Minicourse - Morse Theory and Floer Homology

Note Title 3/12/2015

Three topics:

- Morse theory (useful for topology, diff geom, alg geom, ...)
- Morse homology (useful for persistent homology)
- Floer homology (∞-dim version of Morse homology, very important in topology, symplectic geometry, math physics, ...)

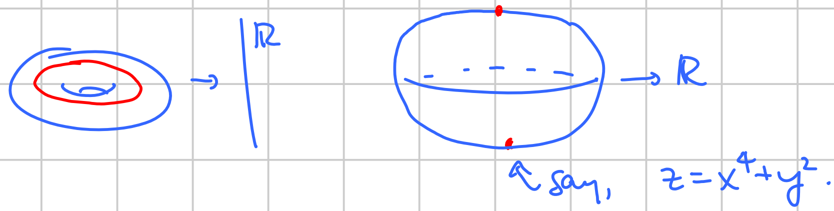
Setting: M = smooth manifold, often compact, dimension n.
$f: M \to \mathbb{R}$ smooth.

<u>Def</u> A <u>critical point</u> of f is a point $p \in M$ with $(df)_p = 0$.
in coords, $\frac{\partial f}{\partial x_i} = 0 \quad i = 1, \dots, n$.

A critical point is <u>nondegenerate</u> if the Hessian $(Hf)_p$ is nondegenerate.
in coord, $\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \neq 0$.

<u>Def</u> f is a <u>Morse function</u> if all critical points are nondegenerate.

<u>Ex</u>: height function on a surface.

Not Morse: 

Existence of Morse Functions

Thm There are Morse functions on any smooth M.

Outline of proof:

Lemma $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}$. For almost all $a = (a_1, \dots, a_n) \in \mathbb{R}^n$,

$$f_a(x) = f(x) - \sum_{j=1}^{n} x_j a_j$$

is Morse.

Pf Define $g: U \to \mathbb{R}^n$, $\quad g(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$.

Sard's Thm: the set of critical values of g (images of critical pts: $x \in U$ with dg_x not surjective.) has measure 0.

Now suppose $\underline{a}$ is a regular value (not a critical value) for g. (true for almost all $\underline{a}$)

If $x \in U$ is a crit pt of f_a, then $g(x) = a$.

Then by assumption, $(Hf_a)_x = dg_x$ is surjective $\Rightarrow$ invertible. $\square$

Now use:

Whitney embedding theorem $\exists$ embedding $M \hookrightarrow \mathbb{R}^N$ for some N (in fact, $N = 2n$).

Embed $M \hookrightarrow \mathbb{R}^N$.

<u>Prop</u> For almost all $a \in \mathbb{R}^N$, the function $f: \underset{\subset \mathbb{R}^N}{M} \to \mathbb{R}$,
$f(x) = a_1 x_1 + \dots + a_N x_N$ is Morse.

<u>Pf</u> Near any $x \in M$, we can choose $J = (j_1, \dots, j_n)$ so that $(x_{j_1}, \dots, x_{j_n})$ form local coordinates for M. ($T_x^*\mathbb{R}^N \to T_x^*M$ surjective)
So we can cover M by open sets $U_J = \{x \in M \mid J \text{ give local coords}\}$.

Let's consider $J = (1, \dots, n)$: $x_1, \dots, x_n$ give local coords.
Fix $(a_{n+1}, \dots, a_N) \in \mathbb{R}^{N-n}$ and define $f: U_J \to \mathbb{R}$,

$$f(x) = x_{n+1}a_{n+1} + \dots + x_N a_N.$$

By the lemma, for almost all $(a_1, \dots, a_n) \in \mathbb{R}^n$,

$$f_a(x) = f(x) + x_1 a_1 + \dots + x_n a_n = \sum_1^N x_j a_j$$

is Morse. So for almost all $(a_1, \dots, a_N) \in \mathbb{R}^N$, $\sum_1^N x_j a_j$ is Morse on U_J.
There are finitely many U_J $\Rightarrow$ $\sum_1^N x_j a_j$ is Morse on <u>M</u> for almost all <u>a</u>. $\square$

This can be easily improved to:

<u>Thm</u> Any function $f: M \to \mathbb{R}$ can be C^k approximated (for any k, on compact subsets) by a Morse function.

More difficult, using transversality:
<u>Thm</u> M compact. Then $\{\text{Morse fns on } M\}$ is open and <u>dense</u> in $C^\infty(M)$.

$f : M \to \mathbb{R}$ Morse, p = critical point.

<u>Def</u> The <u>Morse index</u> of f at p is

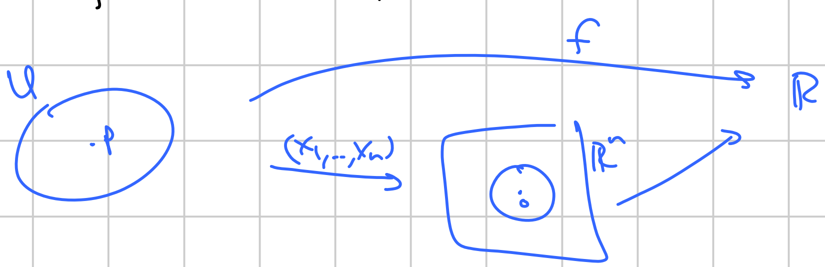
$$\mathrm{ind}_f p = \#(\text{negative eigenvalues of } Hf_p).$$

<u>Ex</u>. $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = c - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$.

$$Hf = \begin{bmatrix} -1 & & & & 0 \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{bmatrix}. \qquad \mathrm{ind}_f 0 = k.$$

<u>Morse Lemma</u> p = critical pt of f of index k. Then $\exists$ nbd U of p and local coords $(x_1, \ldots, x_n)$ on U st. $p = (0, \ldots, 0)$ and on U,

$$f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$



Use:

<u>Hadamard's Lemma</u> $f : U \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}$ smooth, $f(0) = 0$. Then $\exists$ smooth $g_1, \ldots, g_n : U \to \mathbb{R}$ with

$$f(x_1, \ldots, x_n) = \sum_{j=1}^n x_j \, g(x_1, \ldots, x_n)$$

and $g_j(0) = \frac{\partial f}{\partial x_j}(0)$.

<u>Pf</u> $f(x_1, \ldots, x_n) = \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt = \int_0^1 \sum x_j \frac{\partial f}{\partial x_j}(tx_1, \ldots, tx_n)\, dt$

So set $g_j(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_j}(tx_1, \ldots, tx_n)\, dt$. $\square$

Pf of Morse Lemma

Can assume $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, $p = 0$, $f(p) = 0$.

Hadamard $\Rightarrow$ $f(x) = \sum_j x_j g_j(x)$, and $g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0$.

Hadamard again $\Rightarrow$ $g_j(x) = \sum_i x_i h_{ij}(x)$

$\Rightarrow f(x) = \sum_{i,j} x_i x_j h_{ij}(x)$ $\qquad$ $h_{ij} : U \to \mathbb{R}$ smooth.

Can assume $h_{ij}(x) = h_{ji}(x)$ (just replace both by $\frac{1}{2}(h_{ij} + h_{ji})$), so $H(x) = (h_{ij}(x))$ is symmetric. Also note $H(0)$ is nonsingular.

Now apply diagonalization of symmetric matrices: Suppose for $r \geq 1$ we can find coords $u_1, \ldots, u_n$ on U st.

$$f = \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \ldots, u_n)$$

for some symm. (H_{ij}). We claim that (after shrinking U) we can do the same for $r+1$.

Apply linear change to $u_r, \ldots, u_n$ to get $H_{rr}(0) \neq 0$.

Let $g(u_1, \ldots, u_n) = \sqrt{|H_{rr}(u_1, \ldots, u_n)|}$ (might need to shrink U).

Define new coords $v_1, \ldots, v_n$, $\quad v_i = u_i \quad i \neq r$,

$$v_r(u_1, \ldots, u_n) = g(u_1, \ldots, u_n)\left(u_r + \sum_{i > r} u_i \frac{H_{ir}(u_1, \ldots, u_n)}{H_{rr}(u_1, \ldots, u_n)}\right)$$

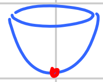
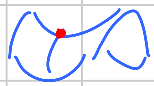
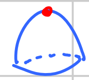
Inverse Fn Thm: near 0, these are coord fns.

These have been designed to satisfy:

$$f = \pm v_1^2 \pm \cdots \pm v_{r-1}^2 \pm v_r^2 + \sum_{i,j > r} v_i v_j \tilde{H}_{ij}(v_1, \ldots, v_n).$$

Finally: # of $\pm$ are determined by index of f at p. $\square$

Cor The critical points of a Morse function are isolated.

Dim 2:  ind 0  ind 1  ind 2

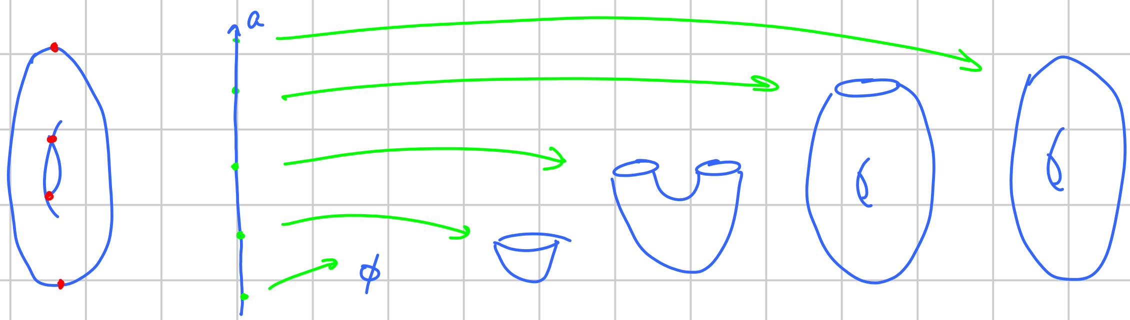
Sublevel Sets

M connected, $f: M \to \mathbb{R}$ Morse, $a \in \mathbb{R}$.

Def The sublevel set $M^a = f^{-1}((-\infty, a])$. (mfd with ∂ if a = regular value)

f is exhaustive if M^a is compact for all $a \in \mathbb{R}$.

ex:



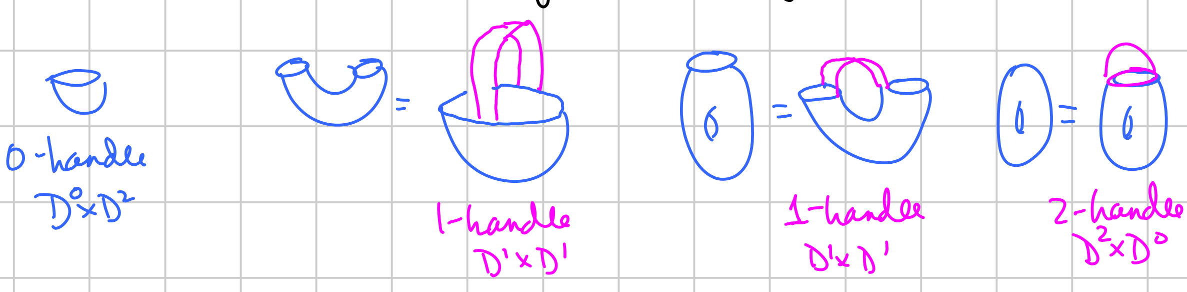
Idea: Study how the topology of M^a changes as a changes.

