

Minicourse - Morse Theory and Floer Homology

Note Title

3/12/2015

Three topics:

- Morse theory (useful for topology, diff geom, alg geom, ...)
 - Morse homology (useful for persistent homology)
 - Floer homology (∞ -dim version of Morse homology, very important in topology, symplectic geometry, math physics, ...)
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Setting: $M = \text{smooth manifold}$, often compact, dimension n .
 $f: M \rightarrow \mathbb{R}$ smooth.

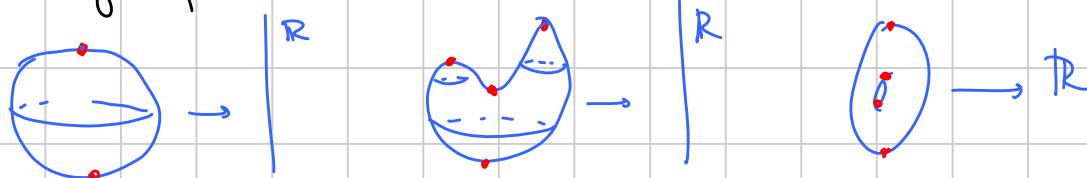
Def A critical point of f is a point $p \in M$ with $(df)_p = 0$.
in coords, $\frac{\partial f}{\partial x_i} = 0 \quad i=1, \dots, n$.

A critical point is nondegenerate if the Hessian $(Hf)_p$ is nondegenerate.

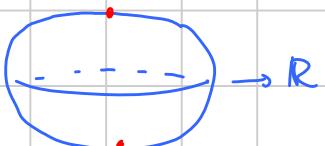
in coords, $\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \neq 0$.

Def f is a Morse function if all critical points are nondegenerate.

Ex: height function on a surface.



Not Morse:



$$\text{say, } z = x^4 + y^2.$$

Existence of Morse Functions

Thus There are Morse functions on any smooth M .

Outline of proof:

Lemma $U \subset \mathbb{R}^n$ open, $f: U \rightarrow \mathbb{R}$. For almost all $(a_1, \dots, a_n) \in \mathbb{R}^n$,

$$f_a(x) = f(x) - \sum_{j=1}^n x_j a_j$$

is Morse.

PF Define $g: U \rightarrow \mathbb{R}^n$, $g(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$.

images of critical pts: $x \in U$ with dg_x not surjective.

Sard's Thm: the set of critical values of g has measure 0.

Now suppose a is a regular value (not a critical value) for g . (true for almost all a)

If $x \in U$ is a crit pt of f_a , then $g(x) = a$.

Then by assumption, $(Hf_a)_x = dg_x$ is surjective \Rightarrow invertible. \square

Now use:

Whitney embedding theorem \exists embedding $M \hookrightarrow \mathbb{R}^N$ for some N (in fact, $N = 2n$).

Embed $M \hookrightarrow \mathbb{R}^N$.

Prop For almost all $a \in \mathbb{R}^N$, the function $f: M \xrightarrow{C(\mathbb{R}^N)} \mathbb{R}$,
 $f(x) = a_1 x_1 + \dots + a_N x_N$ is Morse.

Pf Near any $x \in M$, we can choose $J = (j_1, \dots, j_n)$ so that
 $(x_{j_1}, \dots, x_{j_n})$ form local coordinates for M . ($T_x^* \mathbb{R}^N \xrightarrow{\text{surjective}} T_x^* M$)

So we can cover M by open sets $U_J = \{x \in M \mid J \text{ gives local coords}\}$.

Let's consider $J = (1, \dots, n)$: x_1, \dots, x_n give local coords.

Fix $(a_{n+1}, \dots, a_N) \in \mathbb{R}^{N-n}$ and define $f: U_J \rightarrow \mathbb{R}$,

$$f(x) = x_{n+1} a_{n+1} + \dots + x_N a_N.$$

By the lemma, for almost all $(a_1, \dots, a_n) \in \mathbb{R}^n$,

$$f_a(x) = f(x) + x_1 a_1 + \dots + x_n a_n = \sum_{j=1}^n x_j a_j$$

is Morse. So for almost all $(a_1, \dots, a_N) \in \mathbb{R}^N$, $\sum_{j=1}^n x_j a_j$ is Morse on U_J .

There are finitely many $U_J \Rightarrow \sum_{j=1}^n x_j a_j$ is Morse on M for almost all a .

□

This can be easily improved to:

Thm Any function $f: M \rightarrow \mathbb{R}$ can be C^k approximated
(for any k , on compact subsets) by a Morse function.

Morse difficult, using transversality:

Thm M compact. Then $\{\text{Morse fns on } M\}$ is open and dense
in $C^\infty(M)$.

$f: M \rightarrow \mathbb{R}$ Morse, p = critical point.

Def The Morse index of f at p is

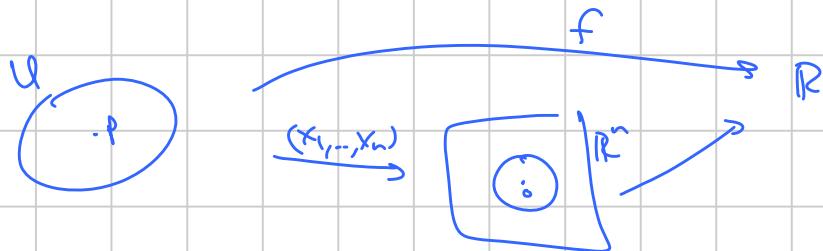
$$\text{ind}_f p = \#(\text{negative eigenvalues of } H_f(p)).$$

Ex. $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $f(x) = c - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$.

$$Hf = \begin{bmatrix} -1 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}. \quad \text{ind}_f 0 = k.$$

Morse Lemma p = critical pt of f of index k . Then \exists $\text{ndd } U$ of p and local coord (x_1, \dots, x_n) on U st. $p = (0, \dots, 0)$ and on U ,

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$



Use:

Hadamard's Lemma $f: U \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}$ smooth, $f(0)=0$. Then \exists

smooth $g_1, \dots, g_n: U \rightarrow \mathbb{R}$ w.s.t

$$f(x_1, \dots, x_n) = \sum_{j=1}^n x_j g(x_1, \dots, x_n)$$

and $g_j(0) = \frac{\partial f}{\partial x_j}(0)$.

$$\text{PF } f(x_1, \dots, x_n) = \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_n) dt = \int_0^1 \sum_{j=1}^n \frac{\partial f}{\partial x_j}(tx_1, \dots, tx_n) dt$$

$$\text{so set } g_j(x_1, \dots, x_n) = \int_0^1 \frac{\partial f}{\partial x_j}(tx_1, \dots, tx_n) dt. \quad \square$$

Pf of Morse Lemma

Can assume $f: U \rightarrow \mathbb{R}$, $p=0$, $f(p)=0$.

Hadamard $\Rightarrow f(x) = \sum_j x_j g_j(x)$, and $g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0$.

Hadamard again $\Rightarrow g_j(x) = \sum_i x_i h_{ij}(x)$

$$\Rightarrow f(x) = \sum_{i,j} x_i x_j h_{ij}(x) \quad h_{ij}: U \rightarrow \mathbb{R} \text{ smooth.}$$

Can assume $h_{ij}(x) = h_{ji}(x)$ (just replace both by $\frac{1}{2}(h_{ij} + h_{ji})$), so $H(x) = (h_{ij}(x))$ is symmetric. Also note $H(0)$ is nonsingular.

Now apply diagonalization of symmetric matrices: Suppose for $r \geq 1$ we can find coords u_1, \dots, u_n on U st.

$$f = \pm u_1^2 \pm \dots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \dots, u_r)$$

for some symm. (H_{ij}) . We claim that (after shrinking U) we can do the same for $r+1$.

Apply linear change to u_{r+1}, \dots, u_n to get $H_{rr}(0) \neq 0$.

Let $g(u_1, \dots, u_n) = \sqrt{H_{rr}(u_1, \dots, u_n)}$ (might need to shrink U).

Define new coords v_1, \dots, v_n , $v_i = u_i \quad i \neq r$,

$$v_r(u_1, \dots, u_n) = g(u_1, \dots, u_n) \left(u_r + \sum_{i > r} u_i \frac{H_{ir}(u_1, \dots, u_n)}{H_{rr}(u_1, \dots, u_n)} \right)$$

Inverse Fn Thm: near 0, these are coord fns.

These have been designed to satisfy:

$$f = \pm v_1^2 \pm \dots \pm v_{r-1}^2 \pm v_r^2 + \sum_{i,j > r} v_i v_j \tilde{H}_{ij}(v_1, \dots, v_n).$$

Finally: # $f \pm$ are determined by index of f at p . \square

Car The critical points of a Morse function are isolated.

Dim 2:



ind 1

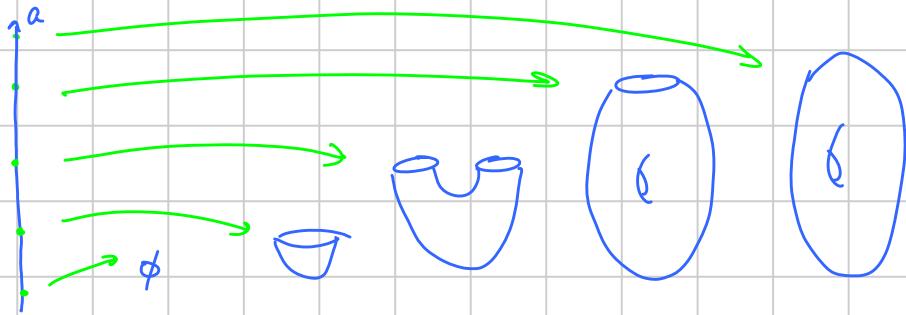
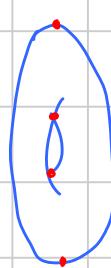
ind 2

Sublevel Sets

M connected, $f: M \rightarrow \mathbb{R}$ Morse, $a \in \mathbb{R}$.

Def The Sublevel set $M^a = f^{-1}((-\infty, a])$. (mf'd with ∂ if $a = \text{regular value}$)
 f is exhaustive if M^a is compact for all $a \in \mathbb{R}$.

Ex:



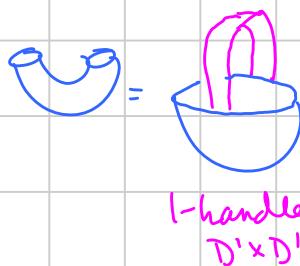
Idea: Study how the topology of M^a changes as a changes.

In fact: say $a < b$.

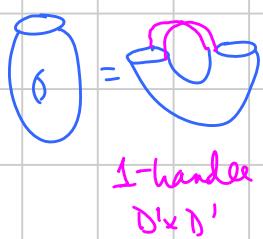
- If no crit values in $[a, b]$, then M^a, M^b are difeo.
- If one crit value of index k , then $M^b = M^a \cup (k\text{-handle})$.
 $k\text{-handle} = D^k \times D^{n-k}$; glue to M^a by $(\partial D^k) \times D^{n-k} \hookrightarrow \partial M^a$.

