Math 621 - Differential Geometry

Note Title 1/11/2017

$\frac{1}{2}$ differential topology: manifolds, vector fields, vector bundle, tensors, diff'l forms

$\frac{1}{2}$ Riemannian geometry: metrics, connections, curvature, geodesics : topics mentioned in qual syllabus

both intrinsic (coord-free) and with coord.

Needs: basic multivar calculus, topology (topl space, subspace topology, covering space, fundamental group)

analysis: differential; inverse/implicit function thm along the lines of Math 532.

- Text: do Carmo, Riemannian Geometry

  Supplement: Lee Smooth manifolds, Gallot-Hulin-Lafontaine Riemannian Geometry.

- My notes will be posted on course webpage math.duke.edu/~ng/math621/
- Grading based on HW and take-home final.
- Office hrs TBD. New office! 216.

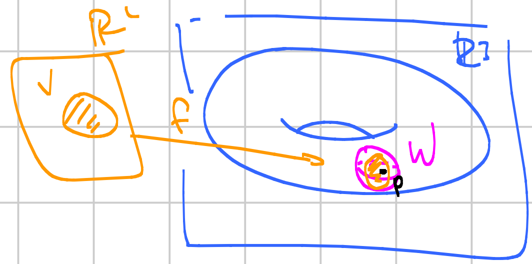
Questions:
- makeup classes on Mondays?
- manifold? tangent space? vector field?
- differential form?
- covering space? fund gp?
- tensor product of vector spaces?

Smooth Manifolds

Ex: surfaces in $\mathbb{R}^3$ or more generally submanifolds of $\mathbb{R}^n$.

Def $M \subset \mathbb{R}^{n+k}$ is a (smooth) submanifold of dimension n if $\forall\, p \in M$, $\exists$ nbhd W of p in $\mathbb{R}^{n+k}$, an open set $V \subset \mathbb{R}^n$, and a map $f : V \to \mathbb{R}^{n+k}$ so that

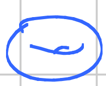
- f is smooth
- $f(V) = M \cap W$ "coord. chart"
- $f|_V$ is a bijection $V \to M \cap W$
- $df(x)$ is injective $\forall\, x \in V$.

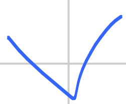


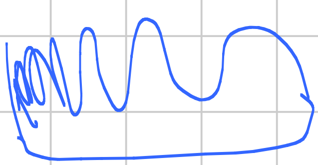
Submfds: $\mathbb{R}^n \subset \mathbb{R}^{n+k}$

 $\subset \mathbb{R}^3$: use spherical coords except at N, S.

$$(\varphi, \theta) \mapsto (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$$



Not submfds:
-  $\subset \mathbb{R}^2$
- $\{y^2 = x^3\} \subset \mathbb{R}^2$  $t \mapsto (t^3, t^2)$
- image of 

$\subset \mathbb{R}^2$
- 
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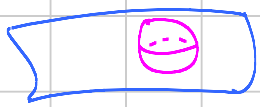
Prop TFAE.

1. M is a smooth submfd of dim n in $\mathbb{R}^{n+k}$
2. $\forall\, p \in M$, $\exists$ nbd W of p in $\mathbb{R}^{n+k}$ and a smooth map $g: W \to \mathbb{R}^k$ such that
   - $W \cap M = g^{-1}(0)$
   - g is a submersion: $dg(x)$ is surjective $\forall\, x \in W$.

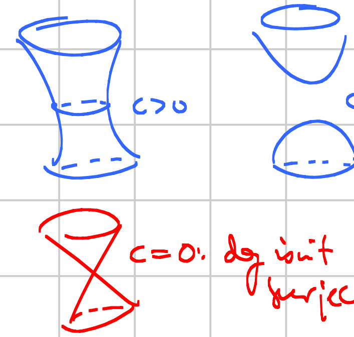


$\longrightarrow \mathbb{R}^k$

Pf implicit function theorem.

Ex: $S^n \subset \mathbb{R}^{n+1}$; $\{x^2+y^2-z^2=c\} \subset \mathbb{R}^3$

$\{x_1^2+\dots+x_{n+1}^2=1\}$ $\quad g=x^2+y^2-z^2-c$



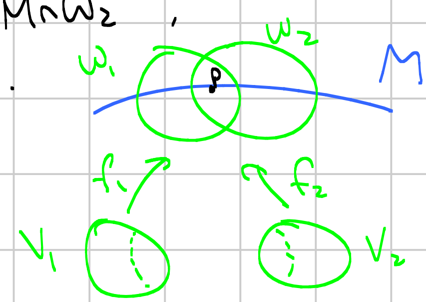
$\{x^3-y^3=0\} \subset \mathbb{R}^2$: $\quad g=x^3-y^3$ doesn't work at $(0,0)$

but this is $=\{x-y=0\}$ which does work.

Prop $M^n \subset \mathbb{R}^{n+k}$ submfd, $p \in M$, two coord charts

$f_1, f_2: V_1, V_2 \to \mathbb{R}^{n+k}$ mapping onto $M \cap W_1$, $M \cap W_2$,

$\hat{\mathbb{R}}^n$ $\qquad W_1, W_2$ nbds of p.



Then

$$f_2^{-1} \circ f_1 : \underset{\subset \mathbb{R}^n}{f_1^{-1}(W_1 \cap W_2)} \to \underset{\subset \mathbb{R}^n}{f_2^{-1}(W_1 \cap W_2)}$$

is a diffeomorphism: smooth, smooth inverse.

This allows us to dispense with the "ambient space" $\mathbb{R}^{n+k}$.

M = set (!), $n \geq 0$ fixed.

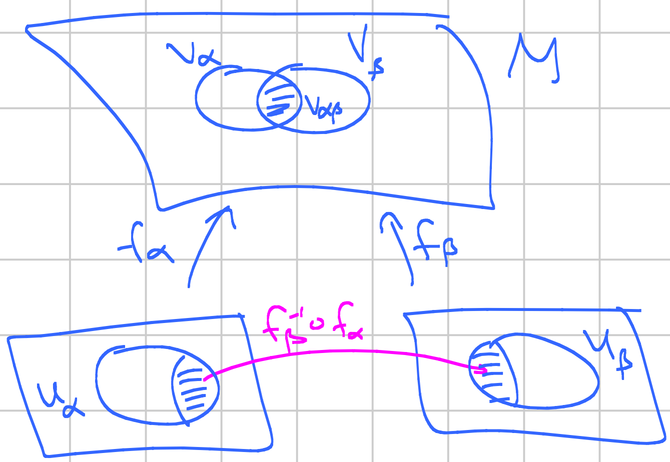
Def A <u>smooth atlas</u> of coord charts on M is a collection

$$f_\alpha : \overset{\subset \mathbb{R}^n}{U_\alpha} \longrightarrow V_\alpha \subset M \qquad \text{(coord chart)}$$

(U_α = open set)

such that:
- f_α is a bijection
- V_α cover M: $\bigcup_\alpha V_\alpha = M$
- $\forall \alpha, \beta$ with $\underbrace{V_\alpha \cap V_\beta}_{\overset{!!}{V_{\alpha\beta}}} \neq \emptyset$,

$$f_\beta^{-1} \circ f_\alpha : \underset{\underset{\mathbb{R}^n}{\cap}}{\underset{U_\alpha}{\cap}} f_\alpha^{-1}(V_{\alpha\beta}) \to \underset{\underset{\mathbb{R}^n}{\cap}}{\underset{U_\beta}{\cap}} f_\beta^{-1}(V_{\alpha\beta}) \qquad \text{is smooth.}$$



Topology on M is induced by atlas:
$V \subset M$ is open $\iff$ $f_\alpha^{-1}(V)$ is open $\forall \alpha$.

Def M is a <u>smooth manifold of dimension n</u> (n-manifold) if it has a smooth atlas, <u>and</u> wrt this topology, M is <u>Hausdorff</u> and <u>second countable</u>.

no

topology has a countable basis.

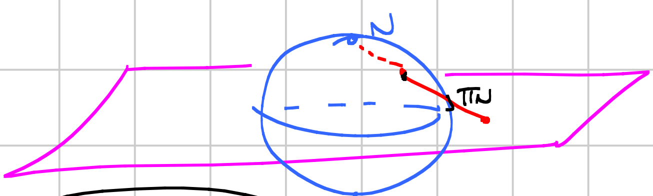
Rmks: · Sometimes useful to consider *maximal* atlases
(not properly contained in any other atlas: any coord chart compatible with the atlas is in the atlas:
if $f_\alpha : U_\alpha \to M$ is a chart then $f : U \to M$ is also a chart for $U \subset U_\alpha$, $f = f_\alpha|_U$.)

· intrinsic vs extrinsic: Whitney embedding theorem:
any smooth n-mfd M can be smoothly embedded in $\mathbb{R}^{2n}$.

Ex of n-manifolds.

- $\mathbb{R}^n$
- any n-dim submfd of $\mathbb{R}^{n+k}$
- $S^n \subset \mathbb{R}^{n+1}$: here's an atlas.

$N = (0, \dots, 0, 1)$ $\quad S = (0, \dots, 0, -1)$



$\underbrace{\pi_N : S^n \setminus N}_{V_1} \to \underbrace{\mathbb{R}^n}_{U_1}$ $\qquad \pi_N(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right)$

$\underbrace{\pi_S : S^n \setminus S}_{V_2} \to \underbrace{\mathbb{R}^n}_{U_2}$ $\qquad \pi_S(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right)$.

$$\begin{array}{ccc} & S^n & \\ V_1 \subset & & \supset V_2 \\ \pi_N \downarrow & & \downarrow \pi_S \\ U_1 & & U_2 \\ \cup & & \cup \\ \pi_N^{-1}(V_1 \cap V_2) = \mathbb{R}^n - 0 & \longrightarrow & \mathbb{R}^n - 0 = \pi_S^{-1}(V_1 \cap V_2) \end{array}$$

this map: $(y_1, \dots, y_n) \mapsto (y_1', \dots, y_n')$

$y_i = \frac{x_i}{1 - x_{n+1}}$, $y_i' = \frac{x_i}{1 + x_{n+1}}$ $\Rightarrow$ $y_i' = \frac{y_i}{y_1^2 + \dots + y_n^2}$

This is a smooth map $\mathbb{R}^n - 0 \to \mathbb{R}^n - 0$.

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· $\mathbb{RP}^n$ Points are equiv. classes $\subset \mathbb{R}^{n+1} \setminus 0$

$(x_1, \dots, x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1})$ $\lambda \neq 0$. Write $[x_1, \dots, x_{n+1}]$ for equiv class.

$1 \le i \le n+1$: $V_i = \{ [x_1, \dots, x_{n+1}] \mid x_i \neq 0 \} \subset \mathbb{RP}^n$

$V_i \longrightarrow U_i = \mathbb{R}^n$ bijective, inverse $f_i : U_i \longrightarrow V_i$

$[x_1, \dots, x_{n+1}] \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right)$ $\quad (y_1, \dots, y_n) \mapsto (y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n)$

$i > j$: $f_j^{-1} \circ f_i (y_1, \dots, y_n) = f_j^{-1}(y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n)$

$$= \left(\frac{y_1}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \dots, \frac{y_n}{y_j} \right).$$

$V_i \cap V_j = \{ x_i, x_j \neq 0 \}$, $f_i^{-1}(V_i \cap V_j) = \{ y_j \neq 0 \}$, $f_j^{-1} \circ f_i$ is a map on $\{ y_j \neq 0 \}$.