In fact: say $a < b$.

- If no crit values in $[a,b]$, then M^a, M^b are diffeo.
- If one crit value of index k, then $M^b = M^a \cup (k\text{-handle})$.

  k-handle $= D^k \times D^{n-k}$; glue to M^a by $(\partial D^k) \times D^{n-k} \hookrightarrow \partial M^a$.

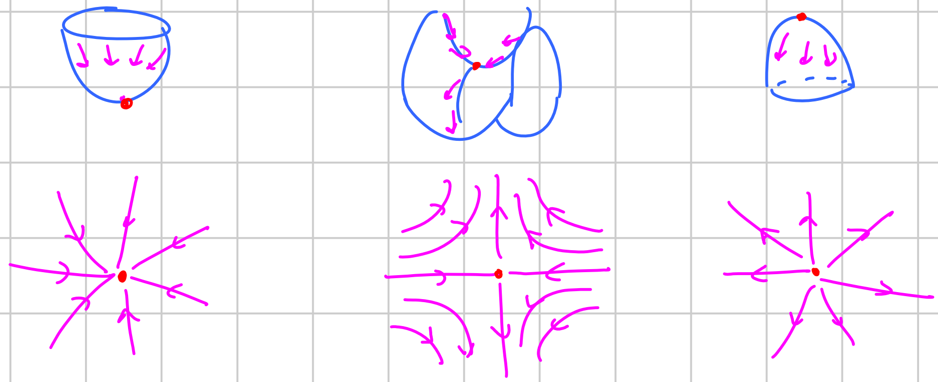
ex:



How to show this? <u>Gradient flow</u>.

$(M, \langle , \rangle)$ Riemannian, $f : M \to \mathbb{R}$. The gradient vector field ∇f is defined by $\langle \nabla_x f, v \rangle = df(v)$, $v \in T_x M$.

Traditional to use <u>negative gradient flow</u>: the flow of $-\nabla f$.



Note: at a critical pt p, $-\nabla_p f = 0$: nearby,

$$-\nabla_x f = 2\left(-x_1 \frac{\partial}{\partial x_1} - \cdots - x_k \frac{\partial}{\partial x_k} + x_{k+1} \frac{\partial}{\partial x_{k+1}} + \cdots + x_n \frac{\partial}{\partial x_n}\right).$$

<u>Def</u> $x \in M$. The <u>gradient flow line</u> $\gamma_x(t)$ is the flow of $-\nabla f$:

$$\gamma_x(0) = x$$
$$\frac{d}{dt} \gamma_x(t) = -\nabla_{\gamma_x(t)} f.$$

<u>Prop</u> M compact, f Morse. Every flow line begins and ends at a crit pt of f: $\exists$ p, q crit pts with

$$\lim_{t \to -\infty} \gamma_x(t) = p, \quad \lim_{t \to \infty} \gamma_x(t) = q.$$

<u>Pf</u> $\gamma_x(t)$ is defined $\forall t$ and

$$\frac{d}{dt} f(\gamma_x(t)) = df_{\gamma_x(t)}(-\nabla_{\gamma_x(t)} f)$$
$$= -\|\nabla_{\gamma_x(t)} f\|^2.$$

Since $\{f(\gamma_x(t))\}_{t\in\mathbb{R}}$ is bounded, $\lim_{t\to\pm\infty} \frac{d}{dt} f(\gamma_x(t)) = 0$
$\Rightarrow \|\nabla_{\gamma_x(t)} f\| \to 0$. If $t_n \to \infty$ then $\{\gamma_x(t_n)\}$ has a limit pt q, which must then be a crit pt of f. Since crit pts are isolated, easy to check that $\lim_{t\to\infty} \gamma_x(t) = q$. Similarly for $t\to-\infty$. □

Define φ_t = time t flow for $-\nabla f$: $\varphi_t : M \to M$, $\varphi_t(x) = \gamma_x(t)$.
Then φ_t is a diffeo for each t. $(\varphi_t \circ \varphi_{-t} = id)$.

Fund. thm. of Morse thy. I: If f is Morse (and $f^{-1}([a,b])$ is compact) and has no critical values in $[a,b]$, then $M^a \cong M^b$ (diffeo).

Pf Define $\rho : M \to \mathbb{R}$ by

$$\rho(x) = \begin{cases} -\dfrac{1}{\|\nabla_x f\|^2} & x \in f^{-1}([a,b]) \\ 0 & \text{outside cpt set containing } f^{-1}([a,b]). \end{cases}$$

Write $\psi_t : M \to M$ for time t flow of $\rho \nabla f$: then
$$\frac{d}{dt} f(\psi_t x) = -1$$
So $f(\psi_t x) = -t + f(x)$. Thus ψ_{b-a} sends M^b onto M^a. □

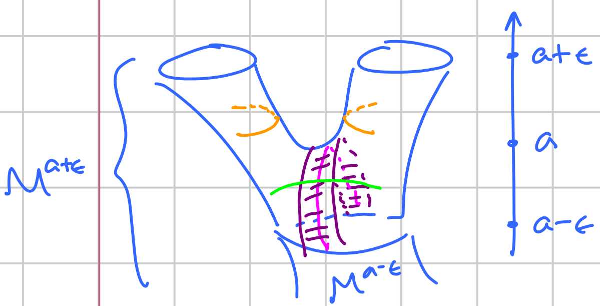
(in fact M^b deformation retracts onto M^a).

Fund Thm of Morse Thy 2: Sppse $f^{-1}([a-\epsilon, a+\epsilon])$ is cpt and contains one critical point p of index k. Then
$$M^{a+\epsilon} \underset{\text{homotopic}}{\sim} M^{a-\epsilon} \cup (k\text{-cell}).$$
D^k

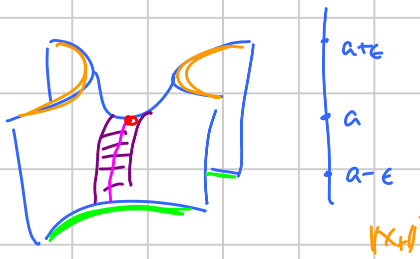
In fact: $M^{a+\epsilon} \underset{\text{diffeo}}{\cong} M^{a-\epsilon} \cup (k\text{-handle})$.

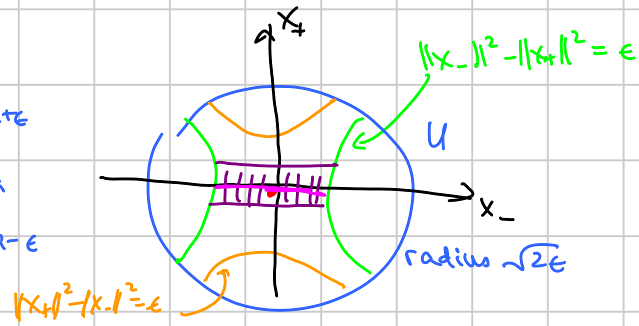
$\uparrow$ $D^k \times D^{n-k}$, glued along $\partial D^k \times D^{n-k}$

Outline of pf. Restrict to a Morse chart around p.



Use gradient flow to retract $M^{a+\epsilon}$ onto $M^{a-\epsilon} \cup k$-cell.





By shrinking ϵ if necessary, suppose the Morse chart is $B(0,\sqrt{2\epsilon}) = U$, on which $f = a - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$, $x_- = (x_1, \ldots, x_k)$, $x_+ = (x_{k+1}, \ldots, x_n)$.

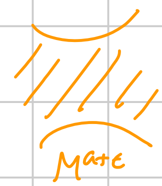
$$= a - \|x_-\|^2 + \|x_+\|^2.$$

Define

$$D^k = \{ x_+ = 0, \|x_-\|^2 < \epsilon \}.$$

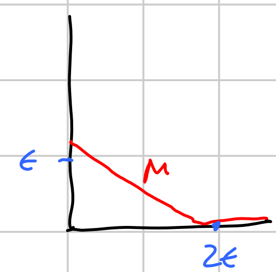
Claim:

$$M^{a+\epsilon} \sim M^{a-\epsilon} \cup D^k.$$



More precisely: define $\mu : [0,\infty) \to [0,\infty)$ by

(need $-1 < \mu'(t) \le 0$)



and then define $F : M \to \mathbb{R}$ by

$$F = \begin{cases} f & \text{outside } U \\ f - \mu(\|x_-\|^2 + 2\|x_+\|^2) & \text{inside } U. \end{cases}$$

Straightforward to check:

- $F = f$ except in nbd of p
- F has the same critical pts as f:

write $s = \|x_+\|^2$, $t = \|x_-\|^2$. Then $F = a + s - t - \mu(t+2s)$

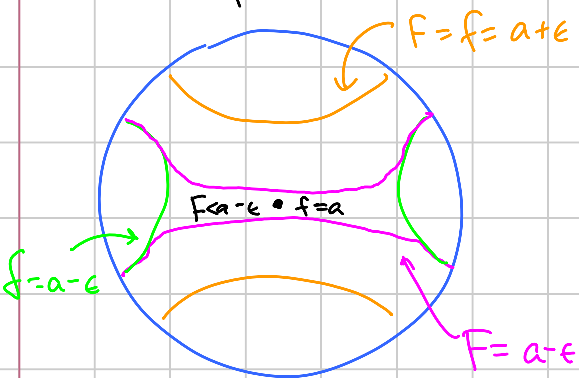
$$\Rightarrow \quad \frac{\partial F}{\partial s} = 1 - 2\mu'(t+2s) \geq 1$$

$$\frac{\partial F}{\partial t} = -1 - \mu'(t+2s) < 0$$

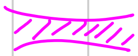
so $\quad dF = \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial t} dt = 0 \Rightarrow ds = dt = 0 \Rightarrow x_+ = x_- = 0$

$\Rightarrow$ in $\mathcal{U}$, only critical pt of F is p.

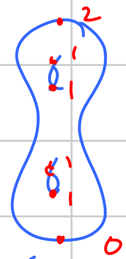
- $F(p) = a - \epsilon$.
- $F^{-1}(a+\epsilon) = f^{-1}(a+\epsilon)$.

