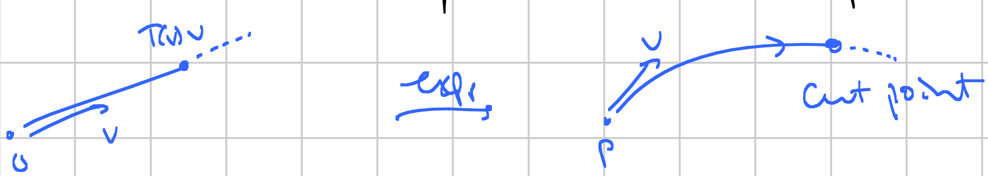
Def The cut point of p along γ is $\gamma(T)$ (if $I = [0,\infty)$, no cut point).
The Cut Locus of p is $Cut(p) = \{\text{cut points over all } \gamma\}$.

Ex
- round sphere: $Cut(p) = \{-p\}$
- $\mathbb{R}^2/\mathbb{Z}^2$: $Cut(p) =$



- Complete, simply connected, $K \le 0 \Rightarrow Cut(p) = \emptyset$. (Hadamard)

Now let the geodesic vary: $v \in T_pM$, $|v| = 1 \rightsquigarrow \exp_p(tv)$



Define $T(v) = \begin{cases} T & \text{if } I=[0,T] \\ \infty & \text{if } I=[0,\infty) \end{cases}$.

The preimage of the geodesic up to the cut point is the ray to $T(v)\,v$.
Write $\mathcal{U}(p) = \bigcup(\text{these open rays})$ i.e.

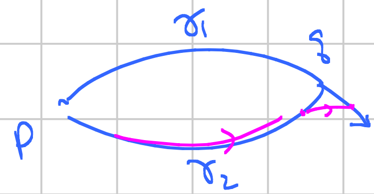
$$\mathcal{U}(p) = \{tv \mid v \in S^{n-1},\ 0 \le t < T(v)\}$$
$$= \{t_0 v \mid |v|=1,\ \exp_p(tv) \text{ is minimizing past } t_0\}.$$

- $\mathcal{U}(p)$ is star-shaped
- Can show $T: S^{n-1} \to \mathbb{R}^+ \cup \{\infty\}$ is continuous $\Rightarrow \mathcal{U}(p) \overset{\text{homeo}}{\cong} D^n$ (in fact, diffeo)
- $\operatorname{Cut}(p) = \exp_p(\partial \mathcal{U}(p))$

Prop M (complete) is the disjoint union $\exp_p(\mathcal{U}(p)) \sqcup \operatorname{Cut}(p)$.

Pf $q \in M \Rightarrow \exists$ min geodesic from p to q (Hopf-Rinow).
Either this stops being minimizing past $q \Rightarrow q \in \operatorname{Cut}(p)$
or not $\Rightarrow q \in \exp_p(\mathcal{U}(p))$.
If $q \in \exp_p(\mathcal{U}(p)) \cap \operatorname{Cut}(p)$ then $\exists\, 2$ minimizing geodesics between p and q
one minimizing past q, one not. Prop now follows from the next result. $\square$

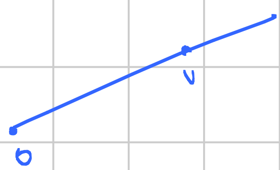
Lemma If $\exists$ 2 minimizing geodesics between p and q, then neither minimizes past q.

Pf. 

$\ell(\gamma_1)=\ell(\gamma_2)$. Extend γ_1 past q.
A shorter path is $\gamma_2 \cup$ cutoff. $\square$

Note: this argument actually shows
$\exp_p$ is an injection $U(p) \to M$
+ a bijection $U(p) \to M \setminus Cut(p)$.

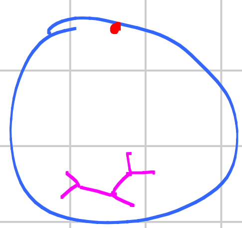
Prop $\exp_p : U(p) \to M \setminus Cut(p)$ is a diffeomorphism.