Ex:

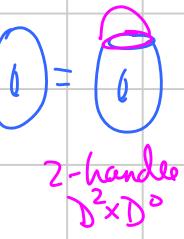
0-handle
 $D^0 \times D^2$



1-handle
 $D^1 \times D^1$



1-handle
 $D^1 \times D^1$

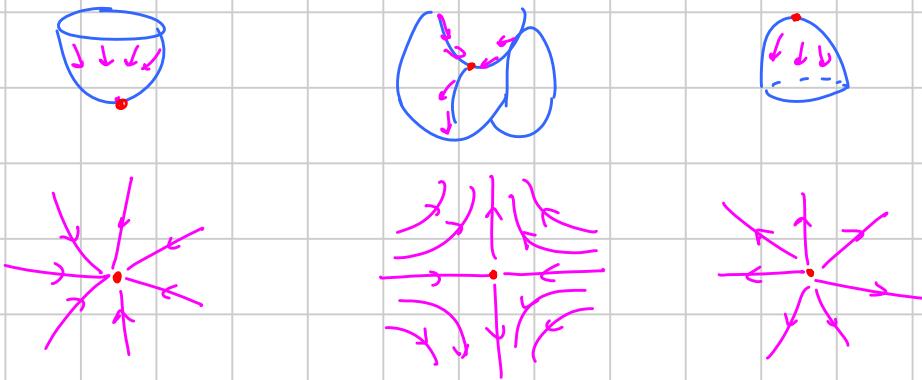


2-handle
 $D^2 \times D^0$

How to show this? Gradient flow.

$(M, <, >)$ Riemannian, $f: M \rightarrow \mathbb{R}$. The gradient vector field ∇f is defined by $\langle \nabla_x f, v \rangle = df(v)$, $v \in T_x M$.

Traditional to use negative gradient flow: the flow of $-\nabla f$.



Note: at a critical pt p , $-\nabla_p f = 0$: nearby,
 $-\nabla_x f = -\left(x_1 \frac{\partial}{\partial x_1} - \dots - x_k \frac{\partial}{\partial x_k} + x_{k+1} \frac{\partial}{\partial x_{k+1}} + \dots + x_n \frac{\partial}{\partial x_n}\right)$.

Def $x \in M$. The gradient flow line $\gamma_x(t)$ is the flow of $-\nabla f$:

$$\gamma_x(0) = x$$

$$\frac{d}{dt} \gamma_x(t) = -\nabla_{\gamma_x(t)} f.$$

Prop M compact. Every flowline begins and ends at a crit pt of f : $\exists p, q$ crit pts with

$$\lim_{t \rightarrow -\infty} \gamma_x(t) = p, \quad \lim_{t \rightarrow \infty} \gamma_x(t) = q.$$

If $\gamma_x(t)$ is defined $\forall t$ and

$$\begin{aligned} \frac{d}{dt} f(\gamma_x(t)) &= df_{\gamma_x(t)}(-\nabla_{\gamma_x(t)} f) \\ &= -\|\nabla_{\gamma_x(t)} f\|^2. \end{aligned}$$

Since $\{f(\gamma_x(t))\}_{t \in \mathbb{R}}$ is bounded, $\lim_{t \rightarrow \pm\infty} \frac{d}{dt} f(\gamma_x(t)) = 0$
 $\Rightarrow \|\nabla_{\gamma_x(t)} f\| \rightarrow 0$. If $t_n \rightarrow \infty$ then $\{\gamma_x(t_n)\}$ has a limit pt q , which must then be a crit pt of f . Since crit pts are isolated, easy to check that $\lim_{t \rightarrow \infty} \gamma_x(t) = q$. Similarly for $t \rightarrow -\infty$. \square

Define φ_t = time t flow for $-\nabla f$: $\varphi_t: M \rightarrow M$, $\varphi_t(x) = \gamma_x(t)$.

Then φ_t is a diffeo for each t . ($\varphi_t \circ \varphi_{-t} = \text{id}$).

From thm. of

Morse Thm. I If f is Morse (and $f^{-1}([a,b])$ is compact) and has no critical values in $[a,b]$, then $M^a \cong M^b$ (diffeo).

Pf Define $p: M \rightarrow \mathbb{R}$ by

$$p(x) = \begin{cases} -\frac{1}{\|\nabla_x f\|^2} & x \in f^{-1}([a,b]) \\ 0 & \text{outside cpt set containing } f^{-1}([a,b]). \end{cases}$$

Write $\psi_t: M \rightarrow M$ for time t flow of $p \nabla f$: then

$$\frac{d}{dt} f(\psi_t x) = -1$$

so $f(\psi_t x) = -t + f(x)$. Thus ψ_{b-a} sends M^b onto M^a . \square

(in fact M^b deformation retracts onto M^a).

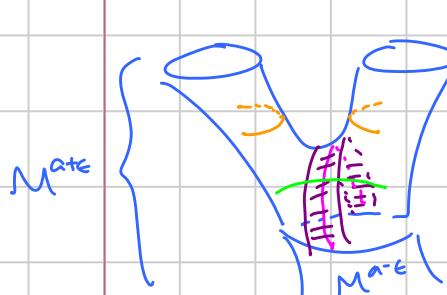
From Thm. of Morse Thm. 2 Since $f^{-1}([a-\epsilon, a+\epsilon])$ is cpt and contains one critical point p of index k . Then

$$M^{a+\epsilon} \underset{\text{homotopic}}{\sim} M^{a-\epsilon} \cup (k-\text{cell}). \quad \hookleftarrow D^k$$

In fact: $M^{a+\epsilon} \underset{\text{diffeo}}{\approx} M^{a-\epsilon} \cup (\text{k-handle})$.

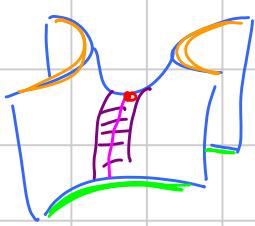
$\approx S^k \times D^{n-k}$, glued along $\partial D^k \times D^{n-k}$

Outline of pf. Restrict to a Morse chart around p.

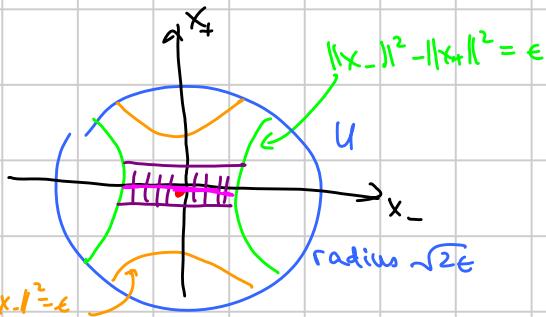


$a+\epsilon$
a
 $a-\epsilon$

Use gradient flow to retract $M^{a+\epsilon}$ onto $M^{a-\epsilon} \cup \text{k-cell}$.



$a+\epsilon$
a
 $a-\epsilon$



$$||x_+||^2 + ||x_-||^2 = \epsilon$$

radius $\sqrt{2\epsilon}$

By shrinking ϵ if necessary, suppose the Morse chart is $B(0, \sqrt{2\epsilon}) = U$, on which $f = a - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$, $x_- = (x_1, \dots, x_k)$, $x_+ = (x_{k+1}, \dots, x_n)$.
 $= a - \|x_-\|^2 + \|x_+\|^2$.

Define

$$D^k = \{ x_+ = 0, \|x_-\|^2 < \epsilon \}.$$

Claim:

$$M^{a+\epsilon} \sim M^{a-\epsilon} \cup D^k.$$



More precisely: define $\mu: [0, \infty) \rightarrow [0, \infty)$ by

$$(\text{need } -1 \leq \mu'(t) \leq 0)$$

and then define $F: M \rightarrow \mathbb{R}$ by

$$F = \begin{cases} f & \text{outside } U \\ f - \mu(\|x_-\|^2 + 2\|x_+\|^2) & \text{inside } U. \end{cases}$$



Straightforward to check:

- $F = f$ except in nbhd of p
- F has the same critical pts as f :

Write $s = \|x_+\|^2$, $t = \|x_-\|^2$. Then $F = a + s - t - \mu(t+2s)$

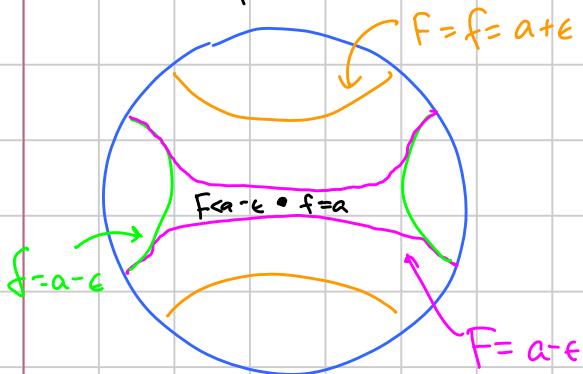
$$\Rightarrow \frac{\partial F}{\partial s} = 1 - 2\mu'(t+2s) \geq 1$$

$$\frac{\partial F}{\partial t} = -1 - \mu'(t+2s) < 0$$

$$\text{so } dF = \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial t} dt = 0 \Rightarrow ds = dt = 0 \Rightarrow x_+ = x_- = 0$$

\Rightarrow in U , only critical pt of F is p .

- $F(p) = a - \epsilon$.



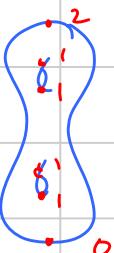
- $F^{-1}(a+\epsilon) = f^{-1}(a+\epsilon)$.

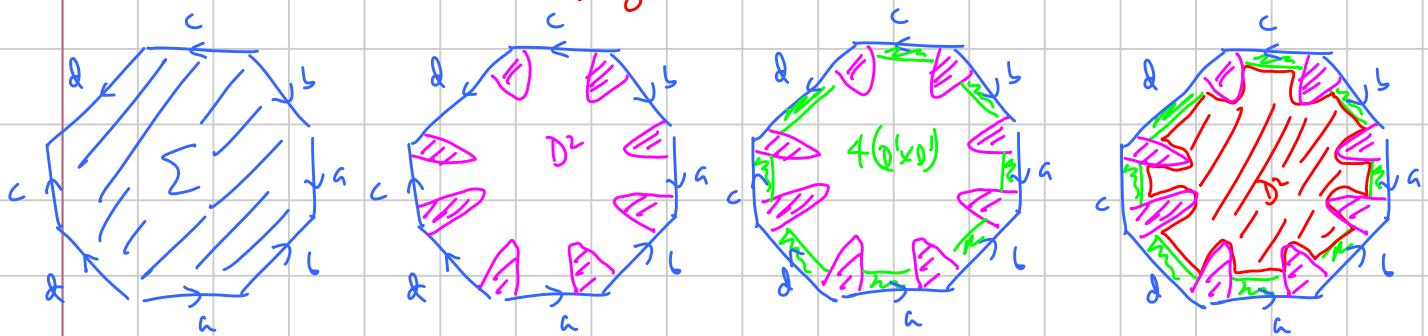


Then F has no critical values in $[a-\epsilon, a+\epsilon]$, so by previous prop, $\{F < a+\epsilon\} = M^{a+\epsilon}$ is diffeo to $\{F < a-\epsilon\}$.

Now $\{F < a-\epsilon\} = \{f < a-\epsilon\} \cup \overset{M^{a-\epsilon}}{\text{=====}}$

and one can construct a def. retract to $M^{a-\epsilon} \cup (k\text{-cell})$ by hand. \square

Another ex: $\Sigma =$  $= (0\text{-handles}) \cup 4(1\text{-handles}) \cup (2\text{-handles}).$

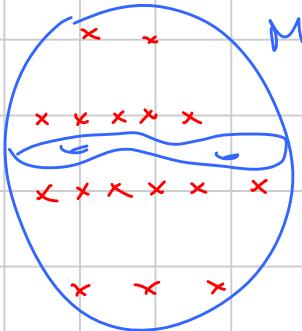


Application to 3-mfds:

Def A Morse function $f: M \rightarrow \mathbb{R}$ is self-indexing if each critical pt p satisfies $f(p) = \text{ind}(p)$.