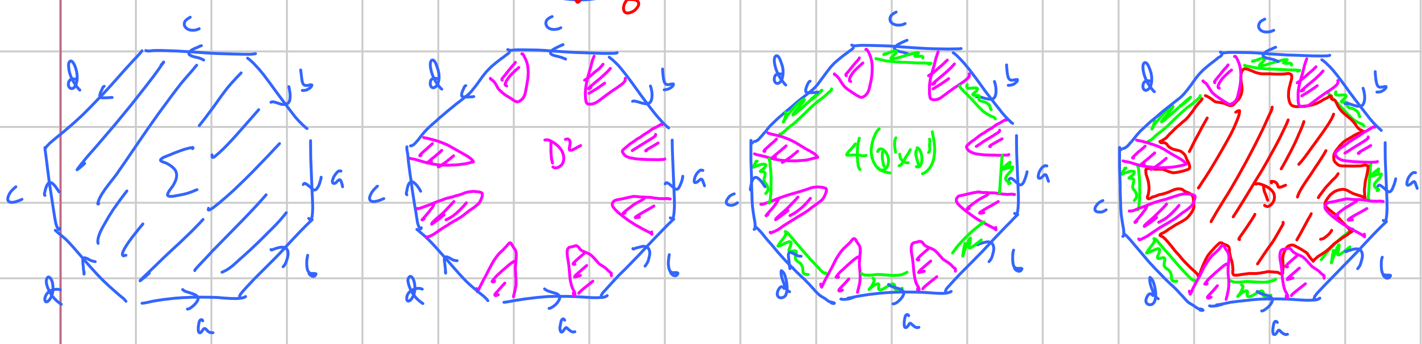


Then F has no critical values in $[a-\epsilon, a+\epsilon]$, so by previous prop, $\{F < a+\epsilon\} = M^{a+\epsilon}$ is diffeo to $\{F < a - \epsilon\}$.

Now $\quad \{F < a-\epsilon\} = \underbrace{\{f < a - \epsilon\}}_{M^{a-\epsilon}} \cup$ 

and one can construct a def. retract to $M^{a-\epsilon} \cup (k\text{-cell})$ by hand. $\square$

Another ex: $\Sigma =$  $= (0\text{-handle}) \cup 4\,(1\text{-handles}) \cup (2\text{-handle})$.

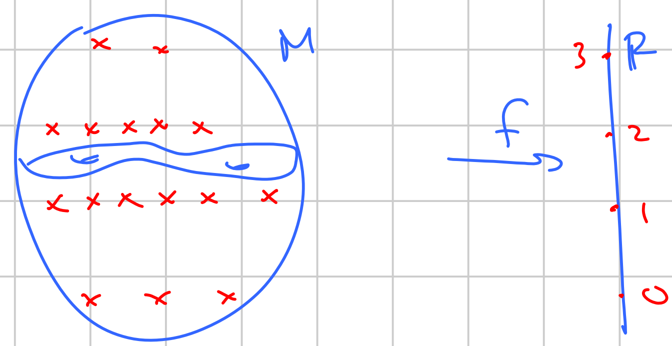


Application to 3-mfds:

<u>Def</u> A Morse function $f: M \to \mathbb{R}$ is <u>self-indexing</u> if each critical pt p satisfies $f(p) = \operatorname{ind}(p)$.

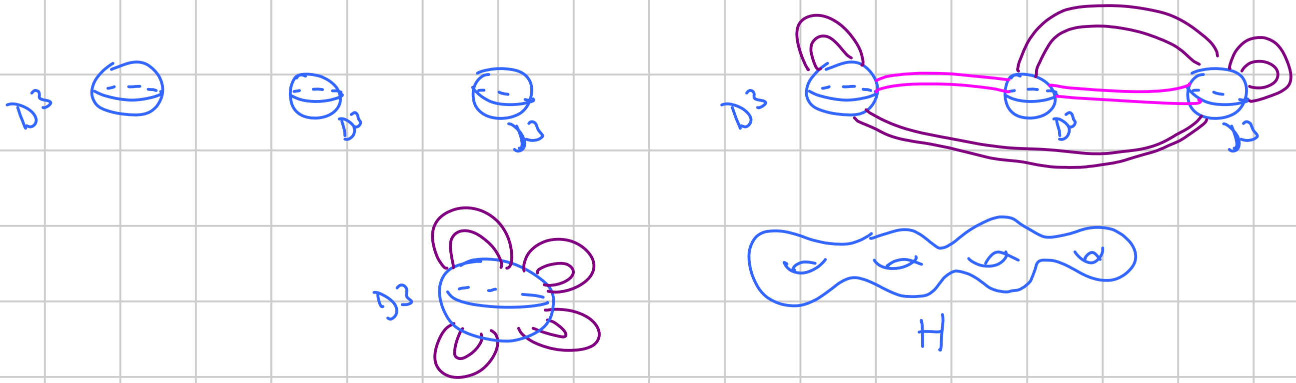
<u>Prop</u> (Smale) M cpt. Then $\exists$ self-indexing Morse fn. on M.
(later)

So: suppose M^3 cpt 3-mfd. Let $f: M \to \mathbb{R}$ be self-indexing.



note $f^{-1}(3/2)$ = smooth surface.

Look at $\{f \le 3/2\} =: H$. This is $(0\text{-handles}) \cup (1\text{-handles})$:



If H is orientable, then it only depends (up to diffeo) on the # of dark purple 1-handles: it's a <u>handlebody with g handles</u>.

<u>Prop</u> Any smooth compact connected orientable 3-mfd M has a <u>Heegaard decomposition</u>:

$$M = H_1 \cup H_2$$

glued along $\partial H_1 = \partial H_2 =$ surface of genus g.

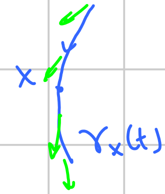
<u>Pf</u> $\{f \leq 3/2\}$, $\{f \geq 3/2\}$ are handlebodies. $\square$

Cor of Fund Thm:

<u>Thm</u> $f: M \to \mathbb{R}$ Morse, exhausting. Then M has the homotopy type of a CW complex with one cell of dim k for each critical pt of index k.

More on gradient flows.

$M = $ cpt.

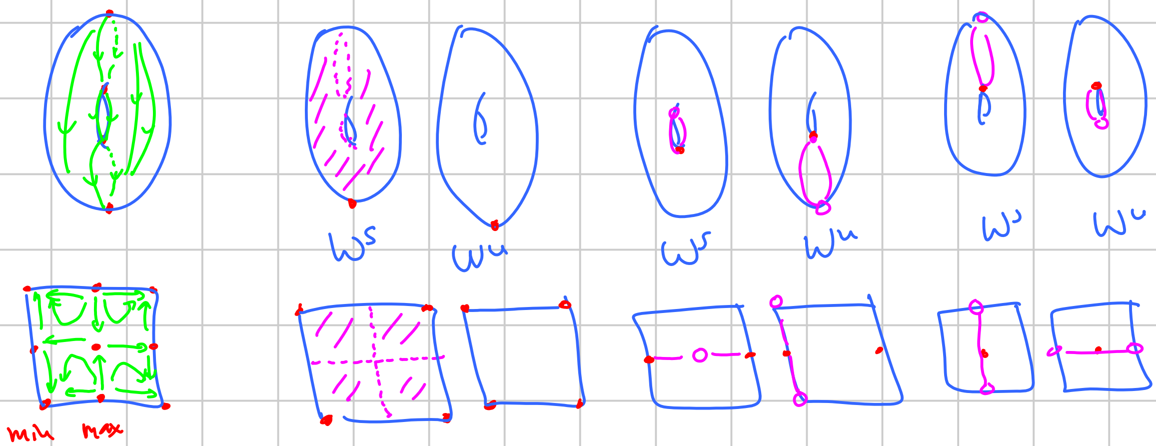


Recall $\varphi_t = $ time t flow for $-\nabla f$: $\varphi_t(x) = \gamma_x(t)$.

Also $\lim\limits_{t\to\pm\infty} \varphi_t(x)$ are critical pts of f.

Def p = critical point of f. The stable/unstable manifolds of p are

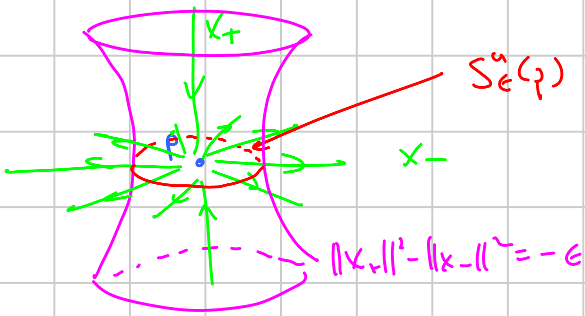
$$W^s(p) = \{x \in M \mid \lim_{t\to\infty} \varphi_t(x) = p\}$$
$$W^u(p) = \{x \in M \mid \lim_{t\to-\infty} \varphi_t(x) = p\}.$$

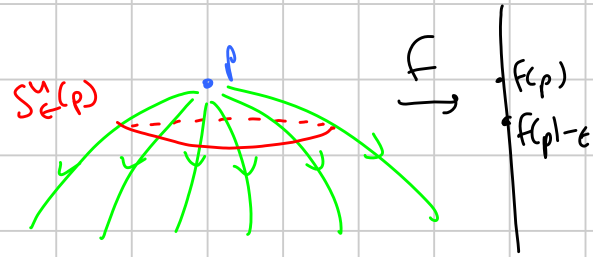


Note $\bigcup\limits_p W^s(p) = \bigcup\limits_p W^u(p) = M$. (and all W^s, W^u are disjoint).

Thm $\mathrm{ind}(p)=k$. $W^s(p)$, $W^u(p)$ are smoothly embedded disks of dim $n-k$, k respectively. (i.e. $W^s(p) \cong \mathbb{R}^{n-k}$, $W^u(p) \cong \mathbb{R}^k$).

Idea of pf: Write $S^u_\epsilon(p) = W^u(p) \cap \{f = f(p) - \epsilon\}$, and use Morse lemma. Then $S^u_\epsilon(p) \cong S^{k-1}$.





Every nonconstant gradient trajectory from p must pass through $S^u_\epsilon(p)$ exactly once $\Rightarrow$ $W^u(p) = p \cup (S^{k-1} \times \mathbb{R}) = D^k$.

More precisely: let (r,θ) be polar coordinates on $\mathbb{R}^k$, $r \in \mathbb{R}^+$, $\theta \in S^{k-1}$.
View $\theta \in S^u_\epsilon(p)$, and define $F: \mathbb{R}^k \to S^u(p)$

$$F(r,\theta) = \varphi_{\log r}(\theta) = \gamma_\theta(\log r).$$

This is a diffeomorphism.