Smooth Maps

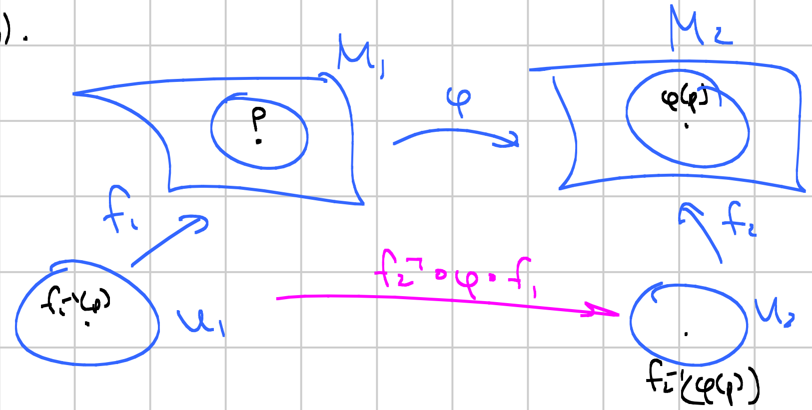
<u>Def</u> M_1^n, M_2^m smooth mfds. A map $\varphi : M_1 \to M_2$ is <u>smooth</u> at $p \in M_1$ if $\exists$ charts $f_1 : \underset{\cap \mathbb{R}^n}{U_1} \to M_1$, $f_2 : \underset{\cap \mathbb{R}^m}{U_2} \to M_2$ with

$p \in f_1(U_1)$, $\varphi(p) \in f_2(U_2)$, $\varphi(f_1(U_1)) \subset f_2(U_2)$,

and

$f_2^{-1} \circ \varphi \circ f_1 : \overset{\subset \mathbb{R}^n}{U_1} \longrightarrow \overset{\subset \mathbb{R}^m}{U_2}$ (or more precisely nbd$(f_1^{-1}(p)) \to$ nbd$(f_2^{-1}\varphi(p))$)

is smooth at $f_1^{-1}(p)$.



φ is smooth if it's smooth at all points.
Rmk: this is independent of coord charts.

Special cases of smooth maps.

Def $\varphi: M_1 \to M_2$ is an immersion if at any $p \in M_1$, there are coord charts f_1 near p, f_2 near $\varphi(p)$ such that

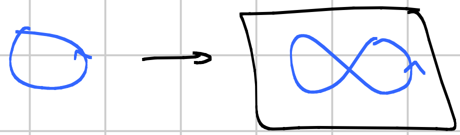
$$f_2^{-1} \circ \varphi \circ f_1 : U_1 \to U_2$$

is an immersion at $f_1^{-1}(p)$: $d(f_2^{-1} \circ \varphi \circ f_1)$ is injective

φ is a submersion .... submersion: $d(f_2^{-1} \circ \varphi \circ f_1)$ is surjective.

Ex: Submersion: $\mathbb{R}^n \to \mathbb{R}^m$ projection, $n \geq m$

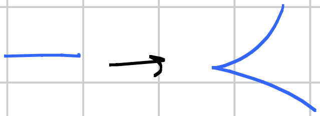
immersion:



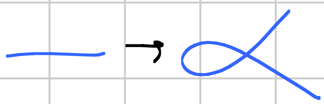
Def cont'd. embedding: immersion and homeomorphism onto its image (with subspace topology); in particular, injective.

diffeomorphism: smooth, smooth inverse (in particular, immersion and submersion)

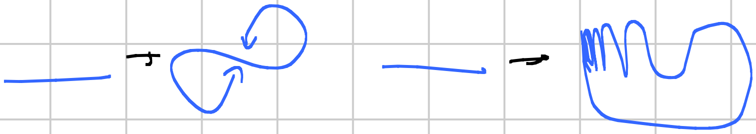
Ex cont'd. not immersion: $\mathbb{R} \to \mathbb{R}^2$
$t \mapsto (t^2, t^3)$



immersion, not injective: $\mathbb{R} \to \mathbb{R}^2$



immersion, injective, not embedding:



Rmks. ① Any immersion is locally an embedding:
$\varphi: M_1 \to M_2$ immersion, $p \in M_1 \Rightarrow \exists$ nbd $V \ni p$ with
$\varphi|_V : V \to M_2$ embedding.
② A map that's both an immersion and a submersion is a <u>local diffeomorphism</u>: restricted to a nbd of any point, it's a diffeo. (e.g. covering maps)
③ if $\varphi: M_1 \to M_2$ is an embedding, then $\varphi(M_1)$ is a <u>submanifold</u> of M_2.

Quotients by group actions

Def A group G <u>acts</u> on a manifold M if each $g \in G$ gives a diffeo. $\varphi_g : M \to M$ st

$$\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2} \qquad (\Rightarrow \varphi_{id} = id).$$

G acts <u>properly discontinuously</u> if $\forall p \in M \ \exists$ nbd V of p s.t. $V \cap \varphi_g(V) = \emptyset \ \ \forall g \neq id$.

G acting on $M \rightsquigarrow$ <u>quotient space</u> M/G of orbits under G $(p \sim \varphi_g(p))$.

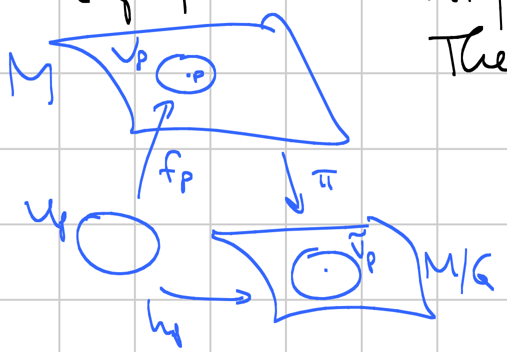
Ex. 1(a) $\mathbb{R}^\times$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$ by scalar mult., <u>not</u> properly disc.
$(\mathbb{R}^{n+1} \setminus 0)/\mathbb{R}^\times = \mathbb{RP}^n$.
1(b) $\mathbb{Z}/2 = \{\pm 1\}$ acts on $S^n \subset \mathbb{R}^{n+1}$ by antipodal map.
$S^n/(\mathbb{Z}/2) = \mathbb{RP}^n$.
2. $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by vector addition, $\mathbb{R}^n/\mathbb{Z}^n = T^n = \overbrace{S^1 \times \cdots \times S^1}^{n}$
3. Möbius strip $\mathbb{R} \times (0,1)/$~~gp~~ gp gen'd by $(x,y) \mapsto (x+1, 1-y))$.

Prop If G acts properly discontinuously on M, then M/G is a smooth mfd and $\pi : M \to M/G$ is a smooth map (in fact, local diffeomorphism).

Pf First construct atlas on M/G.

$p \in M$: choose a coord chart $f_p : U_p \to M$, $f_p(U_p) = V_p \ni p$, such that $V_p \cap \varphi_g(V_p) = \phi$ $\forall g \neq id$.

(if V_p doesn't satisfy this, intersect it with V from def.).



Then $\pi|_{V_p}$ is injective so

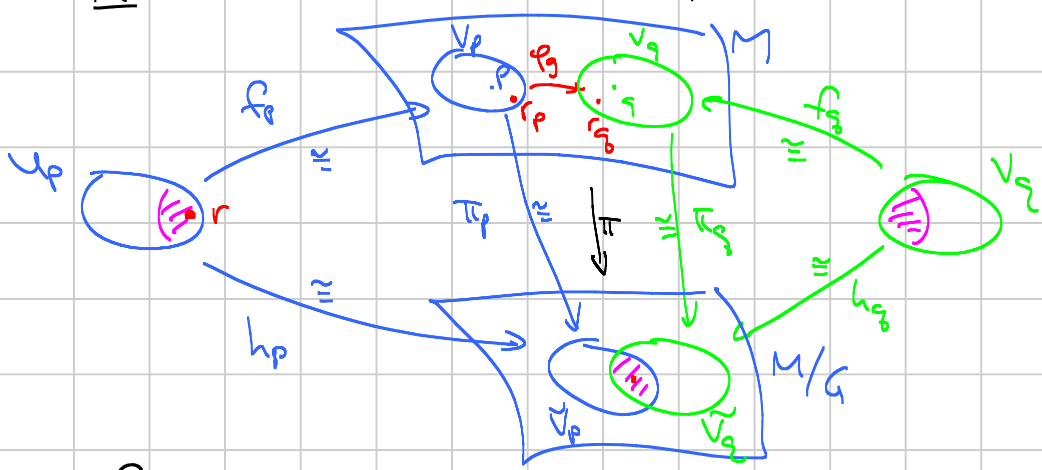
$h_p := \pi \circ f_p : U_p \to M/G$ is injective.

Write $\tilde{V}_p := \pi(V_p) = h_p(U_p)$.

Then $h_p : U_p \to \tilde{V}_p$ is bijective.

Claim $\{(h_p, U_p, \tilde{V}_p)\}$ is an atlas for M/G.

Pf Need to check transition functions.



The following are bijections:

$f_p : U_p \to V_p$ $\qquad$ $f_q : U_q \to V_q$

$\pi_p : V_p \to \tilde{V}_p$ $\qquad$ $\pi_q : V_q \to \tilde{V}_q$

$h_p : U_p \to \tilde{V}_p$ $\qquad$ $h_q : U_q \to \tilde{V}_q$

Near $r \in h_p^{-1}(\tilde{V}_p \cap \tilde{V}_q)$ we have

$$h_q^{-1} \circ h_p = f_q^{-1} \circ \pi_q^{-1} \circ \pi_p \circ f_p .$$

Write $r_p = f_p(r)$, $r_q = \pi_q^{-1}(\pi_p(r_p))$ so $\pi(r_p) = \pi(r_q)$

$\Rightarrow \exists\, g$ with $r_q = \varphi_g(r_p)$.

Near r_p, $\pi = \pi \circ \varphi_g \Rightarrow \pi_p = \pi_g \circ \varphi_g \to \pi_g^{-1} \circ \pi_p = \varphi_g$.

So $h_q^{-1} \circ h_p = f_q^{-1} \circ \varphi_g \circ f_p$ is smooth. $\square$

smooth map

coord charts

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Orientations

M = n-mfd, atlas $\{(f_\alpha, U_\alpha, V_\alpha)\}$.

$f_1 : U_1 \to V_1 \subseteq M$

$f_2 : U_2 \to V_2 \subseteq$

If $V_1 \cap V_2 \neq \emptyset$ then "transition function"

$$f_2^{-1} \circ f_1 : \underset{\underset{\mathbb{R}^n}{\cap}}{f_1^{-1}(V_1 \cap V_2)} \to \underset{\underset{\mathbb{R}^n}{\cap}}{f_2^{-1}(V_1 \cap V_2)}$$

is smooth with smooth inverse.

So for all $x \in f_1^{-1}(V_1 \cap V_2)$,

$d(f_2^{-1} \circ f_1)(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a <u>nonsingular</u> linear map.

$$\det\left(d(f_2^{-1} \circ f_1)(x)\right) \neq 0 .$$

Say f_1, f_2 determine the $\left\{\begin{matrix}\text{same} \\ \text{opposite}\end{matrix}\right\}$ orientation at $p = f_1(x) \in V_1 \cap V_2$

if $\det d(f_2^{-1} \circ f_1)(x) \left\{\begin{matrix} > 0 \\ < 0 \end{matrix}\right\}$.

f_1, f_2 determine $\left\{\begin{matrix}\text{same} \\ \text{opp}\end{matrix}\right\}$ orientation if they determine $\left\{\begin{matrix}\text{same} \\ \text{opp}\end{matrix}\right\}$ orientation at all points: note if $V_1 \cap V_2$ connected then one of these must happen.

An atlas for M is <u>oriented</u> if all coord charts determine the same orientations.

An <u>orientation</u> for M is a choice of oriented atlas, mod saying that 2 atlases are equiv if their union is oriented (i.e. all charts determine same orientation).

<u>Note</u>: Switching orientation. $f_1 : U \to V \subset M$, $h : \mathbb{R}^n \to \mathbb{R}^n$
$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n)$

$\rightsquigarrow$ chart $f_2 = f_1 \circ h : h(U) \to V$ has opposite orientation to f_1:
$f_2^{-1} \circ f_1 : U \to h(U)$ is h, and $\det h = -1$.