Prop (Smale) M cpt. Then \exists self-indexing Morse fn. on M .
(later)

So: Suppose M^3 cpt 3-mfd. Let $f: M \rightarrow \mathbb{R}$ be self-indexing.

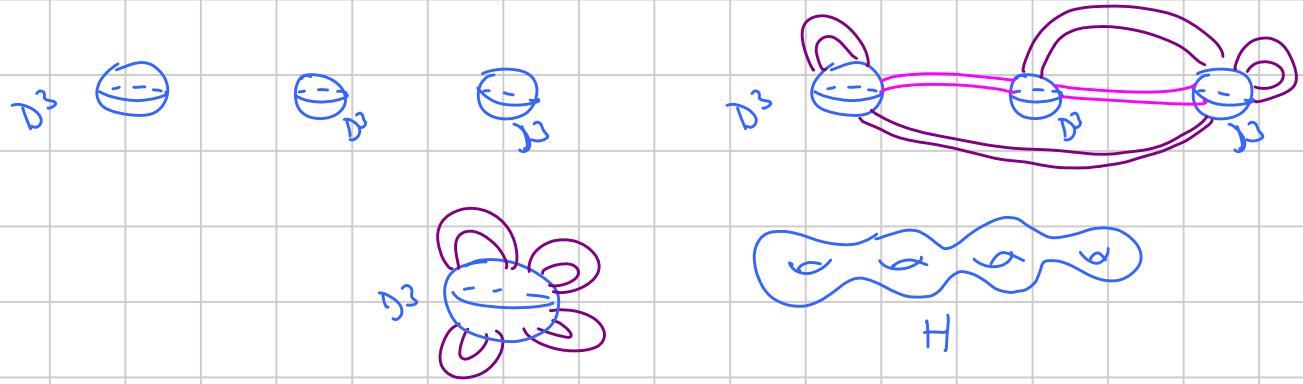


$$f \rightarrow$$

note $f^{-1}([3]) = \text{Smooth Surface}$.



Look at $\{f \leq \frac{3}{2}\} =: H$. This is $(0\text{-handles}) \cup (1\text{-handles})$:



If H is orientable, then it only depends (up to diffeo) on the # of dark purple 1-handles: it's a handlebody with g handles.

Prop Any smooth compact connected orientable 3-mfd M has a Heegaard decomposition:

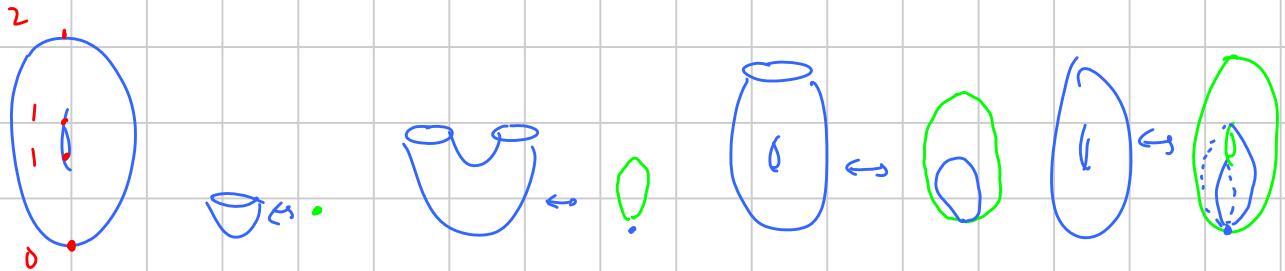
$$M = H_1 \cup H_2$$

↑ glued along $\partial H_1 = \partial H_2 =$ surface of genus g .

Pf $\{f \leq \frac{3}{2}\}$, $\{f \geq \frac{3}{2}\}$ are handlebodies. \square

Car of Fund Thm:

Thm $f: M \rightarrow \mathbb{R}$ Morse, exhausting. Then M has the homotopy type of a CW complex with one cell of dim k for each critical pt of index k .

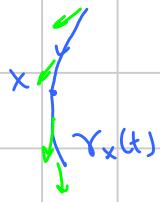


More on gradient flows.

$$M = \mathbb{C}^n -$$

Recall $\varphi_t = \text{time } t \text{ flow for } -\nabla f$: $\varphi_t(x) = \gamma_x(t)$.

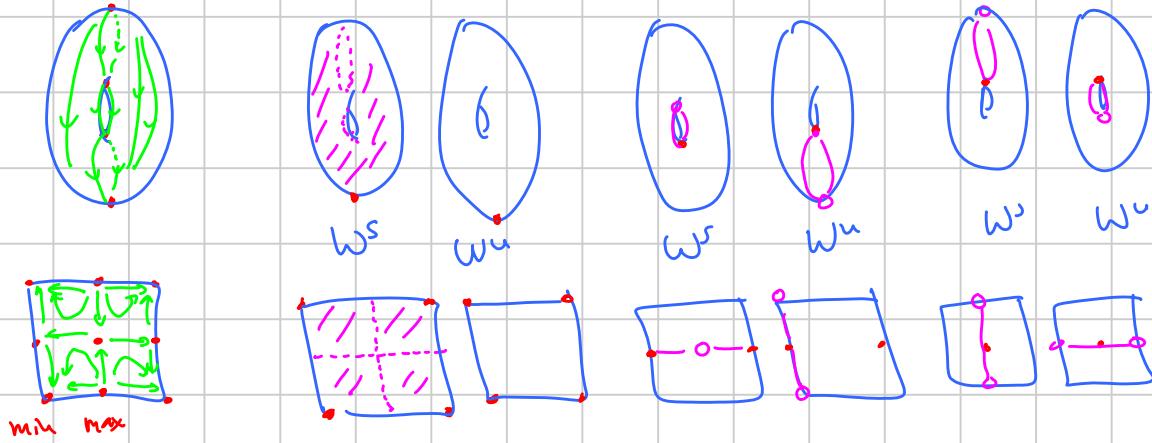
Also $\lim_{t \rightarrow \infty} \varphi_t(x)$ are critical pts of f .



Def p = critical point of f . The stable/unstable manifolds of p are

$$W^s(p) = \{x \in M \mid \lim_{t \rightarrow \infty} \varphi_t(x) = p\}$$

$$W^u(p) = \{x \in M \mid \lim_{t \rightarrow -\infty} \varphi_t(x) = p\}.$$



Note $\bigcup_p W^s(p) = \bigcup_p W^u(p) = M$. (and all W^s, W^u are disjoint).

Then $W^s(p), W^u(p)$ are smoothly embedded disks of dim $n-k, k$ respectively. (i.e. $W^s(p) \cong \mathbb{R}^{n-k}, W^u(p) \cong \mathbb{R}^k$).

Idea of pf: Write $S_\epsilon^u(p) = W^u(p) \cap \{f = f(p) - \epsilon\}$, and use Morse lemma. Then $S_\epsilon^u(p) \cong S^{k-1}$.



Every nonconstant gradient trajectory from p must pass through $S^u_{\epsilon}(p)$ exactly once $\Rightarrow W^u(p) = p \cup (S^{k-1} \times R) = D^k$.

More precisely: Let (r, θ) be polar coordinates on R^k , $r \in R^+$, $\theta \in S^{k-1}$.

View $\theta \in S^u_{\epsilon}(p)$, and define $F: R^k \rightarrow S^u(p)$

$$F(r, \theta) = \theta \log_r(\theta) = \gamma_\theta(\log r).$$

This is a diffeomorphism.

Same argument upside down for $W^s(p)$. \square

Note: $W^u(p)$, $W^s(p)$ intersect at p , and they intersect transversely: $T_p W^u(p) = T_p^{u^*} M$, $T_p W^s(p) = T_p^{s^*} M$

where $T_p M = T_p^{u^*} M \oplus T_p^{s^*} M$.

Hf_p is $-$ definite Hf_p is $+$ definite.

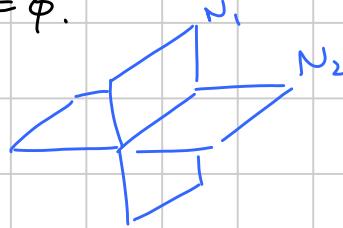
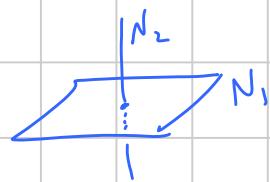
Def $N_1, N_2 \subset M$ submanfs. N_1 intersects N_2 transversely, written $N_1 \pitchfork N_2$, if $\forall x \in N_1 \cap N_2$,

$$T_x N_1 + T_x N_2 = T_x M.$$

(alt: $(n - \dim T_x N_1) + (n - \dim T_x N_2) = (n - \dim (T_x N_1 \cap T_x N_2))$)

Note: in particular, if $\dim N_1 + \dim N_2 < n$, then

$$N_1 \pitchfork N_2 \Leftrightarrow N_1 \cap N_2 = \emptyset.$$



Prop If $N_1 \pitchfork N_2$ then $N_1 \cap N_2$ is a smooth submanifold of dimension $\dim N_1 + \dim N_2 - n$.

(follows from: inverse image of a regular value is a smooth submfld).

Note gradient flow \Rightarrow stable/unstable mfd's depend on the metric.

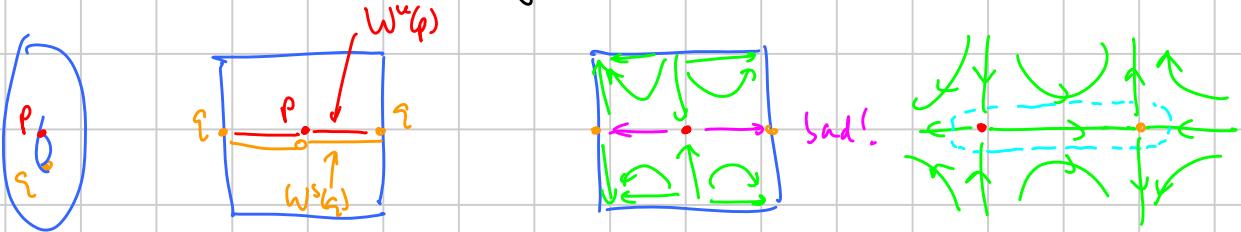
Def A pair (f, g) on M , $f: M \rightarrow \mathbb{R}$, g = metric, is Morse-Smale if

- f = Morse
- $\forall p, q$ critical pts of f ,
 $W^u(p) \pitchfork W^s(q)$.

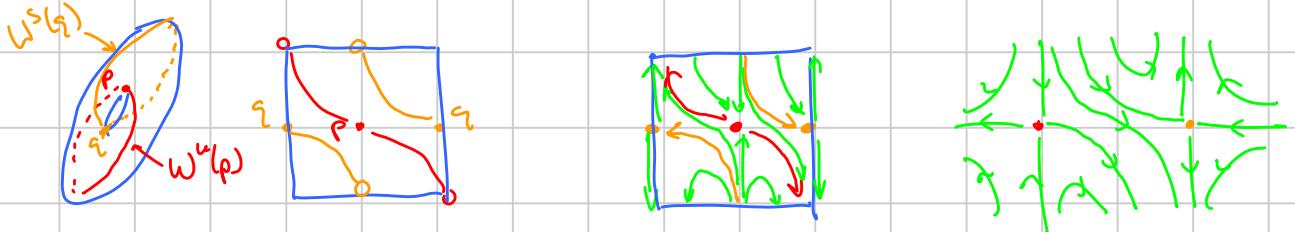
Then $W^u(p) \cap W^s(q)$ is a submanifold of dim
 $\text{ind}(p) + (n - \text{ind}(q)) - n = \text{int}(p) - \text{int}(q)$.