Same argument upside down for $W^s(p)$. $\square$

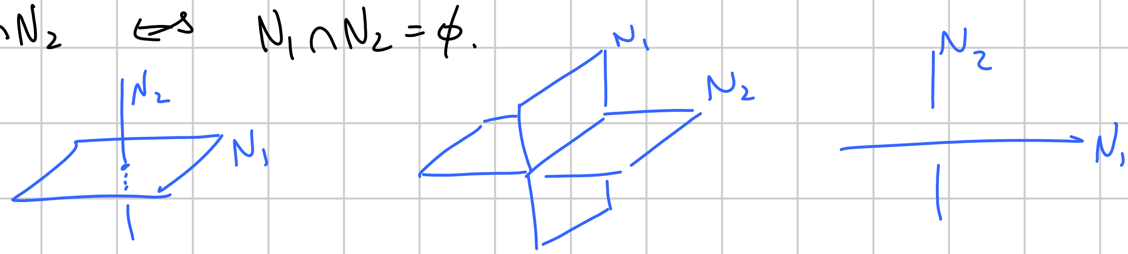
Note: $W^u(p)$, $W^s(p)$ intersect at p, and they intersect transversely: $T_p W^u(p) = T_p^u M$, $T_p W^s(p) = T_p^s M$
where $T_p M = \underset{\substack{\uparrow \\ Hf_p \text{ is } - \text{definite}}}{T_p^u M} \oplus \underset{\substack{\uparrow \\ Hf_p \text{ is } + \text{definite.}}}{T_p^s M}$.

Def $N_1, N_2 \subset M$ submfds. N_1 intersects N_2 transversely, written $N_1 \pitchfork N_2$, if $\forall x \in N_1 \cap N_2$,

$$T_x N_1 + T_x N_2 = T_x M.$$

(alt: $(n - \dim T_x N_1) + (n - \dim T_x N_2) = (n - \dim(T_x N_1 \cap T_x N_2))$).

<u>Note</u>: in particular, if $\dim N_1 + \dim N_2 < n$, then
$N_1 \pitchfork N_2 \iff N_1 \cap N_2 = \phi$.



<u>Prop</u> If $N_1 \pitchfork N_2$ then $N_1 \cap N_2$ is a smooth submanifold of dimension $\dim N_1 + \dim N_2 - n$.

(follows from: inverse image of a regular value is a smooth submfd).

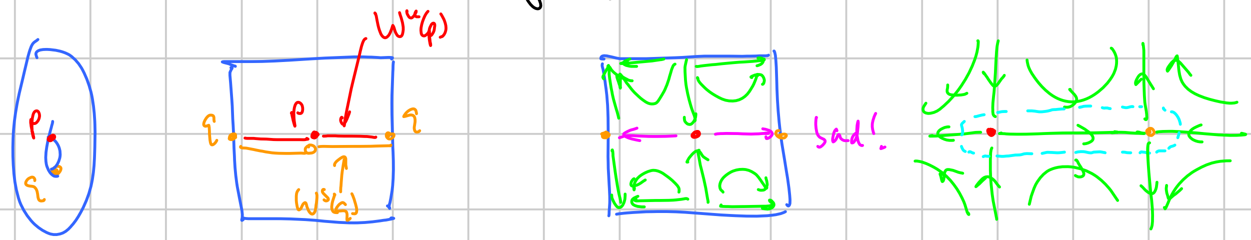
Note gradient flow $\Rightarrow$ stable/unstable mfds depend on the metric.

<u>Def</u> A pair (f, g) on M, $f: M \to \mathbb{R}$, g = metric, is <u>Morse-Smale</u> if
- f = Morse
- $\forall\, p, q$ critical pts of f,
  $W^u(p) \pitchfork W^s(q)$.

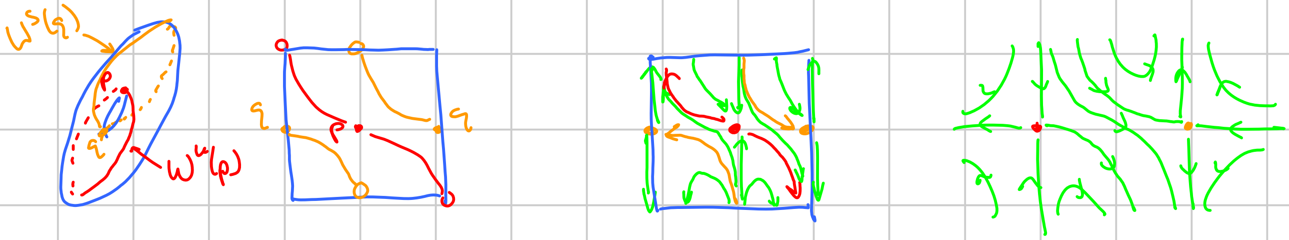
Then $W^u(p) \cap W^s(q)$ is a submanifold of dim
$\mathrm{ind}(p) + (n - \mathrm{ind}(q)) - n = \mathrm{ind}(p) - \mathrm{ind}(q)$.

<u>Note</u>: always true when $p = q$; trivially true if $f(p) < f(q)$.

With the standard metric: height function on T^2 is <u>not</u> Morse-Smale.



But perturb slightly:



<u>Thm</u> (Smale, Kupka 1963) The set of Morse-Smale pairs (f,g) is dense in $C^\infty(M) \times \mathrm{Met}(M)$.

So in what follows, we'll start with a Morse-Smale pair.

One consequence:

<u>Prop</u> Any compact M has a self-indexing Morse function.

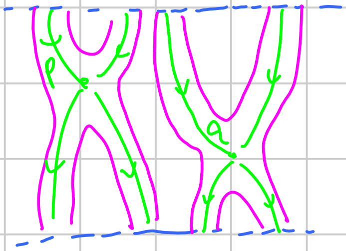
The key:

<u>Rearrangement Lemma</u> (f,g) Morse-Smale, critical points p,q with $a < f(q) < f(p) < b$, p,q only crit pts in $f^{-1}([a,b])$,

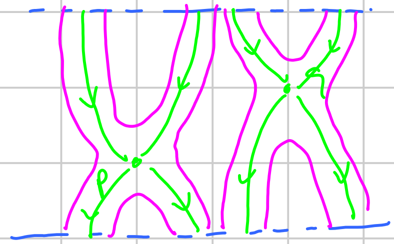
$$W^u(p) \cap W^s(q) \cap f^{-1}([a,b]) = \phi.$$

Then for any $a', b' \in [a,b]$, $\exists\ \tilde{f}$ s.t.

- $(\tilde{f}, g)$ is Morse-Smale
- $\tilde{f} = f$ outside $f^{-1}([a,b])$
- p,q only crit pts of $\tilde{f}$ in $f^{-1}([a,b])$
- $g(p) = a'$, $g(q) = b'$.



only change f in $\mathrm{nbd}(W^u(p)\cup W^s(p))$ and $\mathrm{nbd}(W^u(q)\cup W^s(q))$ $\longrightarrow$



Then by Morse-Smale Condition, whenever $\mathrm{ind}(p) > \mathrm{ind}(q)$ but $f(p) \leq f(q)$, we have $W^s(p) \cap W^u(q) = \emptyset$ so we can change f to swap $f(p)$ and $f(q)$.

$\Rightarrow$ self-indexing.

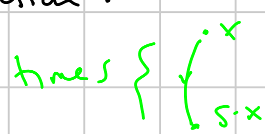
Toward Morse Homology

(f,g) Morse-Smale, $p,q \in \mathrm{Crit}\, f$. Define

$$W(p,q) = W^u(p) \cap W^s(q)$$
$$= \{x \in M : \lim_{t\to-\infty} \gamma_x(t) = p,\ \lim_{t\to\infty}\gamma_x(t) = q\}.$$

Note: $\mathbb{R}$ acts on $W(p,q)$ by time translation:

$$s \cdot x = \varphi_s(x) = \gamma_x(s).$$



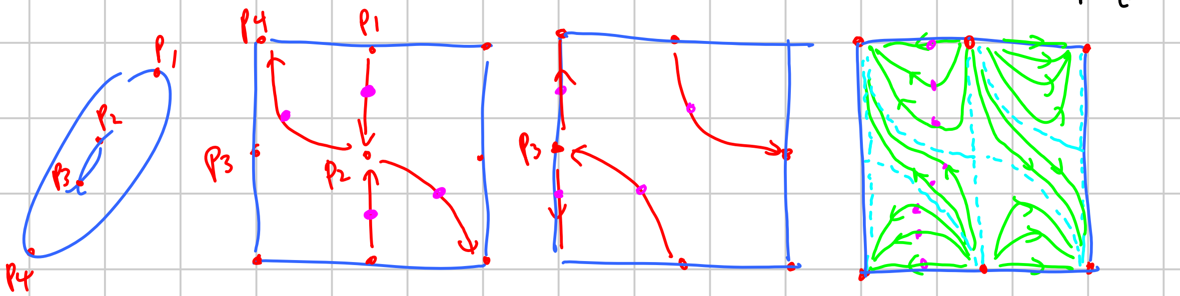
$s \mapsto \varphi_s$.

Furthermore, if $p \neq q$ then this action is free.

Def $\mathcal{M}(p,q) = W(p,q)/\mathbb{R}$. (moduli space of flow lines from p to q, mod translation).

This is a smooth mfd of dim $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$. (except $p=q$)

Ex.



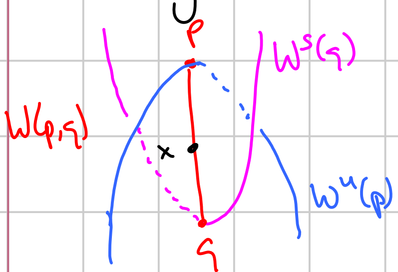
$\mathcal{M}(p_1,p_2) = \{2 \text{ pts}\}$ $\quad \mathcal{M}(p_1,p_3) = \{2 \text{ pts}\}$ $\quad \mathcal{M}(p_1,p_4) = 4\times \longleftrightarrow$

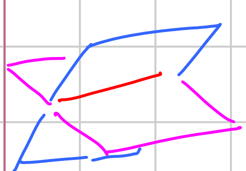
$\mathcal{M}(p_2,p_4) = \{2 \text{ pts}\}$ $\quad \mathcal{M}(p_3,p_4) = \{2 \text{ pts}\}$

Orientations

We can orient $\mathcal{M}(p,q)$ in a "consistent" manner $\forall\, p,q$ by choosing an orientation of $W^u(p)$ $\forall\, p \in \operatorname{Crit} f$, and then:

for $x \in W(p,q)$, γ = flowline through $x \in \mathcal{M}(p,q)$,



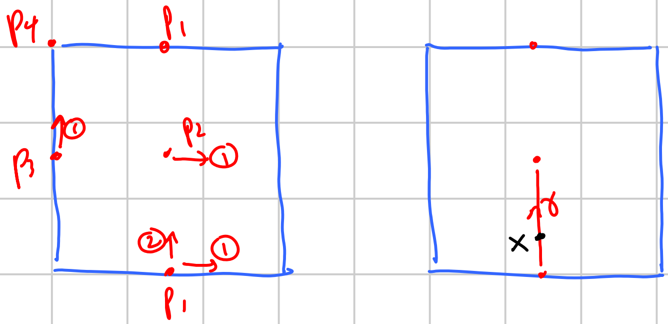


$$
\begin{aligned}
T_pW^u(p) &\cong T_xW^u(p) \\
&\cong T_x\left(W^u(p)\cap W^s(q)\right) \oplus \left(T_x W^s(q)\right)^\perp \\
&\qquad \text{(by transversality)} \\
&\cong \left(T_\gamma \mathcal{M}(p,q) \oplus T_x\gamma\right) \oplus \left(T_q W^s(q)\right)^\perp \\
&\cong T_\gamma \mathcal{M}(p,q) \oplus \mathbb{R} \oplus T_qW^u(q).
\end{aligned}
$$

Orient $\mathcal{M}(p,q)$ so that this isomorphism is orient-preserving.