<u>Ex</u> S^n. 2 coord charts $f_1, f_2 : \mathbb{R}^n \to S^n$.
Recall $f_2^{-1} \circ f_1 : \mathbb{R}^n - 0 \circlearrowleft$
$$(y_1, \ldots, y_n) \mapsto \left(\frac{y_1}{y_1^2 + \cdots + y_n^2}, \ldots, \frac{y_n}{y_1^2 + \cdots + y_n^2} \right).$$

exercise: $\det d(f_2^{-1} \circ f_1) = -\dfrac{1}{\|y\|^{2n}}$.

So an oriented atlas is f_1 and $f_2 \circ h$, or $f_1 \circ h$ and f_2.

<u>Ex</u> Möbius strip $\quad M = [0,1] \times (0,1) \,/\, (0,y) \sim (1, 1-y)$

Atlas: $f_1 : (0,1)^2 \to M$ $\quad (x,y) \mapsto (x,y)$

$f_2 : (0,1)^2 \to M$ $\quad (x,y) \mapsto (x + \frac{1}{2}, y)$ where $(x,y) \sim (x+1, 1-y)$ if $x < \frac{1}{2}$.

on $(0, \frac{1}{2}) \times (0,1)$, $\quad f_2^{-1} \circ f_1 (x,y) = (x + \frac{1}{2}, 1-y)$

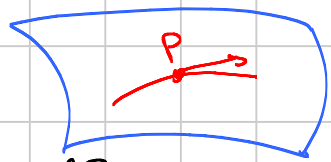
on $(\frac{1}{2}, 1) \times (0,1)$ $\quad f_2^{-1} \circ f_1 (x,y) = (x - \frac{1}{2}, y)$

So these charts agree on one part and disagree on the other.

Tangent Vectors

What's a tangent vector to a mfd M at a point $p \in M$?

If $M \subset \mathbb{R}^{n+k}$:



velocity vector of a curve passing through p.

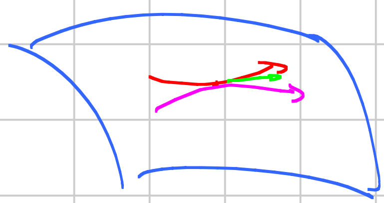
Intrinsically / in general?

APPROACH 1

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve (a smooth map) with $\gamma(0) = p$.

Any such curve gives rise to a tangent vector at p.

When are two curves the same?



these should give same tangent vector.

Use coord chart

$$\begin{tikzcd} & (-\epsilon, \epsilon) \arrow[dl, dashed, "f^{-1}\circ\gamma"'] \arrow[dr, "\gamma"] & \\ \mathbb{R}^n \supset U \arrow[rr, "f"] & & V \subset M \end{tikzcd}$$

Say $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to M$ with $\gamma_1(0) = \gamma_2(0) = p$ are equivalent if

$$\frac{d}{dt}\Big|_{t=0} (f^{-1} \circ \gamma_1) = \frac{d}{dt}\Big|_{t=0} (f^{-1} \circ \gamma_2).$$

Need to check indep of coord chart.

$$\begin{tikzcd}
& (-\epsilon,\epsilon) \arrow[dl, "f_1^{-1}\circ\gamma"'] \arrow[d, "\gamma"] \arrow[dr, "f_2^{-1}\circ\gamma"] & \\
\mathbb{R}^n \supset U_1 \arrow[r, "f_1"'] \arrow[rr, bend right, "f_2^{-1}\circ f_1"'] & V \ni p & U_2 \subset \mathbb{R}^n \arrow[l, "f_2"]
\end{tikzcd}$$

$$\frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ\gamma\right) = \frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ f_1\circ f_1^{-1}\circ\gamma\right)$$

$$= \underbrace{d\left(f_2^{-1}\circ f_1\right)\left(f_1^{-1}(p)\right)}_{\text{nonsingular } n\times n \text{ matrix}} \cdot \underbrace{\frac{d}{dt}\Big|_{t=0}\left(f_1^{-1}\circ\gamma\right)}_{\text{vector in } \mathbb{R}^n}$$

(Chain Rule)

So
$$\begin{bmatrix} \frac{d}{dt}\Big|_{t=0}\left(f_1^{-1}\circ\gamma_1\right) \\ \frac{d}{dt}\Big|_{t=0}\left(f_1^{-1}\circ\gamma_2\right) \end{bmatrix} \xrightarrow{d(f_2^{-1}\circ f_1)(f_1^{-1}(p))} \begin{bmatrix} \frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ\gamma_1\right) \\ \frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ\gamma_2\right) \end{bmatrix}$$

and these are equal $\Leftrightarrow$ these are equal.

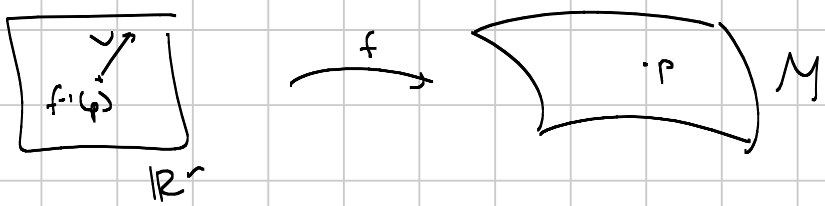
Def A <u>tangent vector</u> to M at p is an equivalence class of curves $\gamma : (-\epsilon,\epsilon)\to M$, $\gamma(0)=p$, with the above equiv. relation.

$T_pM = \{\text{tangent vectors to } M \text{ at } p\}$.

Now: fix a coord chart $f : U \to M$ around p. Every curve γ through p gives rise to a vector in $\mathbb{R}^n$, $\frac{d}{dt}\Big|_{t=0}(f^{-1}\circ\gamma)$, and this induces a map

$$T_pM \longrightarrow \mathbb{R}^n.$$

This is injective by construction, and also surjective:



for $v\in\mathbb{R}^n$, let $c(t)$ be the straight line in $\mathbb{R}^n$: $c(t) = f^{-1}(p) + tv$;
then if $\gamma = f\circ c$ then $\frac{d}{dt}(f^{-1}\circ\gamma) = \frac{d}{dt}c = v$.

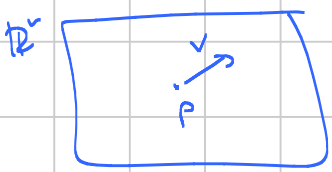
$\Rightarrow$ $T_pM \cong \mathbb{R}^n$; this gives T_pM the structure of a vector space.
Let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ denote the basis of T_pM corr. to the standard basis under this isom:

$$\begin{array}{ccc} T_pM & \xrightarrow{\cong} & \mathbb{R}^n \\ v_1\frac{\partial}{\partial x_1} + \dots + v_n \frac{\partial}{\partial x_n} & & (v_1, \dots, v_n) \end{array}.$$

(for now, just formal notation). Note depends on coord chart.

APPROACH 2 (Coord-free)

vector v in $\mathbb{R}^n$ gives a directional derivative: $f:\mathbb{R}^n \to \mathbb{R} \rightsquigarrow v(f)$.



$\gamma: \mathbb{R} \to \mathbb{R}^n$, $\gamma(0) = p$, $\gamma'(0) = v$ $\Rightarrow$ $v(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)$.

Similarly, any curve γ in M through p gives rise to a map

$$\begin{array}{ccc} \{\text{Smooth functions } M \to \mathbb{R}\} & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \frac{d}{dt}\Big|_{t=0}(f\circ\gamma) \end{array}.$$

Note: · f only has to be defined in a nbd of p
· only depends on infinitesimal behavior of f near p.

Def M mfd, $p \in M$. The germ of smooth functions at p is

$$C_p^\infty(M) := \{f: V \to \mathbb{R} \mid V \ni p \text{ open in } M\}/\sim$$

$$(f_1: V_1 \to \mathbb{R}) \sim (f_2 : V_2 \to \mathbb{R}) \iff \exists\, V_3 \subset V_1 \cap V_2 \text{ nbd of } p \text{ with } f_1|_{V_3} = f_2|_{V_3}.$$

Note: $C_p^\infty(M)$ is a vector space over $\mathbb{R}$ (add, scalar mult as usual).
Then any curve γ through p gives a map

$$\begin{array}{cccc} \gamma'(0) : & C_p^\infty(M) & \longrightarrow & \mathbb{R} \\ & [f] & \longmapsto & \frac{d}{dt}\Big|_{t=0}(f\circ\gamma) \end{array}$$

and this map is linear.

Def A <u>tangent vector</u> to M at p is a linear map $C_p^\infty(M) \to \mathbb{R}$ that's equal to $\gamma'(0)$ for some curve γ.

(note: two curves can give the same linear map)

$\rightarrow$ Note this makes T_pM naturally a vector space.

Claim: this is the same as the previous def. For now let T_pM = first def.
Define $T_p'M$ = second def.

$$\Phi : \underset{\substack{\uparrow \\ \text{equiv class of } \gamma}}{T_pM} \longrightarrow \left(C_p^\infty(M) \to \mathbb{R}\right)$$

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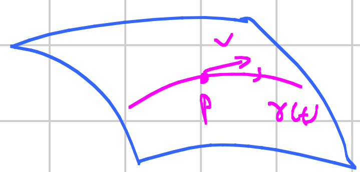
by $\Phi([\gamma])(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)$ where $f : V \to \mathbb{R}$ for some V.

Review: <u>tangent vectors</u>.

M = smooth mfd, $p \in M$, $T_pM = \{\text{tangent vectors to } p \text{ at } M\}$.

Def 1 $T_pM = \{\text{curves in } M \text{ through } p\}/\sim$
$\gamma : (-\epsilon, \epsilon) \to M$, $\gamma(0) = p$

Coord chart $\underset{\subseteq \mathbb{R}^n}{U} \xrightarrow{f} \underset{\subseteq M}{\overset{p}{V}}$, $\gamma_1 \sim \gamma_2 \iff \frac{d}{dt}\Big|_{t=0} (f^{-1} \circ \gamma_1) = \frac{d}{dt}\Big|_{t=0} (f^{-1} \circ \gamma_2)$.



Notation: sometimes write $\gamma'(0)$ for $[\gamma] \in T_pM$.

$v \in T_pM \rightsquigarrow$ <u>directional derivative</u> $v(\cdot) : C_p^\infty(M) \to \mathbb{R}$
$f \longmapsto v(f)$

$v(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma(t))$
for $v = \gamma'(0)$.

Why well-defined? choose coord chart $F: U \to V \subset M \xrightarrow{f} \mathbb{R}$

$F^{-1}\circ\gamma \nwarrow \quad \uparrow \gamma$ $(-\epsilon,\epsilon)$

$$\frac{d}{dt}\Big|_{t=0}(f\circ\gamma) = \frac{d}{dt}\Big|_{t=0}\left(f\circ F\circ F^{-1}\circ\gamma\right)$$
$$= d(f\circ F)(F^{-1}(p)) \cdot \underbrace{\frac{d}{dt}\Big|_{t=0}(F^{-1}\circ\gamma)}_{\text{only depends on } [\gamma]}$$

<u>Def 2</u> $T_p'M = \{$linear maps $C_p^\infty(M) \to \mathbb{R}$ that are equal to $v(\cdot)$ for some $v \in T_pM\}$.

Note there's an obvious map

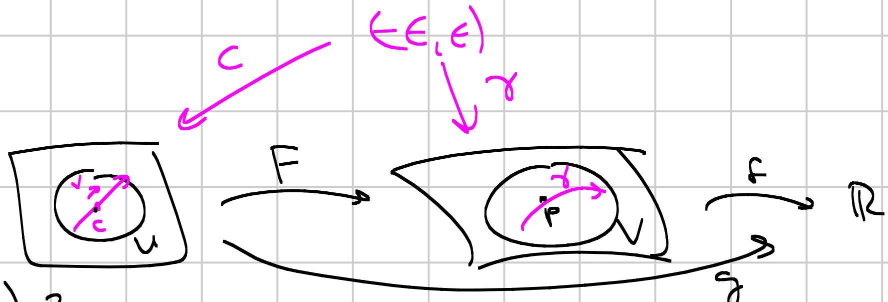
$$\Phi: T_pM \to T_p'M,$$

surjective by definition.

To prove $T_pM \cong T_p'M$, need Φ injective.