Note: always true when $p = q$; trivially true if $f(p) < f(q)$.

With the standard metric: height function on T^2 is not Morse-Smale.



But perturb slightly:



Then (Smale, Kupka) ¹⁹⁶³ The set of Morse-Smale pairs (f, g) is dense in $C^\infty(M) \times \text{Met}(M)$.

So in what follows, we'll start with a Morse-Smale pair.

One consequence:

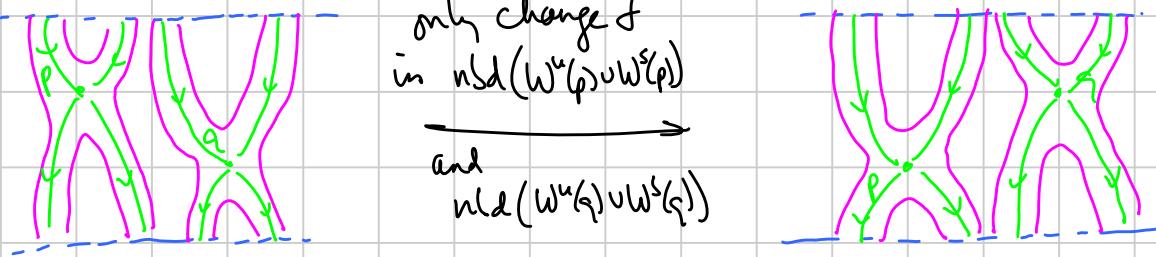
Prop Any compact M has a self-indexing Morse function.

The key:

Rearrangement lemma (f, g) Morse-Smale, critical points p, q with $a < f(q) < f(p) < b$, p, q only crit pts in $f^{-1}([a, b])$,
 $W^u(p) \cap W^s(q) \cap f^{-1}([a, b]) = \emptyset$.

Then for any $a', b' \in [a, b]$, $\exists \tilde{f}$ st.

- (\tilde{f}, g) is Morse-Smale
- $\tilde{f} = f$ outside $f^{-1}([a, b])$
- p, q only crit pts of \tilde{f} in $f^{-1}([a, b])$
- $g(p) = a'$, $g(q) = b'$.



Then by Morse-Smale Condition, whenever $\text{ind}(p) > \text{ind}(q)$ but $f(p) \leq f(q)$, we have $W^s(p) \cap W^u(q) = \emptyset$ so we can change f to swap $f(p)$ and $f(q)$.

⇒ self-indexing.

Toward Morse Homology

(f, g) Morse-Smale, $p, q \in \text{Crit } f$. Define

$$\begin{aligned} W(p, q) &= W^u(p) \cap W^s(q) \\ &= \left\{ x \in M : \lim_{t \rightarrow -\infty} \gamma_x(t) = p, \lim_{t \rightarrow \infty} \gamma_x(t) = q \right\}. \end{aligned}$$

Note: \mathbb{R} acts on $W(p, q)$ by time translation :

$$s \cdot x = \gamma_s(x) = \gamma_x(s).$$

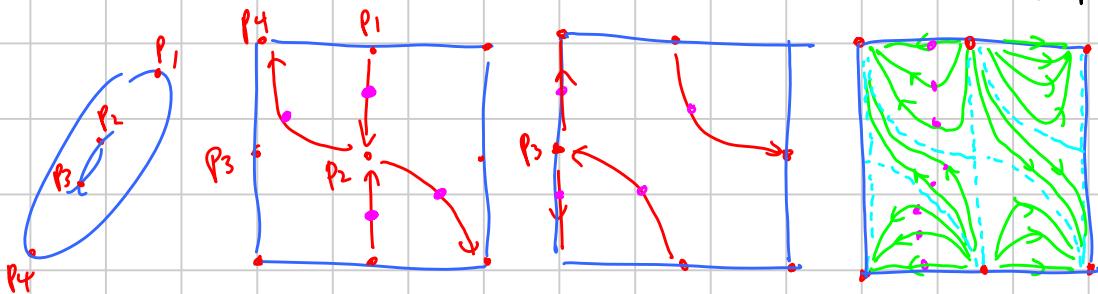
times { $s \cdot x$ } $s \mapsto \gamma_s$.

Furthermore, if $p \neq q$ then this action is free.

Def $M(p, q) = W(p, q) / \mathbb{R}$. (Moduli space of flow lines from p to q , mod translation).

This is a smooth mfd of dim $\text{ind}(p) - \text{ind}(q) - 1$. (except $p = q$)

Ex.



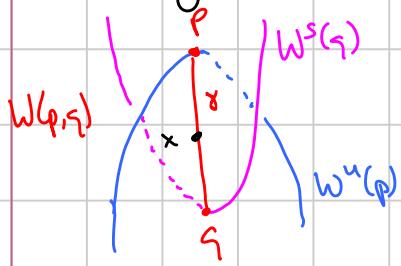
$$\begin{aligned} M(p_1, p_2) &= \{2\text{pt}\} & M(p_1, p_3) &= \{2\text{pt}\} & M(p_1, p_4) &= 4x \leftarrow \\ M(p_2, p_4) &= \{2\text{pt}\} & M(p_3, p_4) &= \{2\text{pt}\} \end{aligned}$$

Orientations

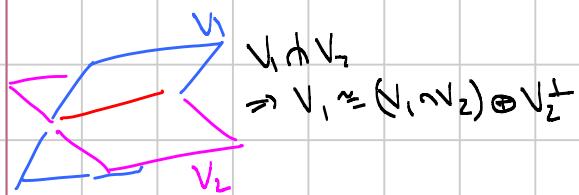
We can orient $M(p, q)$ in a "consistent" manner $\pitchfork p, q$ by

Choosing an orientation of $W^u(p)$ $\pitchfork p \in \text{Crit } f$, and then:

for $x \in W(p, q)$, $\gamma = \text{flowline thru } x \in M(p, q)$,



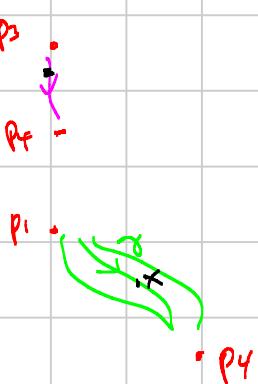
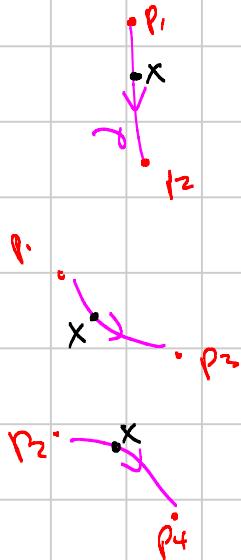
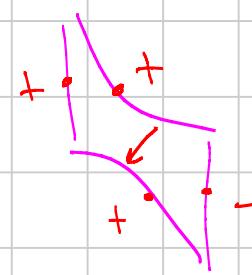
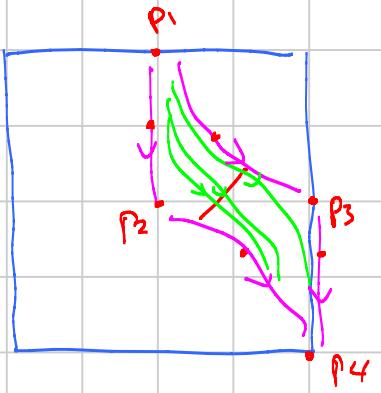
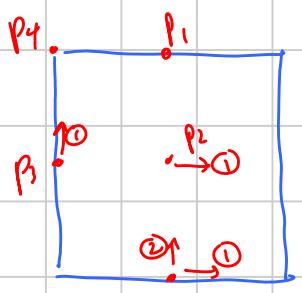
$$\begin{aligned} T_x W^u(p) &\cong T_x (W^u(p) \cap W^s(q)) \oplus (T_x W^s(q))^{\perp} \\ &\cong T_x M(p, q) \oplus T_x \gamma \oplus T_x W^u(q). \end{aligned}$$



$$(T_x W^s(q))^{\perp} \cong (T_x W^s(q))^{\perp} \cong T_x W^u(q)$$

Orient $M(p, q)$ so that this isomorphism is orient-preserving.

Note: doesn't require an orientation on M .



$$T_x W^u(p_1) \cong T_{\gamma} \mathcal{M}(p_1, p_2) \oplus T_x \gamma \oplus (T_x W^s(p_2))^{\perp}$$



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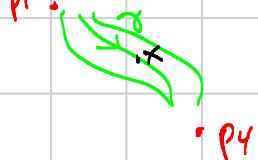
$T_x W^s(p_4)^{\perp}$

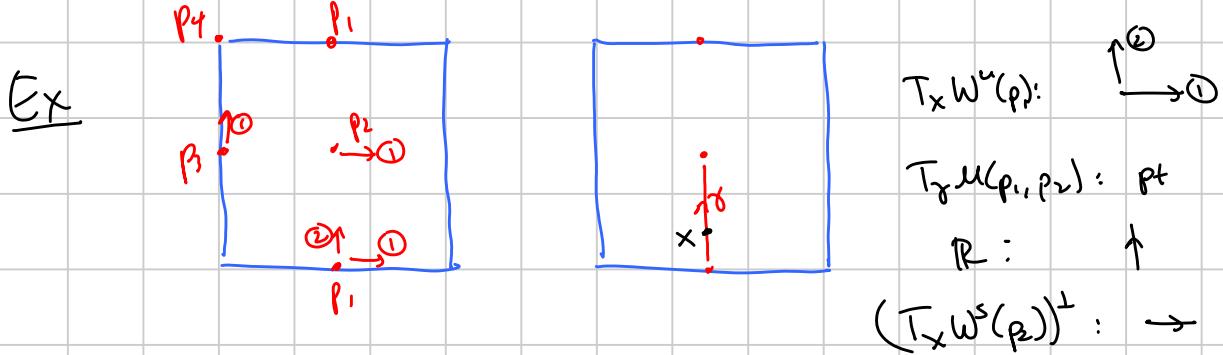
$T_x W^u(p_1)$

?

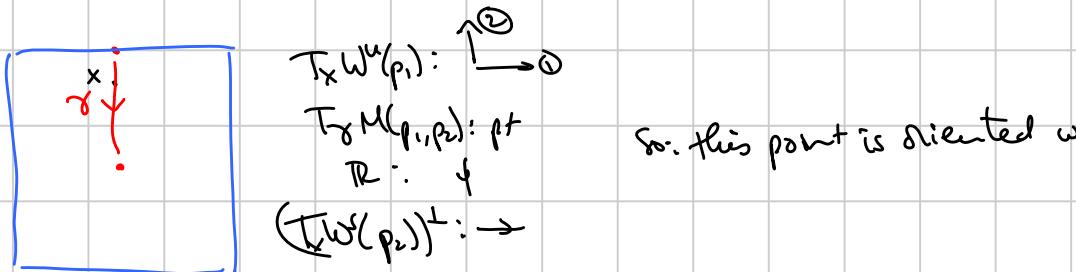
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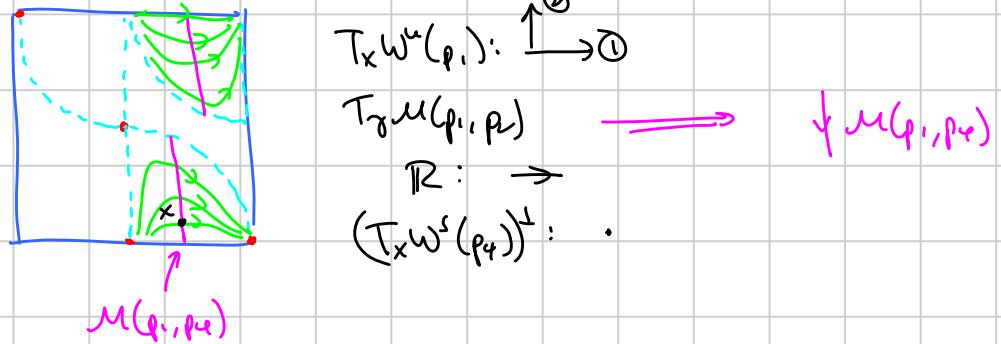




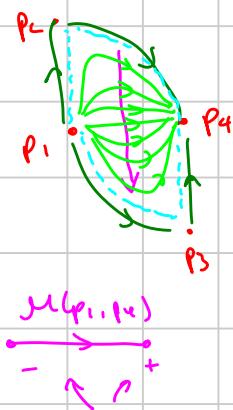
So: this point in $M(p_1, p_2)$ is oriented with -.