Note: doesn't require an orientation on M.

Ex



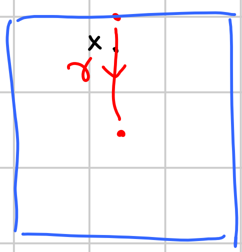
$T_x W^u(p_1)$: ② ①

$T_\gamma \mathcal{M}(p_1,p_2)$: pt

$\mathbb{R}$: ↑

$(T_x W^s(p_2))^\perp$: →

So: this point in $\mathcal{M}(p_1,p_2)$ is oriented with $-$.



$T_x W^u(p_1)$: ② ①

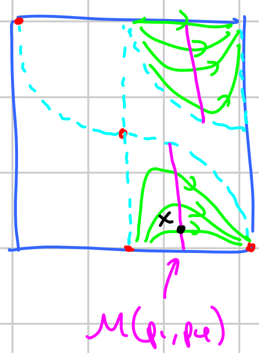
$T_\gamma \mathcal{M}(p_1,p_2)$: pt

$\mathbb{R}$: ↓

$(T_x W^s(p_2))^\perp$: →

So: this point is oriented with $+$.

One more:



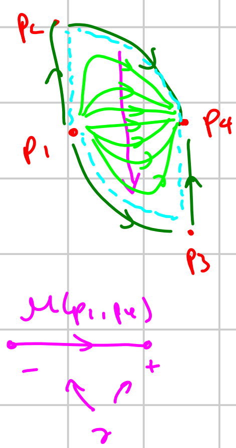
$T_x W^u(p_1)$: ② ①

$T_\gamma \mathcal{M}(p_1,p_2) \Longrightarrow \downarrow \mathcal{M}(p_1,p_4)$

$\mathbb{R}$: →

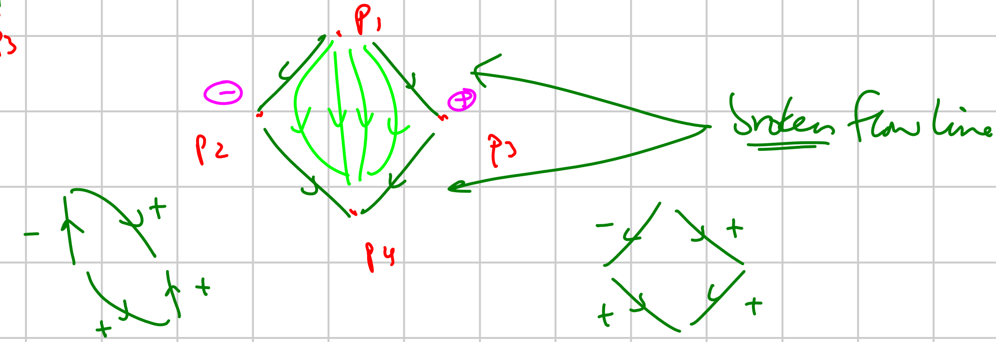
$(T_x W^s(p_4))^\perp$: .

The key to Morse homology: compactification by broken flow lines.



$\mathcal{M}(p_1,p_4) = \cup$ 4 open intervals; can compactify:

$\overline{\mathcal{M}}(p_1,p_4) = \cup$ 4 closed intervals.



Thm. 1. If $\operatorname{ind} p = \operatorname{ind} q + 1$ then
$\mathcal{M}(p,q)$ is compact (= finite union of points).

2. If $\operatorname{ind} p = \operatorname{ind} q + 2$ then $\mathcal{M}(p,q)$ has a compactification $\overline{\mathcal{M}}(p,q)$ = cpt 1-mfd with ∂ (= finite union of closed intervals and S^1's)
and

$$\partial \overline{\mathcal{M}}(p,q) = \bigcup_{\substack{\operatorname{ind} r = \operatorname{ind} q + 1 \\ = \operatorname{ind} p - 1}} \mathcal{M}(p,r) \times \mathcal{M}(r,q).$$

(even true with orientations!)

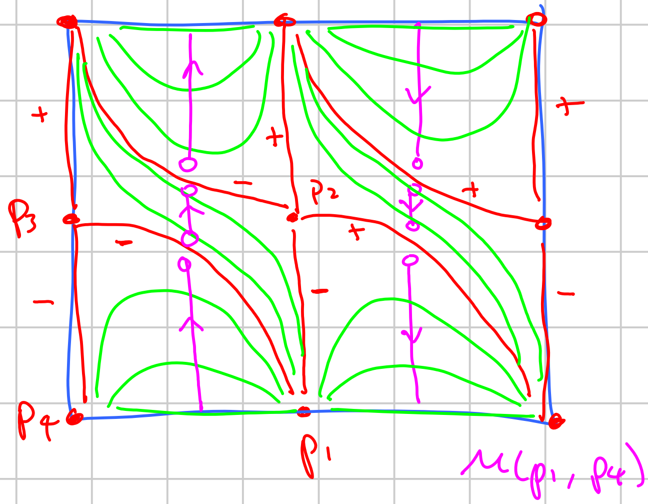
(In fact, in general $\overline{\mathcal{M}}(p,q)$ has a compactification to a mfd with corners whose strata are broken flow lines.)

Components of proof:

① any sequence of flowlines in $\mathcal{M}(p,q)$ has a subsequence converging to a broken flow line

② any broken flow line can be perturbed to an honest flow line: if the broken flow line has k intermediate crit pts, these perturbations are parametrized by $(\mathbb{R}^+)^k$.

③ orientations.



Morse Complex

$M, (f,g)$ Morse-Smale $\Rightarrow (CM_*(M;f,g), \partial)$.

Def $CM_k = \mathbb{Z}\,\mathrm{Crit}_k(f)$ $\qquad \mathrm{Crit}_k(f) = \{\text{crit pts of index } k\}$

$\partial : CM_k \to CM_{k-1}$ defined by

$$\partial(p) = \sum_{q \in \mathrm{Crit}_{k-1}(f)} \underbrace{(\#\mathcal{M}(p,q))}_{\in \mathbb{Z}:\ \#\text{ of pts in } \mathcal{M}(p,q),\text{ counted with sign.}} q$$

This is the Morse complex of M associated to (f,g).

Prop $\partial^2 = 0$.

Pf. $p \in \mathrm{Crit}_k f$, $q \in \mathrm{Crit}_{k-2}(f) \Rightarrow$

$$\underbrace{\langle \partial p, q \rangle}_{\text{coeff of } q \text{ in } \partial p} = \sum_{r \in \mathrm{Crit}_{k-1}(f)} \langle \partial p, r \rangle \langle \partial r, q \rangle$$

$$= \# \bigcup_{r \in \mathrm{Crit}_{k-1}(f)} \mathcal{M}(p,r) \times \mathcal{M}(r,q)$$

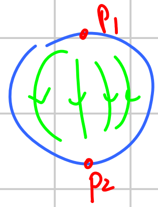
$$= \# \partial \overline{\mathcal{M}}(p,q)$$

$$= 0. \qquad \square$$

Def The Morse homology $HM_*(M;f,g) = H_*(CM_*(M;f,g), \partial)$.

(note depends on (f,g)).

Ex. 1. $M = S^2$. $\qquad CM_* = \mathbb{Z}\langle \overset{2}{p_1}, \overset{0}{p_2} \rangle$, $\partial = 0$



$$HM_* = \begin{cases} \mathbb{Z} & * = 2 \\ \mathbb{Z} & * = 0 \\ 0 & \text{else.} \end{cases}$$

$\qquad\qquad\qquad\qquad\qquad\quad 2 \quad 1 \quad 1 \quad 0$

Ex 2. $M = T^2$. $\quad CM_* = \mathbb{Z}\langle p_1, p_2, p_3, p_4 \rangle$

$\partial(p_1) = p_2 - p_2 + p_3 - p_3$

$\partial(p_2) = p_4 - p_4$

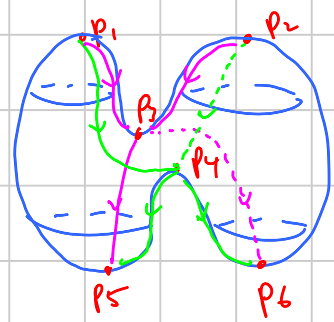
$\partial(p_3) = p_4 - p_4$

$\partial(p_4) = 0$

$$HM_* = \begin{cases} \mathbb{Z} & * = 2 \\ \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 0 \end{cases}.$$

note $\partial^2(p_1)$ has 8 cancelling terms corresponding to 4 components of $\mathcal{M}(p_1, p_4)$

Ex 3 S^2 again.



$CM_* = \langle p_1, p_2 \rangle \oplus \langle p_3, p_4 \rangle \oplus \langle p_5, p_6 \rangle$

$\partial p_1 = p_3 + p_4, \quad \partial p_2 = p_3 + p_4$

$\partial p_3 = p_5 + p_6, \quad \partial p_4 = -p_5 - p_6$

$HM_* = \mathbb{Z} \oplus 0 \oplus \mathbb{Z}$

Thm M = closed smooth mfd, (f, g) Morse-Smale. Then

$$HM_*(M; f, g) \cong H_*^{sing}(M).$$

There's a generalization to mfds with boundary.

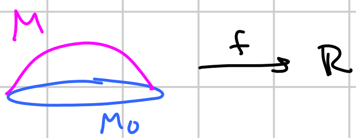
Say $\partial M = M_0 \amalg M_1$ and there's a Morse function $f: M \to \mathbb{R}$ st. $f(M) \subset [0,1]$, $f^{-1}(0) = M_0$, $f^{-1}(1) = M_1$, and all crit pts are nondeg. and in $\text{int}(M_0)$.