Pf 

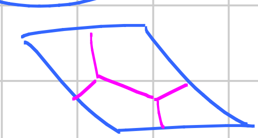
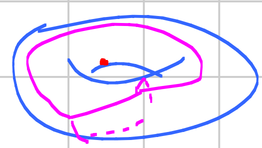
$v \in U(p) \subset T_pM \Rightarrow \gamma(t) = \exp_p(tv)$ is minimizing past $t=1$. Then γ has no conjugate points to p between p and $\exp_p((1+\epsilon)v)$ (from HW) so $\exp_p v$ isn't conjugate to p $\Rightarrow$ $\exp_p$ is a local diffeomorphism at v.
$\Rightarrow$ $\exp_p$ is a local diffeo on $U(p)$
$\Rightarrow$ diffeo since bijective. $\square$

All the interesting topology in M is in $Cut(p)$:

$$\exp_p : \quad D^n \cong U(p) \xrightarrow{\cong} M \setminus Cut(p)$$
$$\partial U(p) \longrightarrow Cut(p).$$

Weird examples: S^n: 

Note $M \setminus p$ deformation retracts onto $Cut(p)$.

$T^2 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}^2/\langle 1, e^{2\pi i/3}\rangle$:  

Prop (see do Carmo Ch 13 Prop 2.2)

q = Cut point of p along γ $\Leftrightarrow$ first point along γ where either $\begin{cases} \cdot \text{ Conjugate to } p \\ \cdot\ \exists \text{ two minimizing geodesic between } p \text{ and } q. \end{cases}$

$q \in M \setminus \mathrm{Cut}(p) \Rightarrow \exists!$ min geodesic from p to q

4/19 So $\exp_p : B_r(0) \to B_r(p)$ is injective $\Leftrightarrow r \leq d(p, \mathrm{Cut}(p))$.

Def The injectivity radius $i(M) := \inf_{p \in M} d(p, \mathrm{Cut}(p))$.

(if $r \leq i(M)$ then $\exp_p : B_r(0) \to M$ is injective $\forall p$).

Intuitively: if K = small then i = large. Sample thm.

Thm (Ch 13 Prop 2.13) If $0 < a \leq K \leq K_{max}$ for some a, K_{max}, then either $\exists$ closed geodesic γ with $i(M) = \frac{1}{2}\ell(\gamma)$ or $i(M) \geq \frac{\pi}{\sqrt{K_{max}}}$.

(Note by Myers that M is cpt.)

Thm (Klingenberg) M simply connected, cpt, dim ≥ 3,

$\frac{K_0}{4} < K \leq K_0$ for constant $K_0 > 0$.

Then $i(M) \geq \frac{\pi}{\sqrt{K_0}}$.

(idea: rule out short geodesics)

Sphere Theorem M simply connected, cpt,

$0 < \frac{K_0}{4} < K \leq K_0$.

Then M is homeomorphic to S^n.

Submanifolds

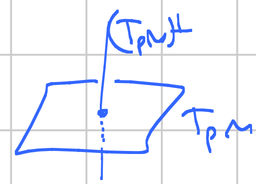
$M^n \subset \overline{M}^{n+1}$ submfd, Riem metric on $\overline{M}$ $\to$ Riem. metric on M.

Levi-Civita connections $\nabla, \overline{\nabla}$.

Recall: $X, Y \in \mathrm{Vect}(M)$ extending to $\overline{X}, \overline{Y} \in \mathrm{Vect}(\overline{M})$

$$\Rightarrow \quad \nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$$

independent of the extensions $\overline{X}, \overline{Y}$ (HW).



where $p \in M \rightsquigarrow$ $T_p\overline{M} = T_pM \oplus (T_pM)^\perp$

$$v \longmapsto v^T \oplus v^N$$

Def $X, Y \in \mathrm{Vect}(M)$

$$\rightsquigarrow \quad B(X,Y) := (\overline{\nabla}_{\overline{X}} \overline{Y})^N = \overline{\nabla}_{\overline{X}} \overline{Y} - \nabla_X Y$$

vector field along M. (in $(T_pM)^\perp$).