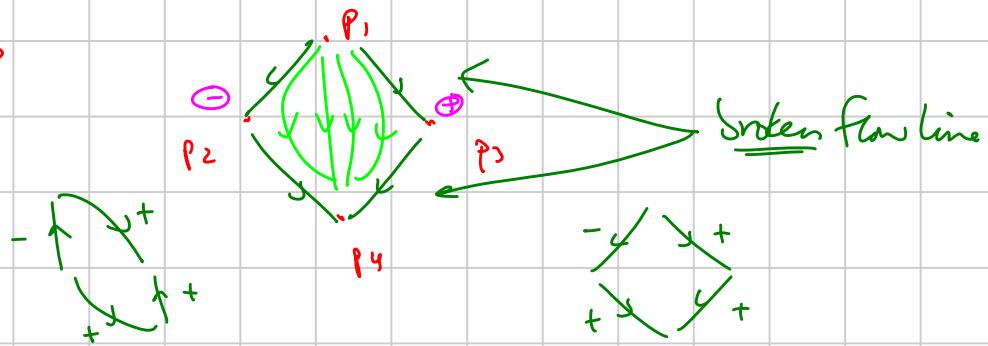
One more:



The key to Morse homology: compactification by broken flow lines.



$M(p_1, p_4) = \cup$ 4 open intervals; can compactify:
 $\bar{M}(p_1, p_4) = \cup$ 4 closed intervals.



Thm. 1. If $\text{ind } p = \text{ind } q + 1$ then
 $M(p, q)$ is compact (= finite union of points).

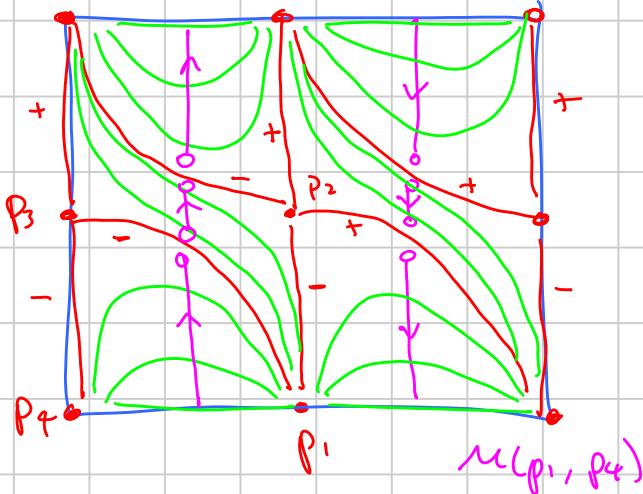
2. If $\text{ind } p = \text{ind } q + 2$ then $M(p, q)$ has a
 Compactification $\overline{M}(p, q) = \text{cpt 1-mfd with } \partial$ (= finite union
 of closed intervals
 and S^1 's)
 And
 $\partial \overline{M}(p, q) = \bigcup_{\text{ind } r = \text{ind } q + 1} M(p, r) \times M(r, q).$

(even true with orientations!)

(In fact, in general $\overline{M}(p, q)$ has a compactification to a mfd with
 corners whose strata are broken flow lines.)

Components of proof:

- ① Any sequence of flowlines in $M(p, q)$ has a subsequence converging to a broken flowline.
- ② Any broken flowline can be perturbed to an honest flowline: if the broken flowline has k intermediate crit pts, these perturbations are parametrized by $(\mathbb{R}^+)^k$.
- ③ Orientations.



Morse Complex

$M, (f, g)$ Morse-Smale $\Rightarrow (CM_*(M; f, g), \partial).$

Def $CM_k = \mathbb{Z} Crit_k(f)$

$Crit_k(f) = \{ \text{crit pt of index } k \}$

$\partial: CM_k \rightarrow CM_{k-1}$ defined by

$$\partial(p) = \sum_{q \in Crit_{k-1}(f)} \underbrace{(\# M(p, q))}_{{\in \mathbb{Z}} \text{.. # of pt in } M(p, q), \text{ counted with sign.}} q$$

This is the Morse complex of M associated to $(f, g).$

Prop $\partial^2 = 0.$

Pf. $p \in Crit_k(f), q \in Crit_{k-2}(f) \Rightarrow$

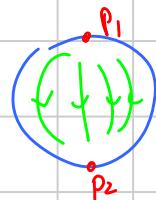
$$\begin{aligned} \langle \partial p, q \rangle &= \sum_{\substack{r \in Crit_{k-1}(f) \\ \text{coeff of } q \text{ in } \partial p}} \langle \partial p, r \rangle \langle \partial r, q \rangle \\ &= \# \bigcup_{r \in Crit_{k-1}(f)} M(p, r) \times M(r, q) \\ &= \# \partial \bar{M}(p, q) \\ &= 0. \quad \square \end{aligned}$$

Def The Morse homology $HM_*(M; f, g) = H_*(CM_*(M; f, g), \partial).$
 (note depends on $(f, g)).$

Ex. 1. $M = S^2.$

$$CM_* = \mathbb{Z} (p_1, p_2), \partial = 0$$

$$HM_* = \begin{cases} \mathbb{Z} & k=2 \\ 0 & k=0 \\ \text{else.} & \end{cases}$$



Ex 2. $M = T^2$. $C\mathcal{M}_* = \mathbb{Z} \langle p_1, p_2, p_3, p_4 \rangle$

$$\partial(p_1) = p_2 - p_1 + p_3 - p_3$$

$$\partial(p_2) = p_4 - p_3$$

$$\partial(p_3) = p_4 - p_4$$

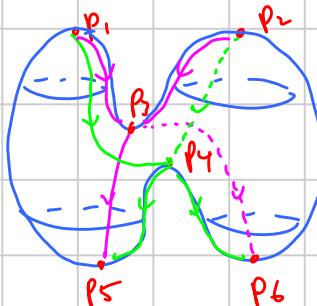
$$\partial(p_4) = 0$$

$$HM_* = \left\{ \begin{array}{ll} \mathbb{Z} & * = 2 \\ \mathbb{Z} & * = 1 \\ \mathbb{Z} & * = 0 \end{array} \right. ,$$

2 1 1 0

note $\partial^2(p_i)$ has 8 cancelling terms
corresponding to 4 components of
 $M(p_1, p_4)$

Ex 3 S^2 again.



$$C\mathcal{M}_* = \langle p_1, p_2 \rangle$$

$$\oplus \quad \partial p_1 = p_3 + p_4, \quad \partial p_2 = p_3 + p_4$$

$$\langle p_3, p_4 \rangle$$

$$\oplus \quad \partial p_3 = p_5 + p_6, \quad \partial p_4 = -p_5 - p_6$$

$$\langle p_5, p_6 \rangle$$

$$HM_* =$$

$$\begin{matrix} \mathbb{Z} \\ \oplus \\ 0 \\ 0 \\ \mathbb{Z} \\ 2 \end{matrix}$$

Thm $M = \text{closed smooth mfd}$, (f, g) Morse-Smale. Then

$$HM_*(M; f, g) \cong H_*^{\text{sing}}(M).$$

There's a generalization to mfs with boundary.

Say $\partial M = M_0 \sqcup M_1$, and there's a Morse function

$$f: M \rightarrow \mathbb{R} \text{ st. } f(M) \subset [0, 1], \quad f^{-1}(0) = M_0, \quad f^{-1}(1) = M_1,$$

and all crit pts are nondeg. and in $\text{int}(M_0)$.

let g be a metric st. (f, g) is Morse-Smale.

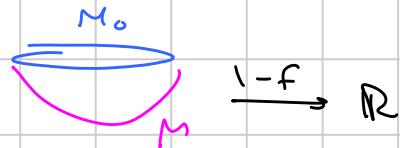
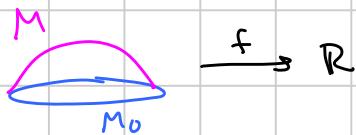


Then

$$HM_*(M; f, g) \cong H_*(M, M_0).$$

Note asymmetry between M_0 and M_1 .

Special case: $M_1 = \emptyset$.



Ex $M = D^n$, $M_0 = S^{n-1}$.



$$\text{ind}(f) = n$$



$$\text{ind}(1-f) = 0$$

$$HM_* = \begin{cases} \mathbb{Z} & k=n \\ 0 & \text{else} \end{cases}$$

$$= H_*(D^n, S^{n-1}).$$

$$HM_* = \begin{cases} \mathbb{Z} & k=0 \\ 0 & \text{else} \end{cases}$$

$$= H_*(D^n)$$

More generally:

If M is orientable, then $(CM_*(M; f, g), \partial)$ and $(CM_*(M; 1-f, g), \partial)$ are duals: same generators, $\deg p \rightarrow n - \deg p$, arrows are reversed.

$$(CM^{n-*}(M; 1-f, g), \partial^*) \cong (CM_*(M; f, g), \partial)$$

$\{\text{homology}\}$ $\{\text{homology}\}$

$$H^{n-*}(M) \qquad \qquad \qquad H_*(M, M_0)$$

Cor (Poincaré duality) M orientable, $\dim n$. Then

$$H^{n-*}(M) \cong H_*(M, \partial M).$$

Several possible approaches.

1. Cellular homology: by Freed Thm, \exists (CW)-complex str for M with $(\text{crit pt of index } k) \longleftrightarrow (k\text{-cell})$

$$\text{so } CM_k \cong C_k \text{ from cellular hom.}$$

If this isom is a chain map, then done.

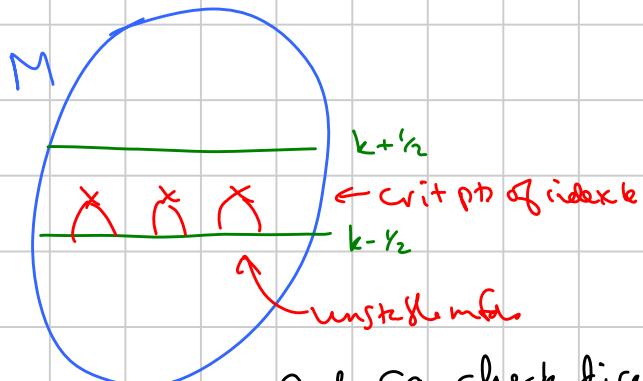
Problem: no clear reason it should be; also the (CW) complex is indirectly defined.

→ Could try to use $(\text{unstable mfld of index } k \text{ crit pt}) \hookrightarrow (k\text{-cell})$

but this isn't a (CW) Complex!