Let g be a metric st. (f, g) is Morse-Smale.

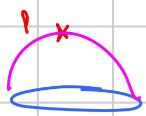
<u>Thm</u> $HM_*(M; f, g) \cong H_*(M, M_0)$.

Note asymmetry between M_0 and M_1.

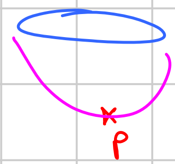
Special case: $M_1 = \phi$.



<u>Ex</u> $M = D^n$, $M_0 = S^{n-1}$.



ind(p) = n



ind(p) = 0

$$HM_* = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases} = H_*(D^n, S^{n-1}).$$

$$HM_* = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases} = H_*(D^n)$$

More generally:
If M is orientable, then $(CM_*(M; f, g), \partial)$ and $(CM_*(M; 1-f, g), \partial)$ are <u>duals</u>: same generators, $\deg p \to n - \deg p$, arrows are reversed.

$$\begin{array}{ccc} (CM^{n-*}(M; 1-f, g), \partial^*) & \cong & (CM_*(M; f, g), \partial) \\ \big\downarrow \text{homology} & & \big\downarrow \text{homology} \\ H^{n-*}(M) & & H_*(M, M_0) \end{array}$$

<u>Cor</u> (Poincaré duality) M orientable, dim n. Then
$$H^{n-*}(M) \cong H_*(M, \partial M).$$

Several possible approaches.

1. Cellular homology: by Fund Thm, $\exists$ CW-complex str for M with (crit pt of index k) $\longleftrightarrow$ (k-cell)

so $CM_k \cong C_k$ from cellular hom.

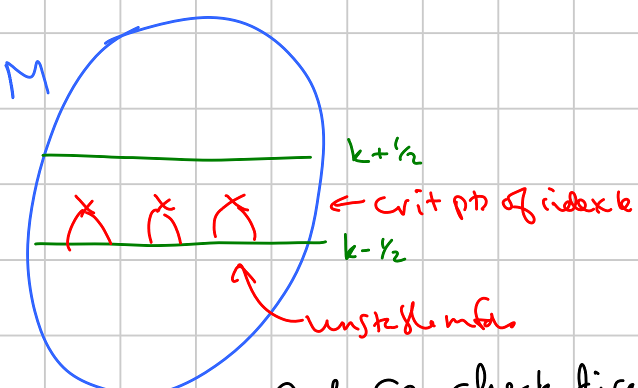
If this isom is a chain map, then done.

Problem: no clear reason it should be; also the CW complex is indirectly defined.

$\to$ Could try to use (unstable mfd of index k crit pt) $\leftarrow$ (k-cell) but this isn't a CW complex!

One correct fix: Let f be self-indexing: argue that we can reduce to this case by enlarging $(f,g) \to (f, X)$, X = gradient like vector field for f, and then (CM_*, ∂) depends only on X and not f.

c_k = # of crit pts of index k



$$M^{k+1/2} = M^{k-1/2} \cup (\text{unstable } k\text{-disks})^{c_k}$$

$$H_k(M^{k+1/2}, M^{k-1/2}) \cong \mathbb{Z}^{c_k} \cong CM_k$$

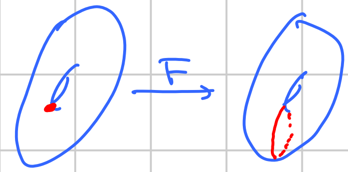
and can check directly that the map

$$H_k(M^{k+1/2}, M^{k-1/2}) \xrightarrow{\partial} H_{k-1}(M^{k-1/2}) \to H_{k-1}(M^{k-1/2, k-3/2})$$

agrees with the Morse differential.

(see Nicolaescu)

2. $CM_*(M;f,g) \underset{G}{\overset{F}{\rightleftarrows}} C_*^{sing}(M)$



$F(p) = W^u(p)$

$G(\Delta) =$ Sum of critical pts (of correct index) that negative gradient flow from Δ limits to.

$G \circ F = id$

$F \circ G - id = \partial H + H\partial$, $\qquad H: C_*^{sing}(M) \to C_{*+1}^{sing}(M)$

$$\Delta \longrightarrow \overline{\{\gamma_x(t) : x \in \Delta,\ t \in [0,\infty)\}}.$$

technicalities: need to restrict to singular simplices transverse to all W^s.

See Hutchings.

Some consequences.

1. $b_k = k$-th Betti number of $M = \dim_{\mathbb{Q}}(H_k(M) \otimes \mathbb{Q})$.

Thm $f =$ Morse function on M. Then

$(\#\text{critical points of index } k) \geq b_k$

$(\text{total } \# \text{ critical points of } f) \geq \sum_{k=0}^{n} b_k$.

Pf Find a metric g s.t. (f,g) is Morse-Smale (needs proof). Then

$$\begin{aligned}(\#\text{critical points of index } k) &= \operatorname{rank} CM_k(f,g) \\ &\geq \operatorname{rank} HM_k(f,g) \\ &= b_k. \qquad \square\end{aligned}$$

$\exists$ strengthening: (strong) Morse inequalities.

f Morse, $c_k = \#$ of crit pts of index k. weak: $c_k \geq b_k$.

Strong: $\sum_{j=0}^{k} (-1)^{k-j} c_k \geq \sum_{j=0}^{k} (-1)^{k-j} b_k$.

Nicer way to say this:

define $P_t(M) = \sum_{k=0}^{n} b_k t^k$ Poincaré polynomial of M

$M_t(f) = \sum_{k=0}^{n} c_k t^k$ "Morse polynomial"

Then $\exists\, Q(t) = \sum_{k=0}^{n-1} q_k t^k$, $q_k \geq 0\ \forall k$, s.t.

$$M_t(f) = P_t(M) + (1+t)Q(t).$$

2. Poincaré-Hopf index thm.

v = vector field on M, isolated zeros.

If $v(p) = 0$ then $v(x) \neq 0$ on $S^{n-1}_\epsilon(p)$

$\Rightarrow$ get a map $S^{n-1} = S^{n-1}_\epsilon(p) \longrightarrow \mathbb{R}^n - 0 \longrightarrow S^{n-1}$.

Define $\mathrm{ind}_v(p)$ = degree of this map.

Thm (Poincaré-Hopf) $\sum_{\text{zeros } p \text{ of } v} \mathrm{ind}_v(p) = \chi(M)$.

$\left(= \sum_k (-1)^k b_k(M)\right)$

Idea of pf: ① LHS indep of v: it's the degree of the Gauss map:

$M \hookrightarrow \mathbb{R}^N \rightsquigarrow$ degree $(\partial(\mathrm{nbd}\, M) \to S^{N-1})$.

② Choose $v = -\nabla f$. Zeros of $v \leftrightarrow$ critical pts of f

$\mathrm{ind}_p(f) = k \Rightarrow \mathrm{ind}_p(v) = (-1)^k$ (local computation)

$\Rightarrow$ $\sum \mathrm{ind}_v(p) = \sum_k (-1)^k \operatorname{rank} CM_k = \sum_k (-1)^k \operatorname{rank} HM_k = \chi(M)$.

$\square$

Generalization: <u>Lefschetz fixed point thm</u>.

$f: M \to M$ continuous.

<u>Lefschetz number</u> $L(f) = \sum (-1)^k \operatorname{Tr}\left(f_*: H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})\right)$.

$p \in M$ is a <u>fixed point</u> of f if $f(p) = p$; <u>nondegenerate</u> if $\Gamma(f) \pitchfork \Delta$ at (p,p) ($\Leftrightarrow \det(df_p - id) \neq 0$).

p fixed point $\rightsquigarrow \operatorname{ind}_f(p) \in \mathbb{Z}$: this is ± 1 for nondeg.

<u>LFPT</u> $\displaystyle\sum_{p \in Fix(f)} \operatorname{ind}_f(p) = L(f)$ M compact smooth oriented.

if all nondeg, then $\# Fix(f) \geq |L(f)|$.

<u>Special case</u>: if $f \sim id$, then $L(f) = L(id) = \chi(M)$

$\Rightarrow \# Fix(f) \geq \chi(M)$.

<u>Special special case</u>: V = vector field on M, $f = \varphi_\epsilon : M \to M$ time ϵ flow. Then fixed pt of f = zero of V, and $\operatorname{ind}_f(p) = \operatorname{ind}_V(p) \Rightarrow$ Poincaré-Hopf.

A priori invariance

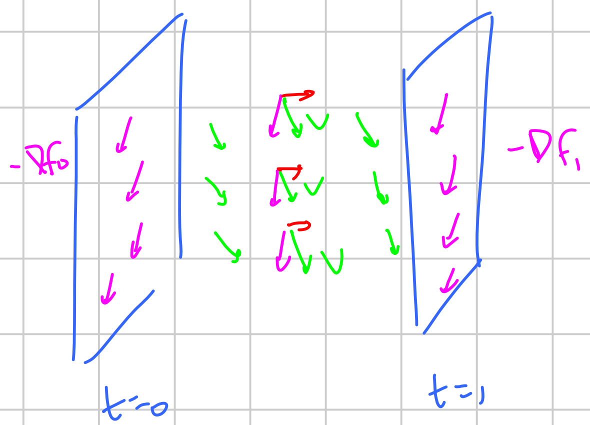
Suppose we didn't know $HM_* \cong H_*^{sing}$. We'd still deduce some interesting inv't of M, provided $HM_*(M;f,g)$ indep of (f,g).

This can be proven directly by
- bifurcation: how CM_* changes
- continuation (Floer).

Suppose (f_0,g_0), (f_1,g_1) are Morse-Smale, and let (f_t,g_t) be a path between them.

Define V = vector field on $[0,1]\times X$ by

$$V = (1-t)t(1+t)\frac{\partial}{\partial t} + V_t \qquad V_t = -\nabla_{g_t} f_t .$$



One can construct "critical pts" = zeros, stable/unstable mfds for V just as for $-\nabla f$. (V is "gradient-like")

$$\mathrm{Crit}(V) = \left(\{0\}\times \mathrm{Crit}(f_0)\right) \cup \left(\{1\}\times \mathrm{Crit}(f_1)\right)$$

index k+1 $\leftarrow$ index k

index k $\leftarrow$ index k

Fact: for generic (f_t, g_t), stable/unstable mfds for V are transverse. (Caution: in general can't avoid times when (f_t, g_t) isn't Morse-Smale.)

$p, q \in \operatorname{Crit} V \rightsquigarrow \mathcal{M}(p,q) = \{\text{flowlines from } p \text{ to } q\}$.