Let $U \xrightarrow{F} V$ be a chart, and let $(v_1,\dots,v_n)\in\mathbb{R}^n$. Define

$c(t) = F^{-1}(p) + t(v_1,\dots,v_n)$ and $\gamma(t) = F(c(t))$. Notation from before: $\gamma'(0) = v_1\frac{\partial}{\partial x_1}+\dots+v_n\frac{\partial}{\partial x_n}$.



What's $\Phi(\gamma'(0))$?

Let $f\in C_p^\infty(M)$, and write $g = f\circ F: U\to\mathbb{R}$.

$$\frac{d}{dt}\Big|_{t=0}(f\circ\gamma) = \frac{d}{dt}\Big|_{t=0}(f\circ F\circ c) = \frac{d}{dt}\Big|_{t=0}(g\circ c) = \begin{bmatrix}\frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n}\end{bmatrix}\begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix} = v_1\frac{\partial g}{\partial x_1}+\dots+v_n\frac{\partial g}{\partial x_n}$$

So diff values of $(v_1,\dots,v_n)$ give diff. maps $C_p^\infty(M)\to\mathbb{R}$ (plug in $g = x_1,\dots,g=x_n$).

With abuse of notation, identify f with g. Then the directional derivative $\quad v(f) = v_1 \frac{\partial f}{\partial x_1} + \cdots + v_n \frac{\partial f}{\partial x_n}$

which explains the notation $\quad v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$.

APPROACH 3 - most abstract.

Def A derivation at p is a linear map
$$\delta : C_p^\infty(M) \to \mathbb{R}$$
s.t. $$\delta(fg) = f(p)\,\delta(g) + \delta(f)\,g(p).$$

note: $v \in T_pM \Rightarrow v(\cdot)$ is a derivation (product rule).

Prop $T_pM \cong \{\text{derivations at } p\}$. ← note: vector space
$$v \longmapsto (f \mapsto v(f))$$

Pf Injective: easy exercise.

Surjective: $\quad U \xrightarrow{F} V \subset M$
$$\begin{tikzcd} U \arrow[r, "F"] \arrow[dr, "x_i"'] & V \arrow[d, "x_i \circ F^{-1}"] \\ & \mathbb{R} \end{tikzcd}$$

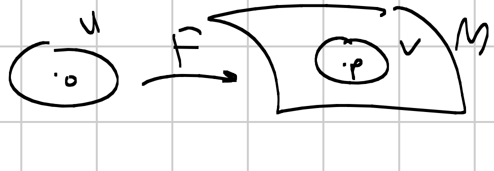
Say $\delta(x_i \circ F^{-1}) = v_i \in \mathbb{R}$.

Claim: $\delta = v(\cdot)$ where $v = \sum v_i \frac{\partial}{\partial x_i}$.

Note: these agree on $x_i \circ F^{-1}$:
$$\sum v_j \frac{\partial}{\partial x_j}(x_i \circ F^{-1}) = \sum v_j \frac{\partial x_i}{x_j} = v_i.$$

Need: $\delta = v(\cdot)$ on $x_i \circ F^{-1}$ means $\delta(f) = v(f)$ for all f.

Say $f \in C_p^\infty(M)$, $F(0) = p$; write $g = f \circ F : \text{nbd}(0) \subset \mathbb{R}^n$



Lemma Can write $g(x_1,\dots,x_n) = c + \sum x_i\, g_i(x_1,\dots,x_n)$ for some $c\in\mathbb{R}$ and smooth $g_i : \text{nbd}(0)\to\mathbb{R}$ with $g_i(\vec{0}) = \frac{\partial g}{\partial x_i}(\vec{0})$.

Pf: $g(\vec{x}) = g(\vec{0}) + \int_0^1 \frac{d}{dt} g(t\vec{x})\,dt = g(\vec{0}) + x_i \underbrace{\int_0^1 \frac{\partial g}{\partial x_i}(t\vec{x})\,dt}_{g_i(\vec{x})}$. $\square$

So:
$$\begin{aligned}\delta(f) &= \delta(c) + \sum \delta\big((x_i\circ F^{-1})\cdot(g_i\circ F^{-1})\big)\\ &= 0 + \sum \Big(\underbrace{\delta(x_i\circ F^{-1})}_{v_i}\cdot g_i(0) + \underbrace{\big(x_i\circ F^{-1}(p)\big)}_{0}\cdot \delta(g_i\circ F^{-1})\Big)\\ &= \sum v_i \frac{\partial g}{\partial x_i}(\vec{0})\\ &= v(f). \qquad \square\end{aligned}$$

Differentials of Smooth maps

Prop/Def $\varphi : M_1 \to M_2$ smooth, $p\in M_1$. There is a well-defined linear map
$$\begin{aligned} d\varphi_p : T_pM_1 &\longrightarrow T_{\varphi(p)}M_2 \\ [\gamma] &\longmapsto [\varphi\circ\gamma] \end{aligned}$$
, the *differential* of φ at p.

Pf

$$\begin{tikzcd}
 & V_1 \subset M_1 \arrow[r, "\varphi"] & V_2 \subset M_2 \\
(-\epsilon,\epsilon) \arrow[ur, "\gamma"] \arrow[dr, "F_1^{-1}\circ\gamma"'] & & \\
 & U_1 \subset \mathbb{R}^n \arrow[uu, "F_1"] \arrow[r] & U_2 \subset \mathbb{R}^m \arrow[uu, "F_2"'] \\
 & (x_1,\dots,x_n) & (y_1,\dots,y_m)
\end{tikzcd}$$

$$F_2^{-1}\circ\varphi\circ F_1(x_1,\dots,x_n) = \big(y_1(x_1,\dots,x_n),\dots,y_m(x_1,\dots,x_n)\big)$$

$(F_1^{-1}\circ\gamma)(t) = (x_1(t),\dots,x_n(t)) \to$ tangent vector ("$[\gamma]$") is

$$x_1'(0)\frac{\partial}{\partial x_1}+\cdots+x_n'(0)\frac{\partial}{\partial x_n}.$$

$$\left(F_2^{-1}\circ(\varphi\circ\gamma)\right)(t) = \left(F_2^{-1}\circ\varphi\circ F_1\right)(F_1^{-1}\circ\gamma)(t)$$
$$= \left(y_1(x_1(t),\dots,x_n(t)),\dots,y_m(x_1(t),\dots,x_n(t))\right).$$

Derivative at $t=0$ is
$$\Big(\underbrace{\frac{d}{dt}\Big|_{t=0} y_1(x_1(t),\dots,x_n(t))}_{=\ \underbrace{\left(\frac{\partial y_i}{\partial x_j}\right)}_{m\times n\text{ matrix}}\cdot\begin{pmatrix}x_1'(0)\\ \vdots\\ x_n'(0)\end{pmatrix}}\Big)\frac{\partial}{\partial y_1}+\cdots$$

and this depends only on $x_1'(0),\dots,x_n'(0)$, not γ:

$$d\varphi_p\left(v_1\frac{\partial}{\partial x_1}+\cdots+v_n\frac{\partial}{\partial x_n}\right) = \sum_i\left(\sum_j\frac{\partial y_i}{\partial x_j}v_j\right)\frac{\partial}{\partial y_i}\quad\left(=\left(\frac{\partial y_i}{\partial x_j}\right)\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\right)$$

linear in $(v_1,\dots,v_n)$. □

Summary: in coords, $d\varphi_p$ is the map $\underset{\cong T_pM_1}{\mathbb{R}^n}\longrightarrow\underset{\cong T_{\varphi(p)}M_2}{\mathbb{R}^m}$ given by the matrix $\left(\frac{\partial y_i}{\partial x_j}\right)$.

Special Case: change of coordinates, $\varphi = id$.

$$\underset{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}}{T_pM}\xrightarrow{id}\underset{\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}}{T_pM}$$

$$\begin{tikzcd} & M & \\ \underset{x_1\cdots x_n}{U_1} \arrow[ur, "f_1"] \arrow[rr] & & \underset{y_1\cdots y_n}{U_2} \arrow[ul, "f_2"'] \end{tikzcd}$$

$$\boxed{\frac{\partial}{\partial x_j} = \sum\frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i}}.$$

Thm (Chain Rule)

$$\begin{tikzcd} \underset{p\in}{M} \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & N \arrow[r, "\psi"] & P \end{tikzcd} \rightsquigarrow \begin{tikzcd} T_pM \arrow[r, "d\varphi_p"] \arrow[rr, bend right, "d(\psi\circ\varphi)_p"'] & T_{\varphi(p)}N \arrow[r, "d\psi_{\varphi(p)}"] & T_{\psi\varphi(p)}P \end{tikzcd}$$

$$\boxed{d(\psi\circ\varphi)_p = d\psi_{\varphi(p)}\circ d\varphi_p}.$$

<u>Rmk</u> $\varphi: M \to N$ diffeomorphism; then $M \xrightarrow{\varphi} N \xrightarrow{\psi} M$ $(\psi = \varphi^{-1})$, with $\psi \circ \varphi = id$

$d(id)_p = id : T_pM \to T_pM$

So $d\psi_{\varphi(p)}$, $d\varphi_p$ are inverse maps

$\Rightarrow d\varphi_p : T_pM \to T_{\varphi(p)}N$ is an <u>isomorphism</u>.

Conversely:

<u>Prop</u> If $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is an isomorphism then φ is a local diffeomorphism at p.

<u>Pf</u> Inverse Function Theorem.

φ is $\begin{cases} \text{immersion} \\ \text{submersion} \\ \text{local diffeo} \end{cases}$ at p $\iff$ $d\varphi_p$ is $\begin{cases} \text{injective} \\ \text{surjective} \\ \text{isomorphism} \end{cases}$.

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<u>Tangent Bundle</u>

M smooth n-mfd. Define $TM := \{(p, v) \mid p \in M, v \in T_pM\}$
$= \coprod_p T_pM$.

This isn't just a set but a smooth $(2n)$-manifold:

$\{(F_\alpha, U_\alpha, V_\alpha)\}$ = atlas for M.

$x_1, \dots, x_n$ coords on U_α. We saw: $p \in U_\alpha \Rightarrow T_pM$ has basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$.

Define

$$\tilde{F}_\alpha : \underbrace{U_\alpha \times \mathbb{R}^n}_{\subset \mathbb{R}^{2n}} \longrightarrow TM$$
$$(x_1, \dots, x_n, v_1, \dots, v_n) \longmapsto (F_\alpha(x_1, \dots, x_n), \sum v_i \tfrac{\partial}{\partial x_i}).$$

Since $\{V_\alpha\}$ cover M, $\{\tilde{F}_\alpha(U_\alpha \times \mathbb{R}^n)\}$ cover TM.

Need to check: transition functions smooth.

Suppose $V_\alpha \cap V_\beta \neq \emptyset$. Say $(p,v) \in \tilde{F}_\alpha(U_\alpha \times \mathbb{R}^n) \cap \tilde{F}_\beta(U_\beta \times \mathbb{R}^n)$.

Then
$$p = F_\alpha(x_1, \ldots, x_n) \quad , \quad v = \sum v_i \frac{\partial}{\partial x_i}$$
$$= F_\beta(y_1, \ldots, y_n) \qquad\qquad = \sum w_i \frac{\partial}{\partial y_i}.$$

$$(y_1, \ldots, y_n) = (F_\beta^{-1} \circ F_\alpha)(x_1, \ldots, x_n) \Rightarrow (w_1, \ldots, w_n) = d(F_\beta^{-1} \circ F_\alpha)(v_1, \ldots, v_n)$$

$$\Rightarrow ((y_1, \ldots, y_n), (w_1, \ldots, w_n)) = ((F_\beta^{-1} \circ F_\alpha)(x_1, \ldots, x_n), d(F_\beta^{-1} \circ F_\alpha)(v_1, \ldots, v_n)).$$

Since $F_\beta^{-1} \circ F_\alpha$ is smooth, so is $d(F_\beta^{-1} \circ F_\alpha)$

so $\tilde{F}_\beta^{-1} \circ \tilde{F}_\alpha$ is a smooth map. $\square$

Vector Fields

Def A vector field X on M is a map $M \to TM$ mapping p to some $X(p) \in T_pM$ for all $p \in M$. (i.e. $M \to TM \xrightarrow{\pi} M$ is id)

Usually assume smooth: the map $M \to TM$ is smooth.