One correct fix: let f be self-indexing: argue that we can reduce to this case by enlarging $(f, g) \rightarrow (f, X)$, $X = \text{gradient like vector field for } f$, and then (CM_*, ∂) depends only on X and not f .

$$C_k = \#\text{ of crit pts of index } k$$



$$M^{k+1/2} = M^{k-1/2} \cup \underset{\downarrow}{(}\text{unstable } k\text{-disks)}^{C_k}$$

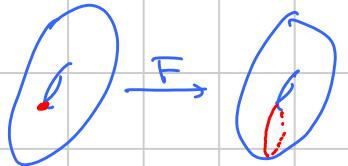
$$\begin{aligned} H_k(M^{k+1/2}, M^{k-1/2}) &\cong \mathbb{Z}^{C_k} \\ &\cong CM_k \end{aligned}$$

$$H_k(M^{k+1/2}, M^{k-1/2}) \xrightarrow{\partial} H_{k-1}(M^{k-1/2}) \rightarrow H_{k-1}(M^{k-1/2}, M^{k-3/2})$$

Agrees with the Morse differential.

(see Nicolaescu)

$$2. \quad CM_*(M; f, g) \xrightleftharpoons[G]{F} C_*^{\text{sing}}(M)$$



$$F(p) = W^u(p) \quad (\text{of correct index})$$

$G(\Delta)$ = Sum of critical pts. that negative gradient flow from Δ limits +.



$$G \circ F = \text{id}$$

$$F \circ G - \text{id} = \partial H + H \partial, \quad H: C_*^{\text{sing}}(M) \rightarrow \underline{C_{k+1}^{\text{sing}}(M)}$$

$$\Delta \rightarrow \{ \gamma_x(t) : x \in \Delta, t \in [0, \infty) \}.$$

technicalities: need to restrict to singular simplices transverse to all W^s .

See Hutchings.

Some Consequences.

$$1. \quad b_k = k\text{-th Betti number of } M = \dim_{\mathbb{Q}} (H_k(M) \otimes \mathbb{Q}).$$

Thm $f =$ Morse function on M . Then

$$(\# \text{critical points of index } k) \geq b_k$$

$$(\text{total # critical points of } f) \geq \sum_{k=0}^n b_k.$$

Pf Find a metric g st. (f, g) is Morse-Smale (needs proof).
Then

$$\begin{aligned} (\# \text{critical points of index } k) &= \text{rank } CM_k(f, g) \\ &\geq \text{rank } HM_k(f, g) \\ &= b_k. \quad \square \end{aligned}$$

3 Strengthening: (Strong) Morse inequalities.

f Morse, $c_k = \# \text{ of crit pts of index } k$. weak: $c_k \geq b_k$.

Strong: $\sum_{j=0}^k (-1)^{k-j} c_k \geq \sum_{j=0}^k (-1)^{k-j} b_k$.

Nicer way to say this:

define

$$P_t(M) = \sum_{k=0}^n b_k t^k$$

$$M_t(f) = \sum_{k=0}^n c_k t^k$$

Poincaré polynomial of M

"Morse polynomial"

Then $\exists Q(t) = \sum_{k=0}^{n-1} q_k t^k$, $q_k \geq 0 \forall k$, s.t.

$$M_t(f) = P_t(M) + (1+t)Q(t).$$

2. Poincaré-Hopf index thm.

v = vector field on M , isolated zeros.

If $v(p) = 0$ then $v(x) \neq 0$ on $S_{\epsilon}^{n-1}(p)$

\Rightarrow get a map $S^n = S_{\epsilon}^{n-1}(p) \longrightarrow \mathbb{R}^n - 0 \longrightarrow S^{n-1}$.

Define $\text{ind}_v(p) = \text{degree of this map}$.

Thm (Poincaré-Hopf)

$$\sum_{\substack{\text{zeros} \\ \text{of } v}} \text{ind}_v(p) = \chi(M). \\ (= \sum_k (-1)^k b_k(M))$$

Idea of pf: ① LHS index of v : iff the degree of the Gauss map:

$$M \hookrightarrow \mathbb{R}^n \rightsquigarrow \text{degree } (\partial(\text{nd } M) \rightarrow S^{n-1}).$$

② choose $v = -\nabla f$. Zeros of $v \iff$ critical pts of f

$$\text{ind}_p(f) = k \Rightarrow \text{ind}_p(v) = (-1)^k \quad (\text{local computation})$$

$$\Rightarrow \sum \text{ind}_v(p) = \sum_k (-1)^k \text{rank } CM_k = \sum_k (-1)^k \text{rank } HM_k = \chi(M). \quad \square$$

Generalization: Lefschetz fixed point thm.

$f: M \rightarrow M$ continuous.

Lefschetz number $L(f) = \sum (-1)^k \text{Tr}(f_*: H_k(M; \mathbb{Q}) \rightarrow H_k(M; \mathbb{Q})).$

$p \in M$ is a fixed point of f if $f(p) = p$; nondegenerate if

$\Gamma(f)$ is Δ at (q, p) ($\Leftrightarrow \det(df_p - id) \neq 0$).

p fixed point $\Rightarrow \text{ind}_f(p) \in \mathbb{Z}$. This is ± 1 for nondeg.

$$\underline{\text{LFTPT}} \quad \sum_{p \in \text{Fix}(f)} \text{ind}_f(p) = L(f) \quad M \text{ compact, smooth, oriented.}$$

If all nondeg, then $\# \text{Fix}(f) \geq |L(f)|$.

Special case: if $f \sim id$, then $L(f) = L(id) = \chi(M)$
 $\Rightarrow \# \text{Fix}(f) \geq \chi(M)$.

Special Special Case: V = vector field on M , $f = \varphi_t: M \rightarrow M$ time t flw. Then fixed pt of f = zero of V , and $\text{ind}_f(p) = \text{ind}_V(p) \Rightarrow$ Poincaré-Hopf.

A priori invariance

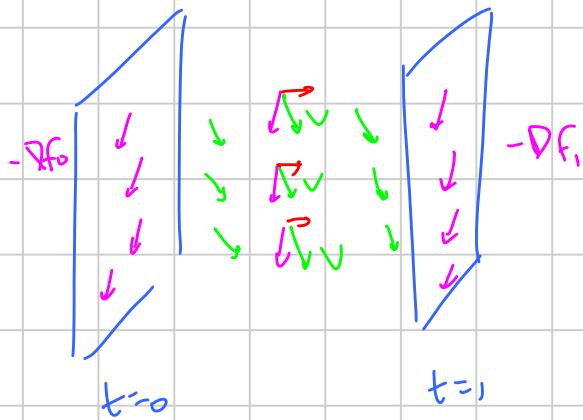
Suppose we didn't know $H_{M^*} \cong H_*^{\text{sing}}$. We'd still deduce some interesting info of M , provided $H_{M^*}(M; f, g)$ indep of (f, g) .

This can be proven directly by -bifurcation: how C_{M^*} changes
-continuation (flow).

Suppose $(f_0, g_0), (f_1, g_1)$ are Morse-Smale, and let (f_t, g_t) be a path between them.

Define V = vector field on $[0, 1] \times X$ by

$$V = (1-t) \cdot (-\nabla f_0) + t \cdot (\nabla f_1) + V_t \quad V_t = -\nabla_{g_t} f_t.$$



One can construct "critical pts" = zeros, stable/unstable mfs for V just as for $-\nabla f$. (V is "gradient-like")

$$\text{crit}(V) = (\{0\} \times \text{crit}(f_0)) \cup (\{1\} \times \text{crit}(f_1))$$

index $k+1$ \leftarrow index k
 index k \leftarrow index k

Fact: for generic (f_t, g_t) , stable/unstable mfs for V are transverse. (Caveat: in general can't avoid times when (f_t, g_t) isn't Morse-Smale.)

$p, q \in \text{Crit } V \rightsquigarrow M(p, q) = \{\text{flowlines from } p \text{ to } q\}.$

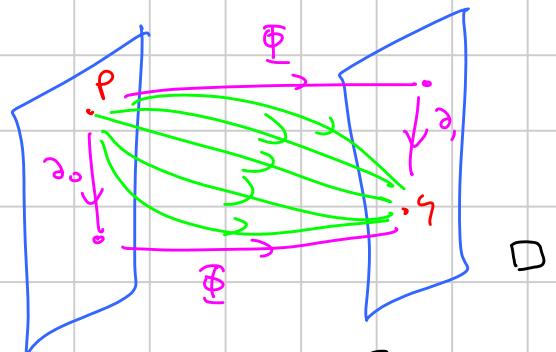
For $p \in \text{Crit}(f_0)$ with $\text{ind}(p) = k$, define

$$\Phi(p) = \sum_{\substack{q \in \text{Crit}(f_1) \\ \text{ind } q = k}} (\# M(p, q)) q.$$

Extend Φ linearly to get a map $\Phi: CM_k(f_0, g_0) \rightarrow CM_k(f_1, g_1)$.

Prop Φ is a chain map: $\Phi \circ \partial_0 = \partial_1 \circ \Phi$.

Pf Usual degeneration argument.



Next: let Γ, Γ' be two generic paths

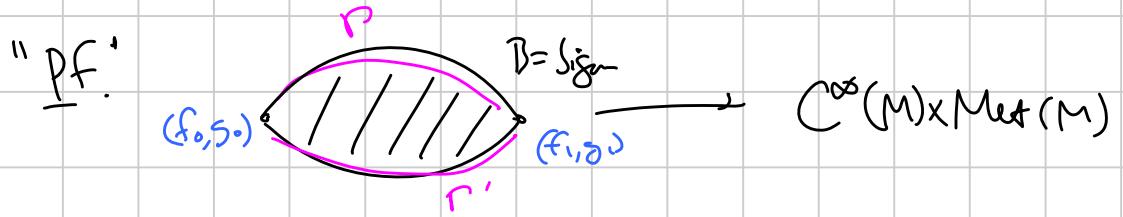
$$(f_0, g_0) \xrightarrow{\Gamma} (f_1, g_1).$$

$$\rightsquigarrow \Phi_\Gamma, \Phi_{\Gamma'}: CM_*(f_0, g_0) \rightarrow CM_*(f_1, g_1).$$

Prop If Γ, Γ' are homotopic then $\Phi_\Gamma, \Phi_{\Gamma'}$ are chain homotopic:

$$\exists K: CM_*(f_0, g_0) \rightarrow CM_{*+1}(f_1, g_1)$$

st. $\partial_1 K + K \partial_0 = \Phi_\Gamma - \Phi_{\Gamma'}$.



Define V on $B \times M$ by $V = V_B + V_b$

$$V_b = -\nabla_g f \text{ at } b \in B.$$

$$V_B =$$



K counts flowlines of V . \square

Cor $\text{HM}_*(f, g)$ index of (f, g) .

PF. $\cdot \xrightarrow{\Gamma} \cdot \xrightarrow{-\Gamma} \cdot$

$$(f_0, g_0) \quad (f_1, g_1) \quad (f_0, g_1)$$

$\Gamma \cup (-\Gamma)$ homotopic
to constant
path.

$$\rightsquigarrow \text{HM}_*(f_0, g_0) \xrightarrow{\Phi_\Gamma} \text{HM}_*(f_1, g_1) \xrightarrow{\Phi_{-\Gamma}} \text{HM}_*(f_0, g_1)$$

\curvearrowright

$\Phi_{\text{const}} = \text{id}$

\square

Floer Homology

General setup: adapt Morse theory to ∞ dim situations.

(Geometric data) \rightsquigarrow infinite-dim mfd M ,
action functional $f: M \rightarrow \mathbb{R}$.

$p, q \in M$ crit pts of f : if $\gamma: (-\epsilon, \epsilon) \rightarrow M$ satisfies $\gamma(0) = p$ the
 $(f \circ \gamma)'(0) = 0$.