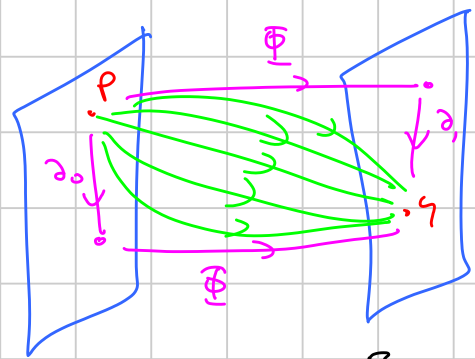
For $p \in \operatorname{Crit}(f_0)$ with $\operatorname{ind}(p) = k$, define

$$\Phi(p) = \sum_{\substack{q \in \operatorname{Crit}(f_1) \\ \operatorname{ind} q = k}} (\# \mathcal{M}(p,q))\, q.$$

Extend Φ linearly to get a map $\Phi : CM_k(f_0, g_0) \to CM_k(f_1, g_1)$.

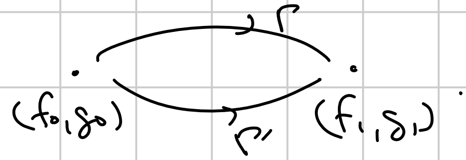
<u>Prop</u> Φ is a chain map: $\Phi \circ \partial_0 = \partial_1 \circ \Phi$.

<u>Pf</u> Usual degeneration argument.



$\square$

Next: let Γ, Γ' be two generic paths

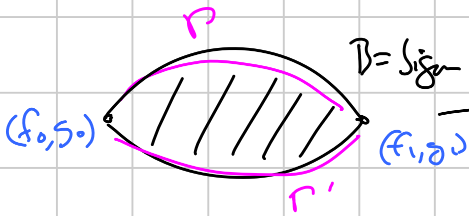


$$\rightsquigarrow \Phi_\Gamma, \Phi_{\Gamma'} : CM_*(f_0, g_0) \to CM_*(f_1, g_1).$$

<u>Prop</u> If Γ, Γ' are <u>homotopic</u> then $\Phi_\Gamma, \Phi_{\Gamma'}$ are chain homotopic:

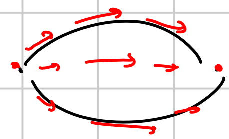
$$\exists K : CM_*(f_0, g_0) \to CM_{*+1}(f_1, g_1)$$
s.t. $$\partial_1 K + K \partial_0 = \Phi_\Gamma - \Phi_{\Gamma'}.$$

"Pf."



$\longrightarrow C^\infty(M) \times \mathrm{Met}(M)$

Define V on $B \times M$ by $V = V_B + V_b$

$V_b = -\nabla_g f$ at $b \in B$. $V_B =$ 

K counts flowlines of V. $\square$

Cor $HM_*(f,g)$ indep of (f,g).

Pf.

$$\underset{(f_0,g_0)}{\cdot} \xrightarrow{\Gamma} \underset{(f_1,g_1)}{\cdot} \xrightarrow{-\Gamma} \underset{(f_0,g_0)}{\cdot}$$

$\Gamma \cup (-\Gamma)$ homotopic to constant path.

$$\rightsquigarrow \quad \begin{tikzcd} HM_*(f_0,g_0) \arrow[r, "\Phi_\Gamma"] \arrow[rr, bend right, "\Phi_{const} = id"'] & HM_*(f_1,g_1) \arrow[r, "\Phi_{-\Gamma}"] & HM_*(f_0,g_0) \end{tikzcd}$$

$\square$

Floer Homology

General setup: adapt Morse theory to ∞ dim situations.

(geometric data) $\rightsquigarrow$ infinite-dim'l mfd $\mathcal{M}$,
action functional $f: \mathcal{M} \to \mathbb{R}$.

$p, q \in \mathcal{M}$ crit pts of f: if $\gamma: (-\epsilon, \epsilon) \to \mathcal{M}$ satisfies $\gamma(0) = p$ then $(f \circ \gamma)'(0) = 0$.

Define "gradient flow" for f on $\mathcal{M}$: note not always possible! Long time existence especially problematic.

$$W(p,q) = \{x \in \mathcal{M} : \lim_{t \to \infty} \gamma_x(t) = q, \lim_{t \to -\infty} \gamma_x(t) = p\}$$
$$= W^u(p) \cap W^s(q).$$

$\dim W(p,q) = \mathrm{ind}(p) - \mathrm{ind}(q)$?

Problem: in general both $W^u(p), W^s(p)$ are ∞ dimensional.

Nevertheless: strategy is to show that
$W(p,q) =$ inverse image of a regular value of some Fredholm operator $\mathcal{F}$ between Banach mfds
$\Rightarrow$ by Inverse Fn Thm, $W(p,q) =$ smooth finite-dim mfd of dimension $\mathrm{ind}\, \mathcal{F}$.

Now note we only need relative indices of crit pts: differences of indices.
In nice cases, can define $\mathrm{ind}: \mathrm{Crit}\, f \to \mathbb{Z}$ (or $\mathbb{Z}/m$)
s.t. $\mathrm{ind}\, \mathcal{F} = \mathrm{ind}\, p - \mathrm{ind}\, q$.

Then: • define $\mathcal{M}(p,q) = W(p,q)/\mathbb{R}$

• study compactification $\overline{\mathcal{M}}(p,q)$

• if $\operatorname{ind} p = \operatorname{ind} q + 1$, $\mathcal{M}(p,q) = \text{cpt}$

$\operatorname{ind} p = \operatorname{ind} q + 2$, $\overline{\mathcal{M}}(p,q) = \mathcal{M}(p,q) \cup \left(\bigcup_r \mathcal{M}(p,r) \times \mathcal{M}(r,q) \right)$

• Construct Floer complex

CF_* : $CF_k = \mathbb{Z}\langle \text{crit pts of index } k \rangle$

$\partial : CF_* \to CF_{*-1}$ $\quad \langle \partial p, q \rangle = \# \mathcal{M}(p,q)$.

• prove invariant under choices made along the way

$\Rightarrow$ $\underline{HF_*}$ <u>Floer homology</u>.

Instanton Floer Homology (Floer '85, Fintushel-Stern '90)

Setup: (Y^3, g) oriented Riemannian, $\begin{matrix} E \\ \downarrow \\ Y \end{matrix}$ any principal $SU(2)$ bundle (in fact, any such is $Y \times SU(2)$). (also works for $SO(3)$)

$$\mathcal{A} = \{\text{connection on } E\}$$
$$\circlearrowleft$$
$$\mathcal{G} = \text{gauge group} = \mathrm{Aut}(E).$$

$\mathcal{M}(\ ) = \mathcal{A}/\mathcal{G}$ = gauge equivalence class of connections on E.
↳ (in fact need to remove 0 connection)

Chern-Simons functional $CS : \mathcal{A} \to \mathbb{R}$: $\mathcal{A} \leftrightarrow \Omega^1(Y, su(2))$

$$CS(A) = \frac{1}{8\pi^2} \int_Y tr\left(\theta \wedge d\theta + \frac{2}{3} \theta \wedge \theta \wedge \theta\right) \qquad \theta \in \Omega^1(Y, su(2))$$

Under gauge transformation ($\leftrightarrow$ map $Y \to SU(2)$),
$$CS(g^*(A)) = CS(A) + \deg g$$
so CS descends to $CS : \mathcal{M}(Y) \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$.

Crit pts of CS = flat connections : $F_A = 0$.

Metric on $\mathcal{A}$: for $\theta_1, \theta_2 \in T_A\mathcal{A} = \Omega^1(Y, su(2))$, define
$$\langle \theta_1, \theta_2 \rangle = \int_Y tr(\theta_1 \wedge * \theta_2)$$
$*$ = Hodge star on Y.

Gradient of CS : $\quad \mathrm{grad}\, CS(A) = * F_A$

So negative gradient flow on $\mathcal{A}$ is
$$\frac{dA}{dt} + * F_A = 0.$$

View this as a connection on $\begin{matrix} E\times\mathbb{R} \\ \downarrow \\ Y\times\mathbb{R}\end{matrix}$: $A(t)+\frac{\partial}{\partial t} = \tilde{A}$.

The gradient flow eq becomes

$$\boxed{F_{\tilde{A}} + * F_{\tilde{A}} = 0}$$

$* =$ Hodge star on $Y\times\mathbb{R}$.

instanton (ASD) equation.

<u>Fact</u>: Given an instanton $\tilde{A}$ on $P\times\mathbb{R}$,
$\lim_{t\to\pm\infty} \tilde{A}$ exist and give flat connections $A^{\pm}$ on P
$\Leftrightarrow$ $\tilde{A}$ has <u>finite energy</u>:

$$\mathcal{E}(\tilde{A}) = \int_{Y\times\mathbb{R}} tr(F_{\tilde{A}} \wedge * F_{\tilde{A}}) < \infty.$$

<u>So</u>: Let $\mathcal{R}(Y) =$ gauge equivalence classes of flat $SU(2)$ connections on Y.
(reps $\pi_1(Y)\to SU(2)$)

$A^+, A^- \in \mathcal{R}(Y) \rightsquigarrow$

$\mathcal{M}(A^+,A^-) =$ gauge equivalence class of connections $\tilde{A}$ on $P\times\mathbb{R}$ such that:

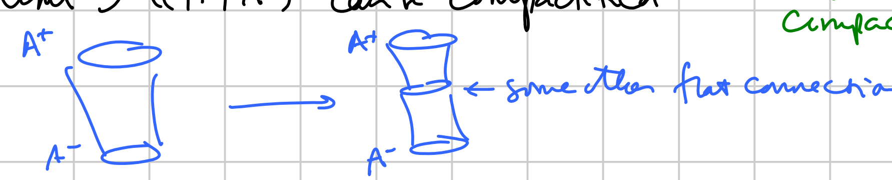
- $F_{\tilde{A}} + * F_{\tilde{A}} = 0$
- $\mathcal{E}(\tilde{A}) < \infty$
- $\lim_{t\to\pm\infty} \tilde{A} = A^{\pm}$.

Then: • $\exists\ \mu : \mathcal{R}(Y) \to \mathbb{Z}$ such that $\mathcal{M}(A^+,A^-)$ is fin-dim'l

(HARD) and $\dim \mathcal{M}(A^+,A^-) = \mu(A^+) - \mu(A^-)$

(roughly speaking, this difference is given by spectral flow:
# of eigenvalues of "Hessian" that change sign as we go from A^+ to A^-)

- $\mathcal{M}(A^+,A^-)$ can be oriented
- 1-dim'l $\mathcal{M}(A^+,A^-)$ can be compactified (Uhlenbeck compactness)



Complications:
- need to perturb everything to get transversality
- grading μ is really only defined in $\mathbb{Z}/8$ for gauge equivalence classes
- issue with reducible connections: $\mathcal{M}(Y)$ isn't a mfd.

  For $Y =$ homology S^3, can just delete 0 from $\mathcal{M}(Y)$.

