Math 545, Stochastic Calculus

Homework 1

To be submitted by **Friday, Jan 16th** at 11:59pm Eastern time on Gradescope.
Explain your reasoning and show your work to ensure full credit.
Write the names of your collaborators at the top (no penalty for collaborating!).

Reading: Klebaner Chapter 2. (Review Chapter 1 as needed.)

Problem 1. Let (X, Y) be uniformly distributed on the right half-disk

$$H = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 + y^2 < 1\}.$$

(a) Let $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan(Y/X)$. Find the joint density of (R, Θ) on $\mathbb{R}^2$. Are Θ and R independent? Explain. Compute the expectation and variance of R and Θ.

(b) Let S be the slope of the line passing through (X, Y) and the origin. Find the density function for S. What is its expectation?

(c) Let Q be a set of n iid copies $\{(X_i, Y_i)\}_{i=1}^n$ of (X, Y). Let $D_n = 1 - \max_{1 \le i \le n}\{(X_i^2 + Y_i^2)^{1/2}\}$ be the distance from Q to the unit circle. Show that nD_n converges in law to an exponential variable of rate 2 as $n \to \infty$, i.e. $\mathbb{P}(nD_n \le z) \to 1 - e^{-2z}$ for every $z \ge 0$.

(d) Is $\mathbb{E}(Y|X)$ a discrete or continuous random variable, or neither? What about $\mathbb{E}(X|Y)$? Find the probability density/mass function (as appropriate) of $\mathbb{E}(Y|X)$ and $\mathbb{E}(X|Y)$.

Problem 2. The proportions of people in a population who received 0, 1, or 2 doses of a vaccine against a certain virus are $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$, respectively. Assume the number of times a person who has received d doses contracts the virus has Poisson distribution with rate λ_d, where $\lambda_0 = 4$, $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$. We can model the number N of times a randomly selected person from the population contracts the virus as follows: Let $D \in \{0, 1, 2\}$ have distribution $\mathbb{P}(D = 0) = 2\mathbb{P}(D = 1) = 2\mathbb{P}(D = 2) = \frac{1}{2}$, and let N_0, N_1, N_2 be Poisson variables of rate λ_0, λ_1 and λ_2, respectively. Assume D, N_0, N_1, N_2 are independent, and set $N = N_D$.

(a) Derive the probability mass function for the random variable $\mathbb{E}(N|D)$.

(b) Derive a formula for $\mathbb{E}(D|N = n)$.

Problem 3. Starting with a stick of unit length, I break a piece off at a uniformly chosen position along the stick, leaving a stick of length L_1. I repeat this procedure with the stick of length L_1 independently to produce a shorter stick of length L_2, and again to finally end up with a stick of length L_3.

(a) Find the joint density function of the vector (L_1, L_2, L_3).

(b) Find the marginal densities of L_1, L_2, L_3 and plot them on the same graph.

(c) Compute the expectation and variance of L_3. (Doing this directly using the marginal density might not be the easiest route.)

(d) Find the conditional density of L_1 given L_3.

Problem 4. Let $\Omega = \{-1, +1\}^n$ and for each $0 \le k \le n$ set $S_k(\omega) = \omega_1 + \cdots + \omega_k$ (with $S_0 = 0$). Define random variables $U = \min_{0 \le k \le n} S_k$ and $V = \max_{0 \le k \le n} S_k$. Do U and V generate the same σ-algebra? Explain.