Define "gradient flow" for f on M : note not always possible!
long time existence especially problematic.

$$\begin{aligned} W(p, q) &= \left\{ x \in M : \lim_{t \rightarrow \infty} \gamma_x(t) = q, \lim_{t \rightarrow -\infty} \gamma_x(t) = p \right\} \\ &= W^u(p) \cap W^s(q). \end{aligned}$$

$$\dim W(p, q) = \text{ind}(p) - \text{ind}(q)?$$

Problem: in general both $W^u(p), W^s(p)$ are ∞ dimensional.

Nevertheless: strategy is to show that

$W(p, q) = \text{inverse image of a regular value of some}$
 $\text{Fredholm operator } \mathcal{F} \text{ between Banach mfd}$
 \Rightarrow by Inverse Fn Thm, $W(p, q) = \text{smooth finite-dim mfd}$
 $\text{of dimension } \text{ind } \mathcal{F}$.

Now note we only need relative indices of crit pts: differences of indices.

In nice cases, can define $\text{ind}: \text{Crit } f \rightarrow \mathbb{Z}$ (or \mathbb{Z}/n)

$$\text{st. } \text{ind } \mathcal{F} = \text{ind } p - \text{ind } q.$$

- Then:
- define $M(p,q) = W(p,q)/R$
 - Study compactifications $\bar{M}(p,q)$
 - if $\text{ind } p = \text{ind } q + 1$, $M(p,q) = \text{cpt}$
 $\text{ind } p = \text{ind } q + 2$, $\bar{M}(p,q) = M(p,q) \cup \left(\bigcup_r M(p,r) \times M(r,q) \right)$
 - Construct Floer complex
 $CF_* : CF_k = \mathbb{Z} \langle \text{crit pts of index } k \rangle$
 $\partial : CF_* \rightarrow CF_{*-1}, \quad \langle \partial p, q \rangle = \# M(p,q).$
 - prove invariant under choices made along the way
 $\Rightarrow \underline{HF_*} \quad \underline{\text{Floer homology}}$.

Instanton Floer Homology

(Floer '85, Fintushel-Stern '90)

Setup: (Y^3, g) oriented Riemannian, $\overset{E}{\downarrow}$ any principal $SU(2)$ bundle (in fact, any such is $Y \times SU(2)$). (also works for $SO(3)$)
 $A = \{\text{connection on } E\}$
 \uparrow
 $G = \text{gauge group} = \text{Aut}(E)$.

$\mathcal{M}() = A/G = \text{gauge equivalence class of connections on } E.$
 \hookrightarrow (in fact need to remove 0 connection)

Chern-Simons functional $CS : A \rightarrow \mathbb{R} : A \mapsto \Omega^1(Y, su(2))$

$$CS(A) = \frac{1}{8\pi^2} \int_Y \text{tr}(\theta \wedge d\theta + \frac{2}{3} \theta \wedge \theta \wedge \theta) \quad \theta \in \Omega^1(Y, su(2))$$

Under gauge transformation (\leftrightarrow map $Y \rightarrow SU(2)$),

$$CS(g^*(A)) = CS(A) + \deg g$$

\therefore CS descends to $CS : \mathcal{M}(Y) \rightarrow \mathbb{R}/\mathbb{Z} = S^1$.

Crit pts of CS = flat connections : $F_A = 0$.

Metric on A : for $\theta_1, \theta_2 \in T_A A = \Omega^1(Y, su(2))$, define

$$\langle \theta_1, \theta_2 \rangle = \int_Y \text{tr}(\theta_1 \wedge * \theta_2)$$

$*$ = Hodge star on Y .

Gradient of CS : $\text{grad } CS(A) = * F_A$

So negative gradient flow on \mathcal{M} is

$$\frac{dA}{dt} + * F_A = 0.$$

View this as a connection on $E \times \mathbb{R}$: $A(t) + \frac{\partial}{\partial t} = \tilde{A}$.

The gradient flow eq becomes

$$F_{\tilde{A}} + * F_{\tilde{A}} = 0$$

$*$ = Hodge star on $Y \times \mathbb{R}$.
instanton (ASD) equation.

Fact: Given an instanton \tilde{A} on $P \times \mathbb{R}$,

$\lim_{t \rightarrow \pm\infty} \tilde{A}$ exist and give flat connections A^\pm on P

$\Leftrightarrow \tilde{A}$ has finite energy:

$$E(\tilde{A}) = \int_{Y \times \mathbb{R}} \text{tr}(F_{\tilde{A}} \wedge * F_{\tilde{A}}) < \infty.$$

So: let $\mathcal{R}(Y)$ = gauge equivalence classes of flat $SU(2)$ connections on Y .
(reps $\pi_1(Y) \rightarrow SU(2)$)

$$A^+, A^- \in \mathcal{R}(Y) \rightsquigarrow$$

$M(A^+, A^-)$ = gauge equivalence class of connections
 \tilde{A} on $P \times \mathbb{R}$ such that:

- $F_{\tilde{A}} + * F_{\tilde{A}} = 0$
- $E(\tilde{A}) < \infty$
- $\lim_{t \rightarrow \pm\infty} \tilde{A} = A^\pm$.



Then: • $\exists \mu: \mathcal{R}(Y) \rightarrow \mathbb{Z}$ such that $M(A^+, A^-)$ is fin-dim

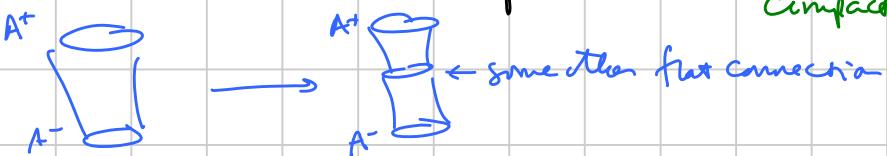
and $\dim M(A^+, A^-) = \mu(A^+) - \mu(A^-)$

(roughly speaking, this difference is given by spectral flow).

of eigenvalues of "Hessian" that change sign as we go from A^+ to A^-)

• $M(A^+, A^-)$ can be oriented

• 1-dim $M(A^+, A^-)$ can be compactified (Whitney
Compactness)



Complications:

- need to perturb everything to get transversality
- grading μ is really only defined in $\mathbb{R}/8$
for gauge equivalence classes
- issue with reducible connections: $\mathcal{M}(Y)$ isn't a mfd.
for $Y = \text{homology } S^3$, can just delete 0 from $\mathcal{M}(Y)$.

Thm (Floer) $Y = \text{homology sphere}$. Can construct a complex

$$(\mathcal{CF}_*, \partial) \quad * \in \mathbb{R}/8, \mathcal{CF}_* = \mathbb{Z} \langle A \in \mathcal{D}(Y) \rangle, \\ \partial \text{ counts } M(A^+, A^-).$$

such that $\partial^2 = 0$ and the homology $\mathcal{HF}_*(Y)$ is independent
of the choice of metric: instanton Floer homology.

The Euler characteristic

$$\sum_{k \in \mathbb{Z}} (-1)^k \text{rk } \mathcal{HF}_k(Y) \text{ is } 2\lambda(M),$$

$\lambda(Y) = \text{Casson invt.}$:

$$Y = H_1 \cup H_2 \text{ Heegaard splitting, } \mathcal{R}(H_1), \mathcal{R}(H_2) \longrightarrow \mathcal{R}(\Sigma) \\ \lambda(Y) = \#(\mathcal{R}(H_1) \cap \mathcal{R}(H_2)). \quad (\text{succ} \leftrightarrow \text{conjugacy})$$

So $\mathcal{HF}_*(Y)$ categorifies the Casson invt.

(but behaves functorially! — Donaldson)

$$Y_0 \times Y_1 \xrightarrow{\exists!} \mathcal{HF}_*(Y_0) \rightarrow \mathcal{HF}_*(Y_1).$$

Kronheimer-Mrowka (published 2007):

Monopole (Seiberg-Witten) Floer homology.

functional: Chern-Simons-Dirac functional

signature of $C\mathcal{M}_*$: Solutions of SW eqns on Y

differential: SW monopoles on $Y \times \mathbb{R}$.

Symplectic Flux Homology

Background: Arnold Conjecture.

(M, ω) compact symplectic mfd: $\dim M = 2n$, $\omega \in \Omega^2(M)$,
closed ($d\omega = 0$) and nondegenerate (ω^n nowhere 0).

$H \in C^\infty(M)$ \rightsquigarrow Hamiltonian vector field $X_H \in \text{Vect}(M)$ defined by

$$dH = X_H \lrcorner \omega.$$

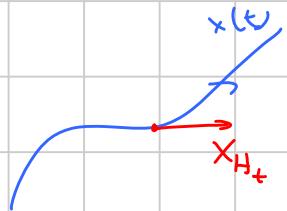
$$\begin{array}{ccc} T_p^*M & \xleftarrow{\cong} & T_p M \\ \omega(v, \cdot) & \longleftarrow & dH \mapsto X_H \end{array} \quad \Omega^*(M) \cong \text{Vect}(M)$$

For $M = T^*N$ (phase space), flow under X_H is the trajectory of a particle subject to the Hamiltonian H .

Now let $H_t =$ time-dependent Hamiltonian $M \rightarrow \mathbb{R}$

with period 1: $H_{t+1} = H_t$.

Flow: $x(0) \rightsquigarrow x(t) = \varphi_t(x) \in M$
 $\frac{d}{dt} x(t) = X_{H_t}(x(t)).$



Time 1 map $\varphi_1: M \rightarrow M$ $\varphi_1(x(0)) = x(1).$

$$\left(\frac{d}{dt} \varphi_t(x) = X_{H_t}(\varphi_t(x)) \right)$$

Fixed points \hookrightarrow periodic orbits

A periodic orbit $x(t)$ is nondegenerate if

$$\det(d\varphi_1(x(0)) - \text{Id}) \neq 0.$$

$$\{\text{periodic orbits}\} = \mathcal{P}(H).$$

$$x = \varphi_1(x)$$

$$x(t+1) = x(t)$$

Arnold Conjecture (1960s) (M, ω) cpt spl, H_t 1-periodic.

If all periodic orbits are nondegenerate, then

$$\# \mathcal{P}(H) \geq \sum b_k(M).$$

Notes: • Lefschetz Fixed Pt Thm: $f: M \rightarrow M$, $f \sim \text{id}$

$$\Rightarrow \# \text{Fix}(f) \geq \chi(M) = \sum (-1)^k b_k(M).$$

• if $H_t = H$ time-independent, then $p \in \text{Crit } H \mapsto X_H(p) = 0$

$\Rightarrow p = \varphi_1(p)$, and nondegeneracy $\Rightarrow H = \text{Morse function}.$

$$\Rightarrow \# \mathcal{P}(H) \geq \# \text{Crit}(H) \geq \sum b_k(M).$$

\uparrow
Morse inequalities

• Conj important for dynamical systems; gave rise to modern symplectic geometry.

• History: proved for monotone symplectic mfd ($[\omega] \propto c_1$)

by Floer; now completely proven, through further work of Hofer-Salamon, Fukaya-Ono, Liu-Tian, Ruan.

$$\underline{\mathcal{L}M} = \{\text{Contractible loops in } M\}: \quad \gamma \in \underline{\mathcal{L}M} \Rightarrow \exists u: D^2 \rightarrow M \text{ with } \gamma = \partial u.$$

M monotone, normalize so $\langle \omega, A \rangle \in \mathbb{Z} \quad \forall A \in H_2(M)$.

Action functional $a_H: \underline{\mathcal{L}M} \rightarrow \mathbb{R}/\mathbb{Z}$ defined by

$$a_H(x) = - \int_{D^2} u^* \omega - \int_0^1 H_t(x(t)) dt.$$

Crit pts of a_H $\longleftrightarrow \mathcal{P}(H)$ (calculus of variations).