Thm (Floer) $Y =$ homology sphere. Can construct a complex

(CF_*, ∂) $\qquad$ $* \in \mathbb{Z}/8$, $CF_* = \mathbb{Z}\langle A \in \mathcal{R}(Y) \rangle$, ∂ counts $\mathcal{M}(A^+, A^-)$.

such that $\partial^2 = 0$ and the homology $HF_*(Y)$ is independent of the choice of metric: <u>instanton Floer homology</u>.

The Euler characteristic $\sum_{k \in \mathbb{Z}/8} (-1)^k \operatorname{rk} HF_k(Y)$ is $2\lambda(M)$,

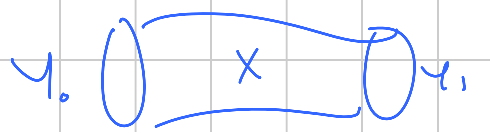
$\lambda(Y) =$ <u>Casson invt</u>:

$Y = H_1 \cup_\Sigma H_2$ Heegaard splitting, $\mathcal{R}(H_1), \mathcal{R}(H_2) \longrightarrow \mathcal{R}(\Sigma)$

$\lambda(Y) = \#(\mathcal{R}(H_1) \cap \mathcal{R}(H_2))$. $\qquad$ ($SU(2)$ reps/conjugacy)

So $HF_*(Y)$ <u>categorifies</u> the Casson invt.

(but behaves functorially! — Donaldson)



$\rightsquigarrow \Phi_X : HF_*(Y_0) \to HF_*(Y_1)$.

Kronheimer–Mrowka (published 2007):

<u>monopole</u> (<u>Seiberg–Witten</u>) Floer homology.

functional: Chern–Simons–Dirac functional

generators of CM_*: solutions of SW eqns on Y

differential: SW monopoles on $Y \times \mathbb{R}$.

Symplectic Floer Homology

Background: Arnold Conjecture.

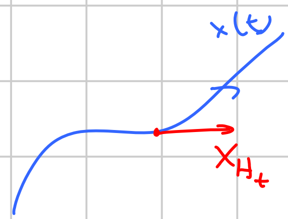
(M,ω) compact symplectic mfd: $\dim M = 2n$, $\omega \in \Omega^2(M)$, closed ($d\omega = 0$) and nondegenerate (ω^n nowhere 0).

$H \in C^\infty(M) \rightsquigarrow$ Hamiltonian vector field $X_H \in \mathrm{Vect}(M)$ defined by $dH = X_H \lrcorner\, \omega$.

$$T_p^*M \xleftarrow{\cong} T_pM \quad \rightsquigarrow \quad \Omega^1(M) \cong \mathrm{Vect}(M)$$
$$\omega(v,\cdot) \longleftarrow\!\shortmid\; v \qquad\qquad dH \longmapsto X_H.$$

For $M = T^*N$ (phase space), flow under X_H is the trajectory of a particle subject to the Hamiltonian H.

Now let H_t = time-dependent Hamiltonian $M \to \mathbb{R}$ with period 1: $H_{t+1} = H_t$.



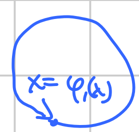
Flow: $x(0) \rightsquigarrow x(t) = \varphi_t(x) \in M$

$$\frac{d}{dt} x(t) = X_{H_t}(x(t)).$$

Time 1 map $\varphi_1 : M \to M$ $\qquad \varphi_1(x(0)) = x(1)$.

$$\left(\frac{d}{dt} \varphi_t(x) = X_{H_t}(\varphi_t(x)) \right)$$

Fixed points $\leftrightarrow$ periodic orbits



$x(t+1) = x(t)$

A periodic orbit $x(t)$ is nondegenerate if
$$\det\left(d\varphi_1(x(0)) - \mathrm{Id} \right) \neq 0.$$

$$\{\text{periodic orbits}\} = \mathcal{P}(H).$$

Arnold Conjecture (1960s) (M,ω) cpt spl, H_t 1-periodic.
If all periodic orbits are nondegenerate, then
$$\#\mathcal{P}(H) \geq \sum b_k(M).$$

Notes: • Lefschetz Fixed Pt Thm: $f: M \to M$, $f \sim id$
$$\Rightarrow \#\mathrm{Fix}(f) \geq \chi(M) = \sum (-1)^k b_k(M).$$
• if $H_t = H$ time-independent, then $p \in \mathrm{Crit}\, H \Rightarrow X_H(p) = 0$
$\Rightarrow$ $p = \varphi_1(p)$, and nondegeneracy $\Rightarrow$ H = Morse function.
$$\Rightarrow \quad \#\mathcal{P}(H) \geq \#\,\mathrm{Crit}(H) \underset{\substack{\uparrow \\ \text{Morse inequalities}}}{\geq} \sum b_k(M).$$

• Conj important for dynamical systems; gave rise to modern symplectic geometry.
• History: proved for monotone symplectic mfds ($[\omega] \propto c_1$) by Floer; now completely proven, through further work of Hofer-Salamon, Fukaya-Ono, Liu-Tian, Ruan.

$\mathcal{L}M$ = {contractible loops in M}: $\quad \gamma \in \mathcal{L}M \Rightarrow \exists\, u: D^2 \to M$ with $\gamma = \partial u$.

M monotone, normalize so $\langle \omega, A \rangle \in \mathbb{Z}$ $\forall$ $A \in H_2(M)$.

action functional $a_H : \mathcal{L}M \longrightarrow \mathbb{R}/\mathbb{Z}$ defined by
$$a_H(x) = -\int_{D^2} u^*\omega - \int_0^1 H_t(x(t))\, dt.$$

Crit pts of $a_H \longleftrightarrow \mathcal{P}(H)$ (calculus of variations).

J_t = almost cx str on M compatible with ω_t,
$g_t = \omega(\cdot, J_t \cdot)$.
Get a metric on $\mathcal{L}M$ by $\xi, \eta \in T_x\mathcal{L}M = \{\text{vector fields along } x(t)\}$
$\rightsquigarrow \langle \xi, \eta \rangle = \int_0^1 g_t(\xi_{x(t)}, \eta_{x(t)})\, dt.$

Gradient: $\nabla a_H(x)(t) = J_t(x(t))\, \dot{x}(t) - \nabla H_t(x(t))$

Negative gradient flow: $u : \mathbb{R}_s \to \mathcal{L}M$ with $\rightarrow u : \mathbb{R}_s \times [0,1]_t \to M$

$$(*) \quad \boxed{\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} - \nabla H_t(u) = 0}$$

this is the perturbed J-hol equation: if H_t = const, this says
$J \circ du = du \circ j$ $\qquad$ j = standard cx str on $\mathbb{R} \times [0,1]$.

$x^\pm \in \mathcal{P}(H)$. Define

$$\mathcal{M}(x^+, x^-) = \Big\{ u : \mathbb{R} \times [0,1] \to M \;\Big|\; u \text{ satisfies } (*),\ \lim_{s \to \pm\infty} u = x^\pm, \text{ and } u \text{ has finite energy } E(u) = \int_0^1 \int_{-\infty}^{\infty} \Big(\Big|\frac{\partial u}{\partial s}\Big|^2 + \Big|\frac{\partial u}{\partial t} - X_t(u)\Big|^2 \Big)\, ds\, dt \Big\}.$$

- for generic J, $\mathcal{M}(x^+, x^-)$ = finite-dim mfd
- $\exists\ \mu : \mathcal{P}(H) \to \mathbb{Z}$ (or $\mathbb{Z}$/minimal Chern number)
  with $\dim \mathcal{M}(x^+, x^-) = \text{ind}\, x^+ - \text{ind}\, x^-$: Conley-Zehnder index
- 1-dim $\mathcal{M}$ compactifies.

$CF_*(M;H,J)$ generated by $\mathcal{P}(H)$,
differential counts points in $\mathcal{M}(x^+,x^-)$.

__Thm__ (Floer) 1. $\partial^2=0$. $\rightsquigarrow HF_*(M;H,J)$

2. If (H',J') is another pair, then
$HF_*(M;H,J)\cong HF_*(M;H',J')$.

} __continuation__

Now set H',J' to be time-independent. Then can set up
$\mathcal{M}(x^+,x^-)$ for H' $\longleftrightarrow$ gradient flows from x^+ to x^-
so $HF_*(M;H',J')=HM_*(M)$.

$$\Rightarrow \#\mathcal{P}(H)\geq \mathrm{rk}\, HF_*(M;H,J) = \mathrm{rk}\, HM_*(M) = \sum b_k(M).$$

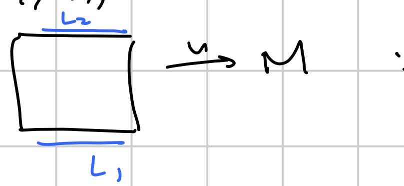
Lagrangian Intersection Floer Homology

$L \subset (M^{2n}, \omega)$ is *lagrangian* if $\omega|_L \equiv 0$, $\dim L = n$.

Given $L_1, L_2 \subset M$, define

$$\mathcal{P}(L_1, L_2) = \{\text{paths in } M \text{ from } L_1 \text{ to } L_2\}.$$

For fixed $\gamma_0 \in \mathcal{P}(L_1, L_2)$ and $\gamma \in \mathcal{P}(L_1, L_2)$, let $u: [0,1] \times [0,1] \to M$ be such that



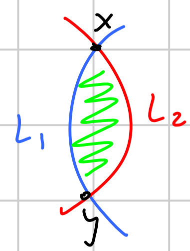
then define action functional

$$a: \mathcal{P}(L_1, L_2) \longrightarrow \mathbb{R}$$
$$\gamma \longmapsto \int_{[0,1]^2} u^*\omega$$

Assuming $\langle \omega, \pi_2(M) \rangle = 0$, this is indep of choice of u.

Crit pts of $\underline{a}$ $\longleftrightarrow$ pts in $L_1 \cap L_2$

gradient flow $\longleftrightarrow$ J-hol disks



$\rightsquigarrow$ Construct $CF_*(L_1, L_2) = \mathbb{Z}\langle p \in L_1 \cap L_2 \rangle$

∂ counts J-hol disks.

$\rightsquigarrow$ $HF_*(L_1, L_2)$ *Lagrangian intersection Floer homology*.

Special case: $Y = H_1 \cup_\Sigma H_2$ Heegaard splitting

$\longleftrightarrow \{\alpha_1, \ldots, \alpha_g\}, \{\beta_1, \ldots, \beta_g\}$ simple closed curves in Σ

$\Rightarrow$ $\mathrm{Sym}^g(\Sigma)$ symplectic, $\alpha_1 \times \cdots \times \alpha_g$, $\beta_1 \times \cdots \times \beta_g$ Lagrangian.

$HF_*(\alpha, \beta)$ = *Heegaard Floer homology* (Ozsváth-Szabó), invariant of Y.