In local coords: can write as
$$X(x_1, \ldots, x_n) = X_1(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} + \cdots + X_n(x_1, \ldots, x_n) \frac{\partial}{\partial x_n}.$$
X_i smooth functions.

Notation: write $\mathrm{Vect}(M) = \{\text{smooth vector fields on } M\}$ ↔ $\mathbb{R}$-vector spaces

$C^\infty(M) = \{\text{smooth functions } M \to \mathbb{R}\}$ ← ring.

tangent vector ⇝ derivation at a pt; vector field ⇝ global derivation

Def A <u>derivation on M</u> is a linear map $\delta : C^\infty(M) \to C^\infty(M)$ such that $f, g \in C^\infty(M) \Rightarrow$

$$\delta(f \cdot g) = f\,\delta(g) + \delta(f)\,g$$

i.e. $\delta(fg)(p) = f(p)\,\delta(g)(p) + \delta(f)(p)\,g(p)$.

$X_p \in T_pM$

$X(f)(p) = X_p(f)$

Prop Any (smooth) vector field X on M gives a derivation $X(\cdot)$ and this gives an isomorphism

$$\mathrm{Vect}(M) \xrightarrow{\cong} \{\text{Derivations on } M\}.$$

Pf Injective: we already proved different tangent vectors give different derivations at a point.

Surjective: let $p \in M$, $U \xrightarrow{F} V$ chart near p.

$\exists$ function $f_i : M \to \mathbb{R}$ s.t. near p, $f_i \circ F(x_1, \ldots, x_n) = x_i$

(to make this smooth, need partition of unity).

Then if $\delta(f_i)(p) = v_i$ then $\delta(\cdot) = v(\cdot)$ where $v = \sum v_i \frac{\partial}{\partial x_i}$.

So from δ we get a tangent vector at p $\forall p$.

(Check: smooth; indep of chart.) $\square$

Lie Bracket

$X, Y \in Vect(M) \rightsquigarrow X(\cdot), Y(\cdot) : C^\infty(M) \to C^\infty(M)$.
Consider the map $f \mapsto X(Y(f)) : C^\infty(M) \to C^\infty(M)$.
This is $\mathbb{R}$-linear; is it a derivation?

$$X(Y(fg)) = X(fY(g) + Y(f)g)$$
$$= fX(Y(g)) + \underbrace{X(f)Y(g) + Y(f)X(g)} + X(Y(f))g$$

No; but

$$f \mapsto X(Y(f)) - Y(X(f)) \ \underline{\text{is}}.$$

Def The Lie bracket $[X,Y] \in Vect(M)$ is the vector field corresponding to the derivation $f \mapsto X(Y(f)) - Y(X(f))$.

In local coords: $X = \sum a_i \frac{\partial}{\partial x_i}$, $Y = \sum b_i \frac{\partial}{\partial x_i}$

$$X(Y(f)) = \sum_{i,j} a_j \frac{\partial}{\partial x_j}\left(b_i \frac{\partial f}{\partial x_i}\right) = \sum a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_j b_i \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

$\Rightarrow$

$$[X,Y](f) = \sum_{i,j} a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i}$$

$\Rightarrow$ $$\boxed{[X,Y] = \sum_i \left(\sum_j \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\right)\frac{\partial}{\partial x_i}}.$$

Properties.

$X, Y \longmapsto [X,Y]$ is:

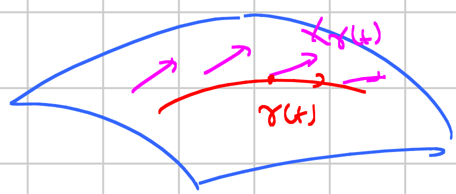
- $\mathbb{R}$-bilinear: $[aX_1 + bX_2, Y] = a[X_1, Y] + b[X_2, Y]$ for $a, b \in \mathbb{R}$.
- antisymmetric: $[Y, X] = -[X, Y]$
- a Lie bracket: satisfies the Jacobi identity
$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Note: for $g \in C^\infty(M)$ it's not true that $[gX, Y] = g[X,Y]$:

$$[gX,Y]f = gX(Y(f)) - Y((gX)(f))$$
$$= gX(Y(f)) - Y(gX(f))$$
$$= gX(Y(f)) - gY(X(f)) - Y(g)X(f)$$

So $\boxed{[gX,Y] = g[X,Y] - Y(g)X}$.

Flow of a vector field



For $X \in \mathrm{Vect}(M)$, $\gamma : (a,b) \to M$ is an integral curve for X if
$$\gamma'(t) = X(\gamma(t)) \in T_{\gamma(t)}M \quad \forall\, t \in (a,b).$$

Prop $p \in M$. $\exists\, (a,b)$ containing 0 and unique integral curve $\gamma : (a,b) \to M$ for X with $\gamma(0) = p$.

Pf Local existence/uniqueness for first order ODE. $\square$

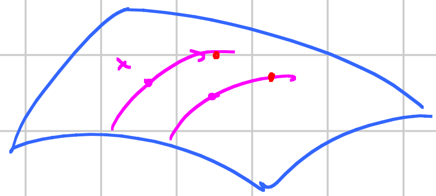
Note maximal (a,b) may depend on p: but:

Prop $X \in \mathrm{Vect}(M)$. $\exists$ nbd V of p, interval $(a,b) \ni 0$ s.t. $\forall x \in V$, $\exists!$ integral curve $\gamma_x : (a,b) \to M$ for X with $\gamma(0) = x$, and the map
$$V \times (a,b) \longrightarrow M$$
$$(x, t) \longmapsto \gamma_x(t)$$
is smooth.

Pf Regularity for ODEs. $\square$

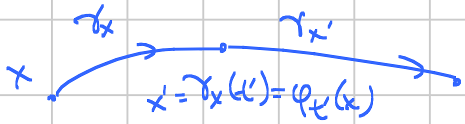
Note: doesn't have to hold on all of M for uniform (a,b).

For $t \in \mathbb{R}$, define local flow of X: $\varphi_t : \text{"}M\text{"} \longrightarrow M$
$$\varphi_t(x) = \gamma_x(t)$$



"time t flow under X" only partially defined for $t \neq 0$.

Where defined: φ_t is a smooth map, and
$$\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}.$$



$\gamma_x(t+t') = \gamma_{x'}(t)$ by uniqueness: these both satisfy $\frac{d}{dt}\gamma(t) = X_{\gamma(t)}$.

Also $\varphi_0 = id$ so $\varphi_{-t} = (\varphi_t)^{-1}$.

If M is compact, φ_t is well-defined $\forall t$ and is a diffeomorphism.

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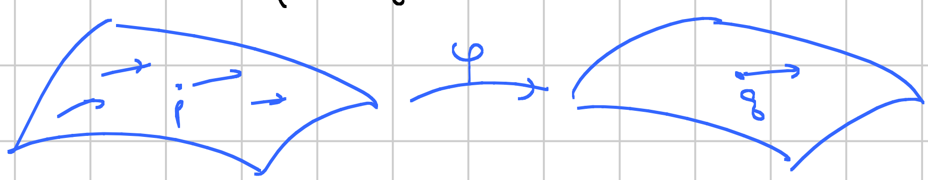
Recall: given a vector field $X \in \mathrm{Vect}(M)$ $(p \in M \Rightarrow X_p \in T_pM)$,
integral curve $\gamma_x(t)$, $\gamma_x'(t) = X_{\gamma_x(t)}$, $\gamma_x(0) = x$.
$\Rightarrow$ time t flow of X, φ_t "$: M$" $\to M$, $\varphi_t(x) = \gamma_x(t)$.

Lie Derivative of a vector field

$X, Y \in \mathrm{Vect}\, M \Rightarrow \mathcal{L}_X Y \in \mathrm{Vect}\, M$

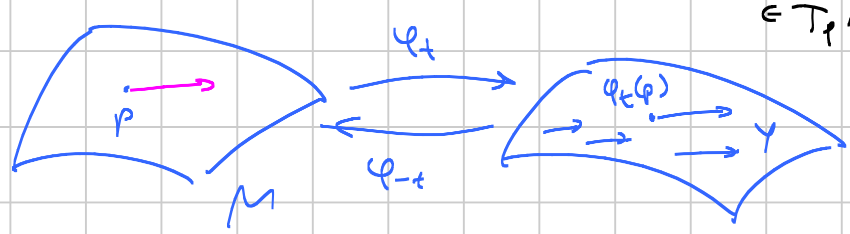
Given diffeo $\varphi : M \to N$ and $X \in \mathrm{Vect}(M)$, define the pushforward $\varphi_* X \in \mathrm{Vect}(N)$ given by
$$(\varphi_* X)_q = (d\varphi)(X_p) \quad \text{where } \varphi(p) = q.$$

(henceforth we'll sometimes write $d\varphi : T_pM \to T_qN$ as φ_* as well)



For $X, Y \in \mathrm{Vect}\, M$, define $\varphi_t =$ flow of $X : M \to M$
$$\Rightarrow (\varphi_{-t})_*(Y) \in \mathrm{Vect}(M).$$

Def $\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_*(Y)$ i.e.
$$(\mathcal{L}_X Y)_p = \frac{d}{dt}\Big|_{t=0} \underbrace{(\varphi_{-t})_*(Y_{\varphi_t(p)})}_{\in T_pM}$$



Prop $\mathcal{L}_X Y = [X, Y]$.

Lemma (similar to last thm) $V = \text{nbd}(p)$, $h : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ smooth, $h(0,x)=0 \ \forall x$. Then $\exists$ smooth $g : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ with

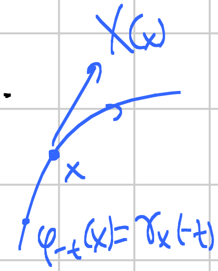
- $h(t,x) = t\, g(t,x)$
- $g(0,x) = \frac{\partial h}{\partial t}(0,x)$.

Pf $h(t,x) = \int_0^1 \frac{\partial}{\partial s} h(st,x)\, ds = t \overbrace{\int_0^1 \frac{\partial h}{\partial t}(st,x)\, ds}^{g(t,x)}$. $\square$

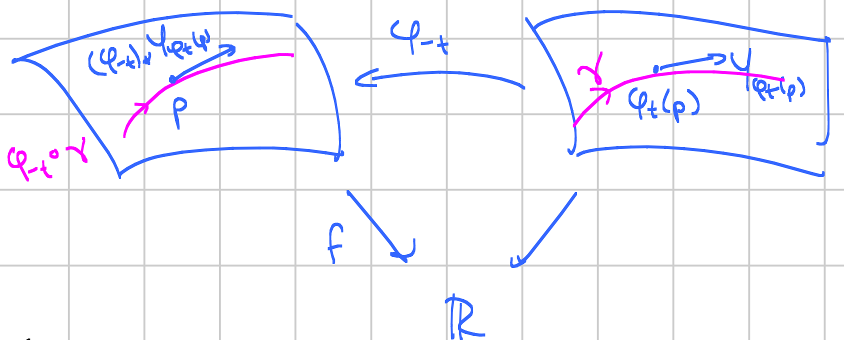
Pf of Prop $f \in C^\infty(M)$: want $[X,Y]f = (\mathcal{L}_X Y)(f)$.