"vector valued second fundamental form"

Facts.

1. Indep of extensions $\overline{X}, \overline{Y}$.

as in HW: if $\overline{X}, \overline{X}'$ are extensions of X, $\overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{X}'} \overline{Y} = \overline{\nabla}_{\overline{X}-\overline{X}'} \overline{Y} = 0$

since $\overline{X} - \overline{X}' = 0$ on M

$\overline{Y}, \overline{Y}'$ extensions of $Y \Rightarrow \overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{X}} \overline{Y}' = \overline{\nabla}_{\overline{X}} (\overline{Y} - \overline{Y}') = 0$ since $\overline{Y} - \overline{Y}' = 0$ on M

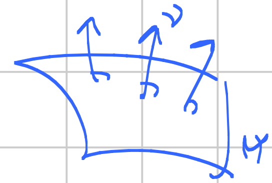
$\overline{X}$ tangent to M.

2. tensor: clearly tensorial in X.

$$B(X, fY) = \overline{\nabla}_{\overline{X}}(f\overline{Y}) - \nabla_X(fY) = f\,B(X,Y) + (\overline{X}f)\overline{Y} - (Xf)Y$$
$$= f\,B(X,Y)$$

3. symmetric:

$$B(X,Y) - B(Y,X) = [\overline{X}, \overline{Y}] - [X,Y] = 0 \quad \text{on } M.$$



Choose a unit normal vector field ν along M.

(if $M, \overline{M}$ orientable, then ν is well-defined globally up to $\pm$).

Def The <u>second fundamental form</u> is the symmetric bilinear form $\mathrm{II} : T_pM \otimes T_pM \to \mathbb{R}$ given by

$$B(X,Y) = \mathrm{II}(X,Y)\,\nu.$$

(Note: 1st fundamental form is just the metric)

Def σ = 2-plane in T_pM generated by X, Y.
The <u>Gaussian curvature</u> $G(\sigma)$ is

$$G(\sigma) := \frac{\mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - \mathrm{II}(X,Y)^2}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

This is independent of choice of X, Y (as in sectional curv.).

Def The <u>shape operator</u> $S : T_pM \to T_pM$ is defined by

$$\langle S(X), Y\rangle = \mathrm{II}(X,Y).$$

Alternate def for S: note

$$\mathrm{II}(X,Y) = \langle B(X,Y), \nu\rangle = \langle \bar\nabla_{\bar X}\bar Y - \nabla_X Y, \nu\rangle = \langle \bar\nabla_{\bar X}\bar Y, \nu\rangle$$
$$= \bar X\langle \bar Y, \nu\rangle - \langle \bar Y, \bar\nabla_{\bar X}\nu\rangle = -\langle \bar Y, \bar\nabla_{\bar X}\nu\rangle$$
$$\Rightarrow S(X) = -(\bar\nabla_X \nu)^T.$$

Eigenvalues of S are <u>principal curvatures</u>.
For surfaces in $\mathbb{R}^3$, $G(\sigma) = \lambda_1\lambda_2$ product of eigenvalues.

<u>Thm</u> (Gauss) $X, Y, U, V \in T_pM$. Then

$$R(X,Y,U,V) = \bar R(X,Y,U,V) + \mathrm{II}(X,U)\,\mathrm{II}(Y,V) - \mathrm{II}(X,V)\,\mathrm{II}(Y,U)$$

sectional, Gaussian curvs. of plane through X, Y $\longrightarrow$ $K(X,Y) = \bar K(X,Y) + G(X,Y).$

Pf Extend X, Y, U, V at p to vector fields X, Y, U, V on M, $\bar{X}, \bar{Y}, \bar{U}, \bar{V}$ on $\bar{M}$.

