Math 631: Real analysis

Homework assignments *(Last updated: 4 Dec)*

- All assignments must be written up in LaTeX and submitted via Gradescope.
- See the course syllabus for homework policies.
- Note that the suggested reading roughly corresponds to the content of the lectures those weeks, which will usually go beyond what's covered by the respective homework sets.
- You should check this page for textbook errata.
- Exercise numbering is as in the published version of the text. Where the numbering is different in the preliminary pdf version on the author's webpage I indicate this with parentheses.

Homework 0. (Due Sept 3rd at 11:59pm.) Submit the course survey (to be posted on the Canvas site).

Homework 1. (Due Sept 9th at 11:59pm.)
From Tao's text: Exercises 0.0.1, 1.1.1, 1.1.3, 1.1.5, 1.1.7, 1.1.8, 1.1.10, 1.1.11, 1.1.18, 1.1.22, 1.2.3, 1.2.4
Suggested reading for weeks 1&2:

- Course syllabus
- Tao Preface, Sections 1.1–1.3
- Stein–Shakarchi Introduction through Section 4.2.

Homework 2. (Due Sept 20th at 11:59pm.)

- Tao section 1.2 exercises 9, 11, 13, 15, 26,
- Tao section 1.3 exercises 3 (i),(iii)[1], 4, 11, 18, 25

Suggested reading for weeks 3&4:

- Tao Sections 1.3, 1.4

Homework 3. (Due Oct 4th at 11:59pm.)
(Numbering for the pdf version of the text is noted in parentheses where it differs from published version.)

- Tao section 1.4 exercises 13, 14, 18, 27, 31, 34 (pdf: 1.4.35), 35 (pdf: 36) (iii,iv,vi), 43 (pdf: 44), 47 (pdf: 48),
- Tao section 1.5 exercise 2(iv,vi,vii)

Suggested reading for weeks 5&6:

- Tao Sections 1.4, 1.5

Homework 4. (Due Friday Oct 25th at 11:59pm.)

- Exercises 2.1 and 2.2 from the probability handout on Canvas
- Tao section 1.5 exercises 3, 6&8,
- Tao section 1.6 exercises 1, 7&8, 16, 20, 28 (check the textbook errata page linked above)[2]

Further recommended exercises:

- Tao section 1.5 exercises 5, 12,
- Tao section 1.6 exercises 6, 15, 27, 31, 36 (pdf 35), 42&45 (pdf 41&44),
- Stein & Shakarchi Chapter 3 exercise 11

Suggested reading:

- Probability handout
- Tao section 1.6 (Section 1.6.1 is optional reading)
- Stein & Shakarchi Chapter 3

[1] See textbook errata page linked above.

[2] In the errata for Exercise 28 the function is changed to $F(x) = \sum_{k=1}^{\infty} A^{-k}\cos(\pi B^k x)$ for $A = 4, B = 16$. It seems any $A > 1$ fixed (such as 2) and B a sufficiently large even integer works. You can write it up for whatever choice of **even integers** A, B you like, as long as it works (and your proof works). In part (b) consider an interval with endpoints $j/B^m, (j+1)/B^m$.

Homework 5. (Due Nov 8th at 11:59pm.)

- Tao section 1.6 exercises 27, 31, 36 (pdf 35), 42&45 (pdf 41&44), 48 (pdf 47), 49(i)–(vi) (pdf 48)
- Tao section 1.7 exercises 13, 14, 15 (i)–(ii), 23

Further recommended exercises:

- Tao Section 1.7 exercises 25, 26

Suggested reading:

- Tao section 1.7
- Folland section 1.4

Homework 6. (Due Nov 26th at 11:59pm.) See footnotes

- Folland 6.1 exercises 3&4, 5 – and show that for all $0 < p < q \leq +\infty$, $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$, and $L^p([0,1]) \supset L^q([0,1])$ (with strict containment in both cases), 9, 14
- Folland 5.5 exercises 55, 56[3], 58, 62,

Suggested reading:

- Folland 5.1 6.1, 6.2, 5.5

Homework 7. (Due Dec 10th at 11:59pm.)

- Tao Section 1.7 exercise 26
- Folland 5.67
- Folland 3.17
- Tao's *Epsilon of Room, vol 1*: 1.12.20 (third part is optional), 1.12.21, 1.12.22
- Show that if X is a non-negative random variable, then $\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq t)dt$.
- From Lawler's probability notes: Exercise 5.7, Exercise 6.3
- A sequence of measure μ_n on $\mathbb{R}$ is said to converge *vaguely* to a measure μ if $\int_{\mathbb{R}} f\,d\mu_n \to \int_{\mathbb{R}} f\,d\mu\ \forall f \in C_c(\mathbb{R})$, while μ_n is said to converge *weakly* to μ if $\int_{\mathbb{R}} f\,d\mu_n \to \int_{\mathbb{R}} f\,d\mu\ \forall f \in BC(\mathbb{R})$.[4] Now supposing μ_n and μ are probability measures on $\mathbb{R}$, show the following are equivalent:
  - (*i*) $\int_{\mathbb{R}} f\,d\mu_n \to \int_{\mathbb{R}} f\,d\mu$ for all $f \in C_c^\infty(\mathbb{R})$.
  - (*ii*) $\mu_n \to \mu$ vaguely.
  - (*iii*) $\mu_n \to \mu$ weakly.
  - (*iv*) $\mu_n((a,b)) \to \mu((a,b))$ for any $-\infty < a < b < +\infty$ such that $\mu(\{a,b\}) = 0$.

  Show that the implication $(ii) \Rightarrow (iii)$ may fail if we drop the requirement that μ be a probability measure.
  An aside: A sequence of random variables X_n is said to *converge in distribution* or *converge weakly* to a random variable X if the associated distributions μ_{X_n} converge weakly to μ_X. Note that X_n can be defined on different probability spaces for each n.

Further recommended exercises:

- Lawler Exercise 3.8

Suggested reading:

- Folland 3.1–3.2
- Folland 8.1–8.4
- Probability notes, through Section 6 (Conditional expectation), not including martingales
- Folland 10.1–10.3

[3] You should spell out the argument for the "immediate" fact that $E^\perp$ is a closed subspace of $\mathcal{H}$ for any $E \subset \mathcal{H}$.
[4] Recall $BC(\mathbb{R})$ is the set of all bounded and continuous functions $f : \mathbb{R} \to \mathbb{C}$.