Apply Lemma to $h(t,x) = f(\varphi_{-t}(x)) - f(x)$:

$$h(t,x) = t\,g(t,x), \qquad g(0,x) = \frac{\partial h}{\partial t}(0,x) = \underbrace{-Xf(x)}.$$



Then



define $\gamma(s)$ to be the integral curve for Y with $\gamma(0) = \varphi_t(p)$

$$\left((\varphi_{-t})_* Y_{\varphi_t(p)}\right)(f) = \frac{d}{ds}\Big|_{s=0} f\left(\varphi_{-t}\circ\gamma(s)\right) = \frac{d}{ds}\Big|_{s=0} (f\circ\varphi_{-t})(\gamma(s))$$

$$= Y_{\varphi_t(p)}(f\circ\varphi_{-t}) = Y_{\varphi_t(p)}(f + tg)$$

$$= (Yf)_{\varphi_t(p)} + t(Yg)_{\varphi_t(p)}$$

$$\Rightarrow (\mathcal{L}_X Y)(f)_p = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* Y_{\varphi_t(p)}(f)$$

$$= \frac{d}{dt}\Big|_{t=0} (Yf)_{\varphi_t(p)} + (Yg)_p$$

$$= X(Yf)(p) - Y(Xf)(p)$$

$$= [X,Y]_p f. \qquad \square$$

Important example: Lie groups

Def A *Lie group* is a group G with the structure of a smooth mfd such that the maps:

- left mult $L_h : G \to G$, $g \mapsto hg$
- right mult $R_h : G \to G$, $g \mapsto gh$
- inverse $\mathrm{inv} : G \to G$, $g \mapsto g^{-1}$

} are smooth.

ex: $\mathbb{R}^n$; quotients like $\mathbb{R}^n/\mathbb{Z}^n = T^n$;

matrix groups
- $GL(n,\mathbb{R})$ = open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
- $O(n) = \{M^T M = I\}$, $SO(n)$
- $GL(n,\mathbb{C})$, $U(n)$, $SU(n)$

$h \in G \Rightarrow L_h, R_h$ induce maps $TG \overset{(L_h)_* = dL_h}{\underset{(R_h)_* = dR_h}{\rightrightarrows}} TG$

Def $X \in \mathrm{Vect}\, M$ is *left/right invariant* if $(L_h)_* X = X$ / $(R_h)_* X = X$ $\forall h \in G$.

X left invariant v.f. is *determined* by $X_e \in T_e M$: (e = identity in G)

$L_g : G \to G$, $e \mapsto g$ satisfies $(L_g)_*(X_e) = X_g$. $\forall g$

Conversely any $X_e \in T_e M$ gives rise to a left invt v.f. X defined by

Check: $(L_h)_*(X_g) \overset{?}{=} X_{L_h g}$ ✓

$(L_h)_*(L_g)_* X_e = (L_{hg})_* X_e = X_{hg}$

So $\{\text{left invt v.f.}\} \overset{1-1}{\longleftrightarrow} T_e G$.

Write $\boxed{\mathfrak{g} := T_e G}$ Lie algebra assoc to G.

Prop X, Y left invt. Then $[X,Y]$ is as well.

Pf. From HW: $\varphi: M \to M$ diffeo $\Rightarrow [\varphi_* X, \varphi_* Y] = \varphi_* [X,Y]$.

Here $(L_h)_* [X,Y] = [(L_h)_* X, (L_h)_* Y] = [X,Y]$. $\square$

So bracket on vector fields induces $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

This is $\mathbb{R}$-bilinear, antisymmetric, and satisfies Jacobi

$\to$ gives $\mathfrak{g}$ the structure of a Lie algebra.

Next define $\Psi_h = R_{h^{-1}} L_h : G \to G$: $\Psi_h(g) = h g h^{-1}$.

Since $\Psi_h(e) = e$, this give a map

$$\begin{aligned} (\Psi_h)_* &: T_e G \to T_e G \\ &\;\; \| \\ \mathrm{Ad}\, h &: \mathfrak{g} \to \mathfrak{g} \end{aligned}$$

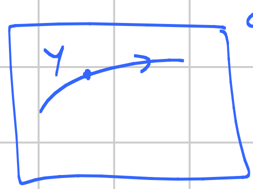
adjoint representation

notes:
- if X is left invt then $(\Psi_h)_* X = (R_{h^{-1}})_* (L_h)_* X = (R_{h^{-1}})_* X$
- $\mathrm{Ad}(h): \mathfrak{g} \to \mathfrak{g}$ is a linear, Lie algebra map:
$$\mathrm{Ad}(h)\, [X,Y] = [\mathrm{Ad}(h) X, \mathrm{Ad}(h) Y]$$
- Ad is a representation: $\mathrm{Ad}(h_1 h_2) = \mathrm{Ad}(h_1)\, \mathrm{Ad}(h_2)$.

Prop $X, Y \in \mathfrak{g}$, $\varphi_t =$ local flow of (left invt v.f.) X.

Then

$$[X,Y] = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\varphi_t(e))\, Y.$$



$\mathfrak{g} = T_e G$

Pf $[X,Y] = \mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* Y$.

From HW: $\varphi_{-t}(g) = g\,\varphi_{-t}(e) = R_{\varphi_{-t}(e)}(g) \Rightarrow \varphi_{-t} = R_{\varphi_{-t}(e)}$

$$\Rightarrow [X,Y] = \frac{d}{dt}\Big|_{t=0} \big(R_{\underbrace{\varphi_{-t}(e)}_{(\varphi_t(e))^{-1} \text{ from HW}}}\big)_* Y = \frac{d}{dt}\Big|_{t=0} Ad(\varphi_t(e))\, Y. \quad \square$$

Vector Bundles

Idea: generalize tangent bundle $TM = \coprod_{p\in M} T_pM \xrightarrow{\pi} M$.

Chart $U \xrightarrow{F} V$ for M gives a chart for TM:

$$U\times\mathbb{R}^n \xrightarrow{\tilde{F}} \pi^{-1}(V)$$

$$(x_1,\dots,x_n,v_1,\dots,v_n) \longmapsto \Big(F(x_1,\dots,x_n),\ \sum v_i \frac{\partial}{\partial x_i}\Big)$$

$$\downarrow$$

$$V\times\mathbb{R}^n$$

think of $\tilde{\tilde{F}}$ as this map.

in particular we get $\{p\}\times\mathbb{R}^n \longrightarrow T_pM$

If we have two charts V_1, V_2 and $p\in V_1\cap V_2$ then we get maps

$$\{p\}\times\mathbb{R}^n \longrightarrow T_pM \longleftarrow \{p\}\times\mathbb{R}^n$$

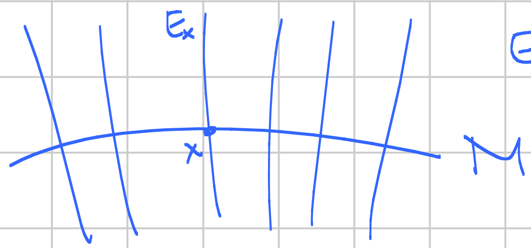
and the induced map $\mathbb{R}^n\to\mathbb{R}^n$ is <u>linear</u>: given by the Jacobian matrix $\left(\frac{\partial y_j}{\partial x_i}\right)$. $\left(\frac{\partial}{\partial x_i} = \sum \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}\right)$.

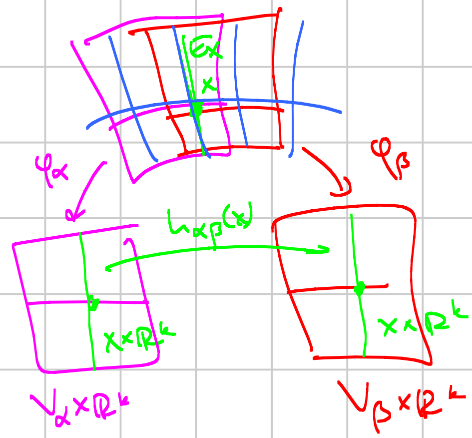
<u>Def</u> M smooth mfd. A rank k <u>vector bundle</u> over M is a smooth mfd $E = \coprod_{x\in M} E_x$ where each E_x = rank k vector space $/\mathbb{R}$, such that:

1. the map $\pi: E\to M$ sending E_x to x is smooth
2. $\exists$ open cover $\{V_\alpha\}$ of M and diffeos
$$\varphi_\alpha:\ \pi^{-1}(V_\alpha) \xrightarrow{\cong} V_\alpha\times\mathbb{R}^k$$ "local trivialization" of E

such that $\varphi_\alpha(E_x) = \{x\} \times \mathbb{R}^k$: i.e.
$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times \mathbb{R}^k \arrow[dl, "\pi"] \\ & V_\alpha & \end{tikzcd}$$
commutes

3. the transition functions
$$h_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^k \longrightarrow (V_\alpha \cap V_\beta) \times \mathbb{R}^k$$
are smooth and <u>linear</u> : that is, for $x \in V_\alpha \cap V_\beta$,
$$h_{\alpha\beta}(x) : \underset{\mathbb{R}^k}{\underset{\parallel}{\{x\} \times \mathbb{R}^k}} \longrightarrow \underset{\mathbb{R}^k}{\underset{\parallel}{\{x\} \times \mathbb{R}^k}} \quad \text{is in } GL(k).$$





<u>Rmks</u>. 1. Write $\begin{array}{c} \mathbb{R}^k \to E \\ \downarrow \\ M \end{array}$ or just $\begin{array}{c} E \\ \downarrow \\ M \end{array}$

2. if $k=1$ this is a <u>line bundle</u> over M.

3. <u>importantly</u>: can reconstruct E from the transition functions.

$$E = \coprod_\alpha (V_\alpha \times \mathbb{R}^k)/\sim \quad \text{where}$$

$$(x, v) \in V_\alpha \times \mathbb{R}^k \sim (y, w) \in V_\beta \times \mathbb{R}^k$$
$$\text{if } x = y \text{ and } w = h_{\alpha\beta}(v).$$

<u>Ex</u>. 1. $M \times \mathbb{R}^k$ "trivial" vector bundle

2. TM: transition fns look like matrix $\left(\frac{\partial y}{\partial x}\right)$.

Def $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ $\begin{matrix} E' \\ \downarrow \\ M \end{matrix}$ vector bundles. A smooth map $\varphi : E \to E'$ is a <u>bundle map</u> if

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr] & E' \arrow[d] \\ & M \end{tikzcd}$$

commute

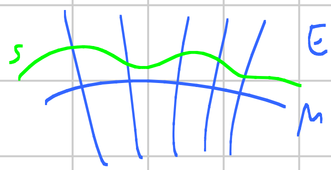
and $\forall\, x \in M$, $\varphi|_{E_x} : E_x \to E'_x$ is a <u>linear</u> map.

A <u>bundle isomorphism</u> is a bundle map that's invertible.
A vector bundle is <u>trivial</u> if it's isomorphic to $M \times \mathbb{R}^k$.

Most things associated to a vector bundle are "invariant" under isom. Eg:

Def A <u>section</u> of a vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a smooth map $s : M \to E$ with $\pi \circ s = id$.

$\Gamma(E) :=$ vector <u>space</u> of sections of E.



$\rightarrow$ rmk: if $E \cong E'$ then $\Gamma(E) \cong \Gamma(E')$.

<u>Ex</u> · Sections of trivial $\mathbb{R}^k$ bundle $= \{$smooth maps $M \to \mathbb{R}^k\}$
· $\Gamma(TM) = \mathrm{Vect}(M)$.

<u>Operations on Vector Bundles</u>

<u>Dual</u>: $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ v.b. $\Rightarrow$ replace each E_x by E_x^* (note $\cong$ but not canonical)

$V_1 \xrightarrow{h_{\alpha\beta}} V_2$ dualizes to $V_1^* \underset{(h_{\alpha\beta})^T}{\overset{(h_{\alpha\beta}^T)^{-1}}{\rightleftarrows}} V_2^*$

Def $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ v.b., transition fns $h_{\alpha\beta}$ $\Rightarrow$ <u>dual</u> $\begin{matrix} E^* \\ \downarrow \\ M \end{matrix}$ v.b. with transition fns $(h_{\alpha\beta}^T)^{-1}$.

<u>Ex</u>: <u>Cotangent bundle</u> T^*M is the dual to TM.

$T_pM = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$ vector space

$T_p^*M = \langle dx_1, \dots, dx_n \rangle$ dual basis: $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$.

element of T_p^*M is a <u>cotangent vector</u>.