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \nabla_X Y + \mathrm{II}(X,Y)\nu \qquad \text{on } M$$

$$\Rightarrow \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{U} = \bar{\nabla}_{\bar{X}}\left(\nabla_Y U + \mathrm{II}(Y,U)\nu\right)$$

$$= \nabla_X\nabla_Y U + \mathrm{II}(X,\nabla_Y U)\nu + \mathrm{II}(Y,U)\bar{\nabla}_{\bar{X}}\nu + \left(\bar{X}\,\mathrm{II}(Y,U)\right)\nu$$

$$\Rightarrow \langle \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{U}, \bar{V}\rangle = \langle \nabla_X\nabla_Y U, V\rangle + \mathrm{II}(Y,U)\langle\bar{\nabla}_{\bar{X}}\nu, \bar{V}\rangle$$

$$= \langle \nabla_X\nabla_Y U, V\rangle - \mathrm{II}(Y,U)\,\mathrm{II}(Y,V). \qquad \square$$

Important special case:

Thm If $\bar{M} = (\mathbb{R}^{n+1}, \text{flat})$ then $K(X,Y) = G(X,Y)$.

Rmk For $n=3$ this is Gauss's Theorema Egregium.
in his language: the extrinsic quantity $G(X,Y)$ (depends on the isometric embedding of M in $\mathbb{R}^3$) is actually an intrinsic quantity.

Ex $S^n(r) \subset \mathbb{R}^{n+1}$.

$x \in S^n(r) \Rightarrow \nu(x) = \frac{x}{r} = \frac{1}{r}(x^i\partial_i)$. For $Y \in T_x(S^n(r))$,

$$\bar{\nabla}_Y \nu = Y\nu \underset{Yr=0}{=} \frac{1}{r}\left(Y(x^i)\partial_i\right) = \frac{Y}{r}$$

So $S(Y) = -\frac{Y}{r} \Rightarrow \mathrm{II}(X,Y) = -\frac{1}{r}\langle X,Y\rangle$

and $G(X,Y) = \frac{1}{r^2} \Rightarrow K = \frac{1}{r^2}$.

Totally geodesic submfds

Def $M \subset \bar{M}$ is totally geodesic if every geodesic in M is also a geodesic in $\bar{M}$.

Prop M is totally geodesic $\iff$ $B \equiv 0$ (vector valued 2nd fund. form).

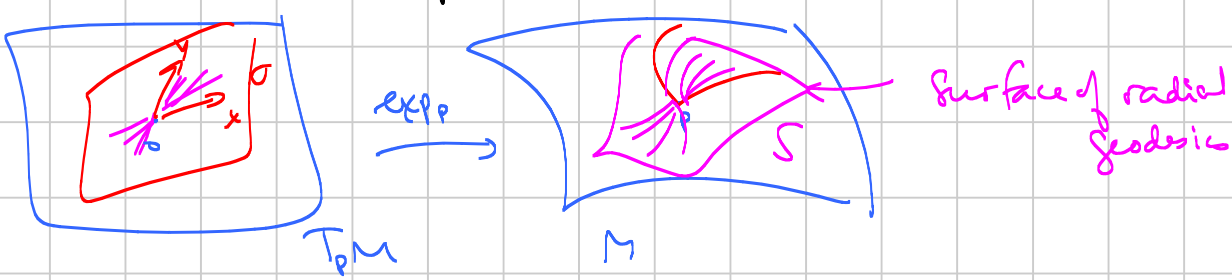
Pf Let γ be a path in M, $\gamma(0) = p$, $\gamma'(0) = x$. Let N be a normal vector field to M, X = extension of x.

$$\langle B(x,x), N\rangle = \langle \bar{\nabla}_x X - \nabla_x X, N\rangle$$
$$= \langle \bar{\nabla}_x X, N\rangle$$

So $B(x,x) = 0 \iff \bar{\nabla}_x X$ has no normal component

$\iff$ geodesics in M (satisfying $\nabla_x X = 0$) are also geodesic in $\bar{M}$ (satisfying $\bar{\nabla}_x X = 0$). $\square$

Riemann's original definition of sectional curvature:



S is geodesic at p (ptwise version of totally geodesic)

$\to$ by above Prop, $B(p) = 0$

$\to$ Give S the metric induced by M; then $R = \bar{R}$

and $K(S) = K(\sigma)$.

(unique) sectional curvature of S ↑ ↑ usual sectional curvature