J_t = almost cx str on M compatible with ω_t ,

$$g_t = \omega(\cdot, J_t \cdot).$$

Get a metric on LM by $\xi, \eta \in T_x LM = \{\text{vector fields along } x(t)\}$
 $\rightsquigarrow \langle \xi, \eta \rangle = \int_0^1 g_t(\xi_{x(t)}, \eta_{x(t)}) dt.$

Gradient: $D_{\partial_H}(x) H = J_t(x(t)) \dot{x}(t) - \nabla H_t(x(t))$

Negative gradient flow: $u: \mathbb{R}_s \rightarrow LM$ with

$$\boxed{u_j \quad \frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} - \nabla H_t(u) = 0}$$

$u: \mathbb{R}_s \times [0, 1] \rightarrow M$

this is the perturbed J-hol equation: if $H_t = \text{const}$, this says

$$J \circ du = du \circ j$$

$j = \text{standard cx str on } \mathbb{R} \times [0, 1]$.

$x^\pm \in \mathcal{P}(H)$. Define

$$M(x^+, x^-) = \left\{ u: \mathbb{R} \times [0, 1] \rightarrow M \mid \begin{array}{l} u \text{ satisfies } (*), \\ \lim_{s \rightarrow \pm\infty} u = x^\pm, \text{ and } u \text{ has finite energy} \end{array} \right. \\ E(u) = \int_0^1 \int_{-\infty}^{\infty} \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} - X_t(u) \right|^2 \right) ds dt \left. \right\}.$$

- for generic J , $M(x^+, x^-) = \text{finite-dim mfd}$
- $\exists \mu: \mathcal{P}(H) \rightarrow \mathbb{Z}$ (or $\mathbb{Z}/\text{minimal Chern number}$)
 with $\dim M(x^+, x^-) = \text{ind } x^+ - \text{ind } x^-$: Conley-Zehnder index
- $1\text{-dim } M$ compactifies.

$CF_*(M; H, J)$ generated by $\mathcal{P}(H)$,
differential counts points in $M(x^+, x^-)$.

Then (Floor) 1. $\partial^2 = 0$. $\rightsquigarrow HF_*(M; H, J)$

2. If (H', J') is another pair, then
 $HF_*(M; H, J) \cong HF_*(M; H', J')$. $\} \text{continuation}$

Now set H', J' to be time-independent. Then can set up

$M(x^+, x^-)$ for $H' \longleftrightarrow$ gradient flows from $x^+ \rightarrow x^-$

so $HF_*(M; H', J') = HM_*(M)$.

$$\Rightarrow \#\mathcal{P}(H) \geq \operatorname{rk} HF_*(M; H, J)$$

$$= \operatorname{rk} HM_*(M)$$

$$= \sum b_k(M).$$

Lagrangian Intersection Floer Homology

$L \subset (M, \omega)$ is Lagrangian if $\omega|_L = 0$, $\dim L = n$.

Given $L_1, L_2 \subset M$, define

$$\mathcal{P}(L_1, L_2) = \{\text{paths in } M \text{ from } L_1 \text{ to } L_2\}.$$

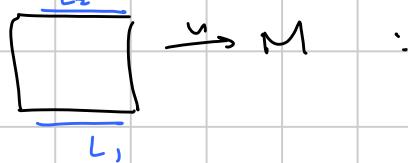
For fixed $\gamma_0 \in \mathcal{P}(L_1, L_2)$ and $\gamma \in \mathcal{P}(L_1, L_2)$, let

$u: [0, 1] \times [0, 1] \rightarrow M$ be such that

then define action functional

$$a: \mathcal{P}(L_1, L_2) \rightarrow \mathbb{R}$$

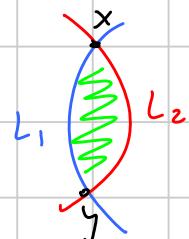
$$\gamma \mapsto \int_{[0, 1]^2} u^* \omega$$



Assuming $\langle \omega, \pi_2(M) \rangle = 0$, this is indep of choice of u .

Crit pt of $a \leftrightarrow$ pts in $L_1 \cap L_2$

Gradient flow \leftrightarrow J-hol disks



\rightsquigarrow Construct $CF_*(L_1, L_2) = \mathbb{Z}\langle p \in L_1 \cap L_2 \rangle$

\supset counts J-hol disks.

$\rightsquigarrow HF_*(L_1, L_2)$ Lagrangian intersection Floer homology.

Special case: $Y = H_1 \cup H_2$ Heegaard splitting

$\leftrightarrow \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\}$ simple closed curves in Σ

$\Rightarrow Sym^g(\Sigma)$ symplectic, $\alpha_1 \times \dots \times \alpha_g, \beta_1 \times \dots \times \beta_g$ Lagrangian.

$HF_*(\alpha, \beta) =$ Heegaard Floer homology (Ozsváth-Szabó), invariant of Y .

Floer Homology

General Setup: adapt Morse theory to ∞ dim situations.

(Geometric data) \rightsquigarrow infinite-dim mfld M ,
action functional $f: M \rightarrow \mathbb{R}$.

$p, q \in M$ crit pts of f : if $\gamma: (-\epsilon, \epsilon) \rightarrow M$ satisfies $\gamma(0) = p$ then
 $(f \circ \gamma)'(0) = 0$.

Define "gradient flow" for f on M : note not always possible!
long time existence especially problematic.

$$\begin{aligned} W(p, q) &= \left\{ x \in M : \lim_{t \rightarrow \infty} \gamma_x(t) = q, \lim_{t \rightarrow -\infty} \gamma_x(t) = p \right\} \\ &= W^u(p) \cap W^s(q). \end{aligned}$$

$$\dim W(p, q) = \text{ind}(p) - \text{ind}(q)?$$

Problem: in general both $W^u(p), W^s(p)$ are ∞ dimensional.
Nevertheless: strategy is to show that

$W(p, q) = \text{inverse image of a regular value of some}$
Fredholm operator \mathcal{F} between Banach mfds
 \Rightarrow by Inverse Fn Thm, $W(p, q) = \text{smooth finite-dim mfd}$
of dimension $\text{ind } \mathcal{F}$.

Now note we only need relative indices of crit pts: differences of indices.

In nice cases, can define $\text{ind}: \text{Crit } f \rightarrow \mathbb{Z}$ (or \mathbb{Z}/m)
st. $\text{ind } \mathcal{F} = \text{ind } p - \text{ind } q$.

- Then:
- define $M(p,q) = W(p,q)/R$
 - study compactification $\bar{M}(p,q)$
 - if $\text{ind } p = \text{ind } q + 1$, $M(p,q) = \text{cpt}$
 $\text{ind } p = \text{ind } q + 2$, $\bar{M}(p,q) = M(p,q) \cup (\bigcup_r M(p,r) \times M(r,q))$
 - Construct Floer complex
- $CF_* : CF_k = \mathbb{Z} \langle \text{crit pts of index } k \rangle$
- $\partial : CF_* \rightarrow CF_{*-1}, \quad \langle \partial p, q \rangle = \# M(p,q).$
- prove invariant under choices made along the way
 $\Rightarrow \underline{HF_*} \quad \underline{\text{Floer homology}}$.
-

Special Case: Lagrangian intersection Floer homology.

Setup: (M^{2n}, ω) symplectic manifold: $\omega \in \Omega^2(M)$ closed and nondegenerate.
(Special case: $M = T^*Q$)

$L \subset (M, \omega)$ is Lagrangian if $\omega|_L = 0$, $\dim L = n$.

$H \in C^\infty(M) \rightsquigarrow$ Hamiltonian vector field $X_H \in \text{Vect}(M)$, $dH = i_{X_H}\omega$.

A Hamiltonian diffeomorphism is the flow of a (time-dep) Hamiltonian vector field.

$\varphi : M \rightarrow M$, $\frac{d}{dt} \varphi_t = X_{H_t}$. Note $\mathcal{L}_{X_H} \omega = 0 \Rightarrow \varphi^* \omega = \omega$:

φ preserves ω . ($\Rightarrow L$ Lagrangian $\Rightarrow \varphi(L)$ Lagrangian).

Then (Floer 1988) (M, ω) cpt spl, L cpt lay., $\varphi : M \rightarrow M$ Ham diffe.

If $L \pitchfork \varphi(L)$ and $\forall u : (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (\omega, L)$, $\int_{\mathbb{D}^2} u^* \omega = 0$ ($\Leftarrow \pi_2(M, L) = 0$)

the

$$\#(L \cap \varphi(L)) \geq \sum \dim H^k(L; \mathbb{Z}_2).$$

Special case: $M = T^*Q$ (not cpt, but ok), $L = \text{zero section}$, $f: Q \rightarrow \mathbb{R}$ Morse.

Can choose φ st. $\varphi(L) = \Gamma_{df} = \{(x, df_x) \mid x \in Q\}$.

$L \cap \varphi(L) = \text{Crit}(f) \Rightarrow$ this generalizes Morse inequality.

→ Application to Arnold Conjecture: $\varphi = \text{Hamilt. diffeo on } (M, \omega)$ s.t.

$$\Rightarrow \# \text{Fix}(\varphi) \geq \sum_{k=0}^{2n} b_k(M). \quad (\text{in the case } \pi_2(M) = 0)$$

How to get this?

Given $L_1, L_2 \subset M$, define

$$\mathcal{P}(L_1, L_2) = \{\text{paths in } M \text{ from } L_1 \text{ to } L_2\}.$$

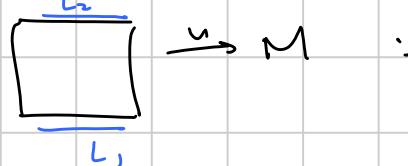
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$$\gamma \mapsto \int_{[0,1]^2} u^* \omega$$



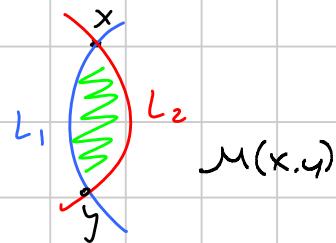
Assuming $\langle \omega, \pi_2(M) \rangle = 0$, this is indep of choice of u .

Crit pts of $a \leftrightarrow$ pts in $L_1 \cap L_2$

Gradient flow \leftrightarrow J-hol disks
(w.r.t. metric on \mathcal{P} induced by J)

Now construct $C_F(L_1, L_2) = \mathbb{Z} \langle x \in L_1 \cap L_2 \rangle$

$$\partial(x) = \sum (\# u(x, y)) y$$



Thm (Floer) Under good conditions: $(CF_*(L_1, L_2), \partial)$ is a complex, indep of choices \rightarrow Lagrangian intersection Floer homology $HF_*(L_1, L_2)$.

- Set up grading, check moduli spaces are manifolds of the expected dimension
 $\rightsquigarrow \partial: CF_* \rightarrow CF_{*-1}$ TRANSVERSALITY
- $\partial^2 = 0$: Compactification of moduli spaces CONTACTNESS
 $\rightsquigarrow HF_*(L_1, L_2) = H_*(CF_*, \partial)$
- Check index of all choices made along the way, e.g. J INVARIANCE

Thm (Floer) If $\varphi =$ Hamilton. diffeo, $L =$ lagr, then
 $HF_*(L, \varphi(L)) \cong H_*(L)$.

Thus $\#(L \cap \varphi(L)) = \text{rk } CF_*(L, \varphi(L)) \geq \text{rk } HF_*(L, \varphi(L)) = \text{rk } H_*(L)$.