Two coord charts x_i, y_i $\Rightarrow$ recall $\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$

$\Rightarrow dy_j\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial y_j}{\partial x_i} \Rightarrow dy_j = \sum_i \frac{\partial y_j}{\partial x_i} dx_i$

i.e. $\begin{pmatrix} dy_1 \\ \vdots \\ dy_n \end{pmatrix} = \left(\frac{\partial y}{\partial x}\right)^T \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$ — compare $\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = \left(\frac{\partial x}{\partial y}\right) \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$.

<u>Other operations on vector spaces/bundles</u>.

<u>$\oplus$</u>: $\begin{matrix} E \\ \downarrow \\ M \\ \text{rank } k \end{matrix}$ $\begin{matrix} F \\ \downarrow \\ M \\ \text{rank } l \end{matrix}$ $\rightsquigarrow$ $\begin{matrix} E\oplus F \\ \downarrow \\ M \\ \text{rank } k+l \end{matrix}$ $(E\oplus F)_x = E_x \oplus F_x$.

transition fns. $h_{\alpha\beta}(x) \in GL(k)$ for E, $j_{\alpha\beta}(x) \in GL(l)$ for F

$\Rightarrow h_{\alpha\beta}(x) \oplus j_{\alpha\beta}(x) \in GL(k+l)$ for $E\oplus F$

<u>$\otimes$</u>: $\begin{matrix} E\otimes F \\ \downarrow \\ M \end{matrix}$ rank kl, $(E\otimes F)_x = E_x \otimes F_x$

transition fns $h_{\alpha\beta}(x) \otimes j_{\alpha\beta}(x) \in GL(kl)$ for $E\otimes F$.

<u>Sym^m</u>: $E \rightsquigarrow \mathrm{Sym}^m E$. $(\mathrm{Sym}^m E)_x = \otimes^m E_x / I$

$I = \langle \cdots \otimes v \otimes w \otimes \cdots - \cdots \otimes w \otimes v \otimes \cdots \rangle$

rank $\binom{k+m-1}{m}$ vector bundle

<u>Λ^m</u>: $E \rightsquigarrow \Lambda^m E$ $(\Lambda^m E)_x = \otimes^m E_x / I'$

$I' = \langle \cdots \otimes v \otimes w \otimes \cdots + \cdots \otimes w \otimes v \otimes \cdots, \cdots \otimes v \otimes v \otimes \cdots \rangle$

In this quotient, usually write $\otimes$ as $\wedge$. So $v \wedge w = -w \wedge v$, $v \wedge v = 0$.

$\Lambda^m E$ has rank $\binom{k}{m}$.

$$\underset{\substack{\| \\ \text{trivial } \mathbb{R}\text{-bundle}}}{\Lambda^0 E} \quad \underset{\substack{\| \\ E}}{\Lambda^1 E} \quad \Lambda^2 E \quad \cdots \quad \Lambda^k E$$

rank 1: line bundles.

2/3

Bundles from TM: Consider the bundle

$$\underbrace{TM \otimes \cdots \otimes TM}_{p} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q} =: T^p_q M.$$

A section of this bundle is called a <u>(p,q)-tensor</u>.

- $(1,0)$ tensor: <u>vector field</u> in $\mathrm{Vect}(M) = \Gamma(TM)$
- $(0,1)$ tensor: <u>1-form</u>, in $\Omega^1(M) := \Gamma(T^*M)$.
- $(0,2)$ tensor: section of $T^*M \otimes T^*M$.
  At x, this is an elt of $T_p^*M \otimes T_p^*M$, i.e. a bilinear map
  $T_pM \otimes T_pM \to \mathbb{R}$ $\qquad (V^* \otimes V^* = (V \otimes V)^*)$
  important future example: <u>Riemannian metric</u>.
- $(0,m)$ tensor: <u>differential forms</u> are important examples. (we'll treat soon).

<u>Operations on tensors</u>

<u>Contraction</u>:

$$C_{ij} : \Gamma(T^p_q M) \to \Gamma(T^{p-1}_{q-1} M) \qquad 1 \le i \le p,\ 1 \le j \le q$$

This is defined fiberwise by the map

$$C_{ij} : V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes(p-1)} \otimes (V^*)^{\otimes(q-1)}$$

$$v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \longmapsto w_j(v_i)\ v_1 \otimes \cdots \hat{v}_i \cdots \otimes w_1 \otimes \cdots \hat{w}_j \cdots$$

<u>ex</u>: $p = q = 1$. $\quad C : V \otimes V^* \to \mathbb{R}$

This is the <u>trace</u> $\quad \mathrm{tr} : \mathrm{End}(V) \to \mathbb{R}$.

Pullback

$M \xrightarrow{\varphi} N \quad \rightsquigarrow \quad T_pM \xrightarrow{\varphi_*} T_qN$
$p \qquad \varphi(p)=q \qquad$ (linear map)

Dualize $\rightsquigarrow$ $T_q^*N \xrightarrow{\varphi^*} T_p^*M$
$\alpha \longmapsto \varphi^*(\alpha) \qquad (\varphi^*\alpha)(v) = \alpha(\varphi_* v)$

This yields a map $\varphi^* : \underbrace{\Gamma(T^*N)}_{= \Omega^1 N} \longrightarrow \underbrace{\Gamma(T^*M)}_{= \Omega^1 M}$

More generally, φ^* gives a map $\underbrace{\Gamma(T^0_k N)}_{(0,k) \text{ tensors on } N} \longrightarrow \underbrace{\Gamma(T^0_k M)}_{(0,k) \text{ tensors on } M}$.

$\alpha \in \Gamma(\otimes^k T^*N)$ means α eats k vectors $w_1, \dots, w_k \in T_qN$: $\alpha(w_1, \dots, w_k) \in \mathbb{R}$.

$$\varphi^*\alpha(\underbrace{v_1, \dots, v_k}_{\in T_pM}) = \alpha(\varphi_* v_1, \dots, \varphi_* v_k)$$

Note this is only a map of $(0,k)$ tensors: recall $\varphi : M \to N$ does not give a map $Vect(M) \to Vect(N)$ (or vice versa).

But: if φ is a diffeo, we can define "pullback"

$$\varphi^* : \Gamma(TN) \longrightarrow \Gamma(TM)$$
$$X \longmapsto (\varphi^{-1})_* X$$

We can extend this to a pullback for any tensors.

$$\varphi^* : \Gamma(T^p_q N) \longrightarrow \Gamma(T^p_q M)$$
$$v_1 \otimes \cdots \otimes w_1 \otimes \cdots \mapsto \varphi^*(v_1) \otimes \cdots \otimes \varphi^*(w_1) \otimes \cdots$$

In particular, suppose $X \in Vect(M) \rightsquigarrow \varphi_t =$ time t flow of X.

Def The Lie derivative associated to X is the linear map

$$\mathcal{L}_X : \Gamma(T^p_q M) \to \Gamma(T^p_q M)$$
$$\mathcal{L}_X(S) = \frac{d}{dt}\Big|_{t=0} \varphi_t^*(S).$$

Ex: $Y \in Vect(M) \to \mathcal{L}_X Y = [X, Y]$. (note here $\varphi_t^* = (\varphi_{-t})_*$)

Local Operators and Tensors

recall section $s \in \Gamma(E)$ is a map $x \in M \mapsto s_x \in E_x$.

Def A *local operator* is a linear map $P : \Gamma(E) \to \Gamma(F)$, E, F vector bundles over M, such that $\forall x \in M$ and $\forall U =$ nbd of x, if $s, s' \in \Gamma(E)$ satisfy $s_y = s'_y \ \forall y \in U$, then $(Ps)_x = (Ps')_x$.

"at a point, the operator depends only on the section near that point"

ex: $X \in \mathrm{Vect}(M)$, $\Gamma(T^p_q M) \to \Gamma(T^p_q M)$ is local.
$$s \mapsto \mathcal{L}_X s$$

Special case of local operator: $P : \Gamma(E) \to \Gamma(F)$ such that if $s_x = s'_x$ then $(Ps)_x = (Ps')_x$.

Then P induces a map $E_x \to F_x$ ($\Leftrightarrow$ elt of $\mathrm{Hom}(E_x, F_x) = E_x^* \otimes F_x$) $\forall x$
$\Rightarrow$ section of $E^* \otimes F$. Call P a *tensor*. Why?

In particular, suppose $E = T^p_q M$, $F = T^r_\ell M$. Such a map P is a section of
$$(\otimes^p TM \otimes \otimes^q T^*M)^* \otimes (\otimes^r TM \otimes \otimes^\ell T^*M)$$
$$\cong \otimes^{q+r} TM \otimes \otimes^{p+\ell} T^*M = T^{q+r}_{p+\ell} M.$$

So P itself is a tensor.

ex: • $\mathrm{tr} : \Gamma(T^1_1 M) \to \mathbb{R}$ (or contractions in general $\Gamma(T^p_q M) \to \Gamma(T^{p-1}_{q-1} M)$)
$(\mathrm{tr}\, s)_x \in \mathbb{R}$ only depends on s_x : tensor.

• $\mathcal{L}_X : T^p_q M \to T^p_q M$ not a tensor.
e.g. $\mathcal{L}_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ $(\mathcal{L}_X Y)_p$ depends on more than Y_p.

Useful characterization of tensors: note $C^\infty(M)$ acts on $\Gamma(E)$ by pointwise scalar multiplication: $(fs)_x = f(x)\, s_x \in E_x$.

<u>Prop</u> $P : \Gamma(T^p_q M) \to \Gamma(T^r_\ell M)$ local operator. TFAE:

1. P is a tensor: if $s_x = s'_x$ then $(Ps)_x = (Ps')_{x}$.
2. P is $C^\infty(M)$-linear: $P(fs) = f P(s)$ $\forall f \in C^\infty(M)$.

<u>Pf</u> <u>$1 \Rightarrow 2$</u>: Given $f \in C^\infty(M)$, $s \in \Gamma(T^p_q M)$, define section $s' \in \Gamma(T^p_q M)$ by $s'_y = f(x) s_y$. Then $s'_x = (fs)_x$ so

$$(P(fs))_x = (P(s'))_x = \underset{\uparrow \text{ constant}}{f(x)}\, (P(s))_x .$$

<u>$2 \Rightarrow 1$</u>: for $x \in M$, $T^p_q M$ is "locally trivial": $\exists$ nbd U of x st.
$\pi^{-1}(U) = U \times \mathbb{R}^N$ $\quad (T^p_q M \xrightarrow{\pi} M)$.
Over U, $\exists$ sections $s_1, \ldots, s_N$ of $T^p_q M$ such that $s_1, \ldots, s_N$ generate $T^p_q M$ pointwise. Now suppose $s, s' \in \Gamma(T^p_q M)$ with $s_x = s'_x$.

Write $\quad s - s' = \sum_{i=1}^N f_i s_i \quad , \; f_i \in C^\infty(U), \; f_i(x) = 0 \; \forall i$.

Then $\quad P(s - s') = P(\sum f_i s_i) = \sum f_i P(s_i)$

So $\quad P(s - s')_x = \sum f_i(x)\, P(s_i)_x = 0$. $\quad \square$

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<u>Ex</u>. $\mathcal{L}_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$.

$$\mathcal{L}_X(fY) = [X, fY] \underset{\uparrow}{=} f[X,Y] + X(f)\, Y \neq f[X,Y] = f \mathcal{L}_X Y$$

$$[X, fY] g = X(fY(g)) - fY(X(g)) = X(f) Y(g) + f X(Y(g)) - f Y(X(g))$$
$$= (X(f) Y + f[X,Y])(g)$$

not a tensor!

Ex Given $\alpha \in \Omega^1 M = \Gamma(T^0_1 M)$, note this gives a map $\alpha(\cdot): \mathrm{Vect}(X) \to C^\infty(X)$ (a tensor). Now define a map

$$d\alpha : \mathrm{Vect}\, M \otimes \mathrm{Vect}\, M \to C^\infty(M)$$

by

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

This is a tensor in each input:

$$d\alpha(X, fY) = X\alpha(fY) - fY\alpha(X) - \alpha([X, fY])$$

$$= \underbrace{X(f\alpha(Y))}_{\cancel{X(f)\alpha(Y)} + fX\alpha(Y)} - f\,Y\alpha(X) - \underbrace{\alpha(f[X,Y] + (Xf)Y)}_{f\alpha[X,Y] + \cancel{(Xf)\alpha(Y)}}$$

$$= f\, d\alpha(X,Y).$$

So in fact $(d\alpha(X,Y))_x$ depends only on X_x and Y_x.

Over x this gives a map $T_xX \otimes T_xX \to \mathbb{R}$, so $d\alpha \in \Gamma(T^0_2 M)$.

$\leftrightarrow$ elt of $(T^*_xX)^{\otimes 2}$

Differential Forms

First: some linear algebra.

$V = $ v.s$/\mathbb{R}$. A k-multilinear form on V is a map

$$\varphi : \overbrace{V \otimes \cdots \otimes V}^{k} \to \mathbb{R}$$ that is linear in each input.

$$v_1 \otimes \cdots \otimes v_k \mapsto \varphi(v_1, -, v_k)$$

($k=2$: bilinear)

Note $\{k\text{-multilinear forms}\} \cong (\underbrace{V \otimes \cdots \otimes V}_{k})^* \cong \underbrace{V^* \otimes \cdots \otimes V^*}_{k}$

under this isom, if $\varphi_1, \dots, \varphi_k \in V^*$ then $(\varphi_1 \otimes \cdots \otimes \varphi_k)(v_1, \dots, v_k) = \varphi_1(v_1) \cdots \varphi_k(v_k)$.

Some forms are antisymmetric: $\varphi(\cdots v_i, v_{i+1}, \cdots) = -\varphi(\cdots v_{i+1}, v_i \cdots)$

so $\varphi(\cdots v_i \cdots v_j \cdots) = -\varphi(\cdots v_j \cdots v_i \cdots)$.

Claim: $\{\text{antisymm } k\text{-multilinear forms}\} \cong \Lambda^k V^*$.

Recall $W = $ v.s. $\rightsquigarrow$ $\Lambda^k W = W^{\otimes k}/I$, I gen'd by $\cdots \otimes w_1 \otimes w_2 \otimes \cdots + \cdots \otimes w_2 \otimes w_1 \otimes \cdots$

Consider the map

$$\mathrm{Alt} : W^{\otimes k} \longrightarrow W^{\otimes k}$$

$$\mathrm{Alt}(w_1 \otimes \cdots \otimes w_k) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

eg $\mathrm{Alt}(w_1 \otimes w_2) = w_1 \otimes w_2 - w_2 \otimes w_1$.

exercise: $I = \ker \mathrm{Alt}$ so $\Lambda^k W \cong \operatorname{Im} \mathrm{Alt}$:

we can think of elts of $\Lambda^k W$ as particular elts of $W^{\otimes k}$.

In particular: if $W = V^*$ then an element of $\Lambda^k W$ is an elt of image

$$\begin{array}{ccc} \otimes^k V^* & \xrightarrow{\mathrm{Alt}} & \otimes^k V^* \\ \shortparallel & & \shortparallel \\ (\otimes^k V)^* & & (\otimes^k V)^* \end{array}$$

: a k-multilinear map.

$k=2$: $\varphi_1, \varphi_2 \in V^* \rightsquigarrow \mathrm{Alt}(\varphi_1 \otimes \varphi_2) = (\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1)$.

note this is antisymmetric: $(\cdot\ -\ )(v_1, v_2) = -(\cdots)(v_2, v_1)$.

In general: $\operatorname{Im} \mathrm{Alt} = \{\text{antisymmetric } k\text{-multilinear maps}\}$

so $\Lambda^k V^* \cong$.

Wedge product

For $\omega \in \Lambda^k W \subset \otimes^k W$, $\eta \in \Lambda^l W \subset \otimes^l W$, define

$$\omega \wedge \eta = \frac{1}{k!\,l!} \mathrm{Alt}(\omega \otimes \eta) \in \Lambda^{k+l} W \subset \otimes^{k+l} W.$$

(note: weird factor is set up so that $\varphi_1 \wedge \cdots \wedge \varphi_k = \mathrm{Alt}(\varphi_1 \otimes \cdots \otimes \varphi_k)$ for $\varphi_i \in W$)

Properties:
- $\wedge$ is bilinear, associative
- $\wedge$ is graded commutative: $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

• $W = V^*$, $\varphi_1, \dots, \varphi_k \in V^* \Rightarrow \varphi_1 \wedge \cdots \wedge \varphi_k$ is the antisymmetric multilinear form

$$(\varphi_1 \wedge \cdots \wedge \varphi_k)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \varphi_{\sigma(1)}(v_1) \cdots \varphi_{\sigma(k)}(v_k).$$

If $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ is a vector bundle then we can define $\begin{array}{c} \Lambda^k E \\ \downarrow \\ M \end{array}$.

Def $\Omega^k(M) := \Gamma(\Lambda^k T^* M)$ <u>space of k-forms on M</u> ($\mathbb{R}$-v.s.; $C^\infty(M)$-module)

Locally a k-form looks like

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \underbrace{a_{i_1 \cdots i_k}(x_1, \dots, x_n)}_{\in C^\infty} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$\Omega^k(M)$ is nonzero for $0 \le k \le n$:

$$\Omega^0(M) = C^\infty(M)$$
$$\Omega^1(M) = \Gamma(T^*M)$$
$$\vdots$$

an element is locally $\leftarrow$ $\Omega^n(M)$

$a(x_1, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$;

<u>volume form</u> if $a \neq 0 \ \forall x_1, \dots, x_n$.

Def $\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$; then $\wedge$ gives $\Omega^*(M)$ the structure of a <u>graded-commutative ring</u>.

A k-form at a point is an antisymm. k-linear form on tangent vectors.

$\omega \in \Omega^k(M)$, $v_1, \dots, v_k \in T_x M \Rightarrow \omega(v_1, \dots, v_k) \in \mathbb{R}$.

A k-form acts on k <u>vector fields</u> to give a function

$\mathrm{Vect}(M)^{\otimes k} \to C^\infty(M)$, tensor in each input.

<u>ex</u>: Coords $x_1, \dots$, $\omega = dx_1 \wedge dx_2$.

$$\omega\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = (dx_1 \otimes dx_2 - dx_2 \otimes dx_1)\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 1 - 0 = 1$$

$$\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \qquad = 0 - 1 = -1$$

Prop (see HW) $\varphi^*(\omega\wedge\eta) = \varphi^*\omega \wedge \varphi^*\eta$

$$\mathcal{L}_X(\omega\wedge\eta) = (\mathcal{L}_X\omega)\wedge\eta + \omega\wedge(\mathcal{L}_X\eta).$$

Exterior derivative

$f\in C^\infty(M) \rightsquigarrow df\in\Omega^1(M)$ defined by $df(X) = X(f)$:

in coordinates, $\quad df = \sum \frac{\partial f}{\partial x_i}\,dx_i$.

(note in particular if $f(x_1,\dots,x_n) = x_i$ then $df = dx_i$: explains "dx_i" notation)

Thm $\exists!$ operator $d : \Omega^k M \to \Omega^{k+1} M$ determined by:

1. for $f\in C^\infty(M)$, $\quad df(X) = X(f)$
2. $d(df) = 0$
3. for $\omega\in\Omega^k M$, $\eta\in\Omega^\ell M$, $\quad d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k\omega\wedge(d\eta)$.

Furthermore, d is local and if $\omega = \sum a_{i_1\cdots i_k}\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ then

(*) $\quad d\omega = \sum (da_{i_1\cdots i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}$.

Rmks. 1. if d satisfies 1, 2, 3 then (*) must hold:

$$d(a_{i_1\cdots i_k}\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}) = (da_{i_1\cdots i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} + a_{i_1\cdots i_k}\,\underbrace{d(dx_{i_1}\wedge\cdots\wedge dx_{i_k})}_{0}.$$

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2. for $M\subset\mathbb{R}^n$, just check that (*) satisfies 1, 2, 3.

To extend to all M:

Lemma $\varphi : U_1\to U_2$, $U_1, U_2\subset\mathbb{R}^n$. Then $\varphi^*\circ d = d\circ\varphi^*$:

$$\begin{tikzcd} \Omega^k(U_2) \arrow[r, "\varphi^*"] \arrow[d, "d"'] & \Omega^k(U_1) \arrow[d, "d"] \\ \Omega^{k+1}(U_2) \arrow[r, "\varphi^*"] & \Omega^{k+1}(U_1) \end{tikzcd}$$

commutes.

Pf $k=0$: this is chain rule.

in general: use $\varphi^*(\omega\wedge\eta) = \varphi^*\omega \wedge \varphi^*\eta$ and induction on k. $\square$

Pf of Thm Atlas $\{(F_i, U_i, V_i)\}$ for M, $\omega \in \Omega^k M$.

On U_i, ω is given by $\omega_i \in \Omega^k U_i$: i.e., $\omega_i = F_i^*\omega$.

$$\begin{tikzcd} & M & \\ U_i \arrow[ur, "F_i"] \arrow[rr, "F_j^{-1}\circ F_i"'] & & U_j \arrow[ul, "F_j"'] \end{tikzcd}$$

The collection $\{\omega_i\}$ agrees on overlaps: i.e.,

$(F_j^{-1}\circ F_i)^*\omega_j = \omega_i$. $\quad$ $((F_j^{-1}\circ F_i)^* = F_i^*(F_j^{-1})^*)$.
Chain Rule

Conversely, a collection $\{\omega_i\}$ that agrees on overlaps gives $\omega \in \Omega^k M$.

But then $\quad (F_j^{-1}\circ F_i)^* d\omega_j = d(F_j^{-1}\circ F_i)^*\omega_j = d\omega_i$

so $\{d\omega_i\}$ agrees on overlaps and give a well-defined $(k+1)$-form $d\omega$. $\square$

Prop
- $d^2 = 0$ $\quad\to$ check locally
- $d\varphi^* = \varphi^* d$ for any smooth map $\varphi: M \to N$
- $\mathcal{L}_X d = d\mathcal{L}_X$. $\quad\to$ differentiate previous result

Cartan's (magic) formula

For $X \in \mathrm{Vect}(M)$, define the interior product

$$i_X : \Omega^k(M) \to \Omega^{k-1}(M) \qquad (\text{sometimes written } X \lrcorner)$$

by
$$i_X\omega = C_{11}(X\otimes\omega)$$
$\uparrow$ viewed as a $(0,k)$ tensor

i.e.
$$i_X\omega(X_1,\dots,X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}).$$

Thm On $\Omega^*(M)$, $\quad \mathcal{L}_X = i_X d + d i_X$.

Pf HW.

Coord-free formula for d.

Prop $\omega \in \Omega^k(M)$, $X_0, \ldots, X_k \in \mathrm{Vect}\, M$. Then

$$(d\omega)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i\left(\alpha(X_0 \cdots \hat{X}_i \cdots X_k)\right) + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots \hat{X}_i \cdots \hat{X}_j \cdots, X_k).$$

Pf HW.

ex: $k=1$: $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$:
we saw this already.

Quick application of Cartan:

Def $\omega \in \Omega^n M$ is a **volume form** if $\forall x \in M$, $\omega_x \in \Lambda^n T^* M \cong \mathbb{R}$ is nonzero.

Given ω = vol form, any elt of $\Omega^n(M)$ can be written(!) as $f\omega$, $f \in C^\infty(M)$.

$X \in \mathrm{Vect}(M) \Rightarrow \quad d(i_X \omega) \in \Omega^n(M)$.

Def The **divergence of X**, $\mathrm{div}\, X \in C^\infty(M)$, is defined by $d(i_X\omega) = (\mathrm{div}\, X)\,\omega$.

(HW: in $\mathbb{R}^n$ this is the usual divergence.)

Prop $\mathrm{div}\, X = 0 \Leftrightarrow X$ is volume-preserving: $\mathcal{L}_X \omega = 0$.