Math 340 / Stat 231
Advanced Introduction to Probability
Fall 2022

Course syllabus

(Updates in blue. Last updated: 22 Nov.)

Instructor: Nicholas Cook

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Office hours: Physics 029A, Tu 11:30–12:30, Th 9–10, Fr 11–noon, or by appointment.
I will try to respond to emails within 24 hours during regular working hours M–F. (So if you email me Friday afternoon you may not get a response until the following Monday. In particular, I cannot monitor my email Friday afternoon for last-minute homework questions!)

Course description.

Official course catalogue description: Advanced introduction to basic, non-measure theoretic probability covering topics in more depth and with more rigor than MATH 230. Topics include random variables with discrete and continuous distributions. Independence, joint distributions, conditional distributions, generating functions, Bayes' formula, and Markov chains. Rigorous arguments are presented for the law of large numbers, central limit theorem, and Poisson limit theorems. Prerequisite: Mathematics 202, 212, or 222. Not open to those who have taken Mathematics 230 or Statistics 230.

This instructor's view of the course: Probability theory is a beautiful area of mathematics that is one of its oldest branches, but continues to be one of the most active areas of research. It is one of the few parts of "pure"[1] math where (most) problems are just a short distance from concrete applications. It provides a foundation and a language for statistics, data science and information theory. It can be used to model very large complicated systems in statistical mechanics, biology and social networks (to name a few) and to shed light on various "universality" phenomena observed in nature.

It is also a pleasure to learn and teach, largely because we all have some intuition for randomness, some of it gained through playing games of chance. In this class we will put probabilistic reasoning in precise mathematical terms, develop some fundamental results and tools such as the law of large numbers and central limit theorem, and use these to study important models such as random walks and Poisson arrivals processes. We'll also see some examples where our intuition might lead us astray.

Ways this course differs from 230 (i.e. what makes this an "advanced" introduction?):

(1) *This class is proof-based.* We will approach the subject as mathematicians, not shying away from abstraction and proofs; we do this in pursuit of clarity and precision. We will cover examples to reinforce concepts and consider applications and modeling, but our primary concern is developing the theory of probability. It will not be assumed that you have previously taken a proof-based course.

(2) *Homework may be challenging.* Math cannot be learned from lectures alone – it is learned by doing! Homework problems will both reinforce and complement lectures. (Really this applies to most subsequent math courses you may take, and college courses in general.) You will learn some new things on the homework – don't expect homework problems to

[1]This is a common term for math that adheres to rigorous proofs; at Duke many of us prefer the term *core* math.

look like small modifications of examples done in the lecture! But you can expect some exam problems to look like homework problems :).

If you have no interest in proofs (and are unwilling to change!) then you are advised to take Math 230 instead of this course.

Texts: 1. *A natural introduction to probability theory, 2nd edition,* Ronald Meester
Freely available online through Duke Libraries here.
2. *Essentials of Stochastic Processes* Freely available online through Duke Libraries here.
3. *Elementary probability for applications, 5th edition,* Rick Durrett
With the author's permission the third text is being made available on the course Sakai page.

We will go through Meester roughly in order, occasionally drawing additional examples from Durrett. Towards the end of the course we will switch to Durrett's books for Poisson processes and Markov chains.

Course structure.

Class meetings. Will be held in person at the scheduled time of 10:15–11:30am EDT on Tuesdays and Thursdays.

Homework. There will be 11 graded homeworks, assigned each week except first week and midterm weeks. They will be due on Fridays at 5pm EDT, to be submitted via Gradescope. In Gradescope you must tag the pages of your uploaded assignment with problem numbers. You should make sure your uploaded writeup is legible. You will have all you need to know to complete the homework after the Tuesday lecture of the week the homework is due. You are free to collaborate with other students on the homework, but you must write up your own solutions, *in your own words*, and acknowledge your collaborators at the top (I emphasize: no penalty whatsoever for collaborating!). Late homework will not be accepted, except as required by the University Policy.

Exams. There will be two midterms and a final exam. Midterms and the final are in person and closed-book. The midterms will be held in class on Oct 6th and Nov 10th. The final will be held on Dec 19th from 2–5pm in Physics 259.

Safety: We will adhere to Duke's COVID safety protocols. As of this writing, masks are required in classrooms. When not required they are still strongly encouraged.

Evaluation: Your grade will be a weighted average of your homework, midterm and final grades, with the following weights: Homework 40%, Midterms $15+15 = 30\%$, Final 30%. Each homework will be graded out of 100%. Your lowest homework score will be dropped, and the remaining homeworks will carry equal weight in your final homework average.

Academic integrity. As members of the Duke community, students are expected to uphold the Duke Community Standard.

Technology requirements and policies: Supporting documentation for Gradescope is available here. Technical problems with Gradescope not addressed there must be communicated to me as early as possible. Technical problems with Sakai should be directed to the Duke OIT Service Desk.

Support Services. The ongoing COVID-19 pandemic is a stressful time for all of us. I urge you to explore the support services available through CAPS and DukeReach.

Tentative rough schedule of topics. Not set in stone – may be updated as we proceed. Some topics may be explored in homework rather than lecture.

- 8/30 Countable and uncountable sets; definition of a discrete probability space; examples
- 9/1 Boolean algebra; uniform probability measure; basic combinatorics; ordered and unordered sets/sequences, with and without repetition
- 9/6 Birthday problem; properties of probability measures;
- 9/8 conditional probability; Bayes's rule
- 9/13 independence
- 9/15 discrete arrivals process, law of small numbers
- 9/20 Law of large numbers for iid trials (LLNv1): statement and proof
- 9/22 Random variables and their distributions
- 9/27 Properties of distribution functions;
- 9/29 independence of random variables;
  Expectation
- 10/4 Variance and correlation; Markov and Chebyshev inequalities; Application: weak law of large numbers (LLNv2); Erdős–Rényi graphs
- 10/6 **Midterm 1**. Based on lectures through 9/27
- 10/13 Random vectors; joint distribution functions; conditional distributions
- 10/18 Conditional expectation
- 10/18 Conditional expectation: further examples
- 10/20 Random walk: definition, reflection principle
- 10/25 Ballot theorem
- 10/27 Arcsine law
- 11/1 Stirling's formula; Finish proof of arcsine law
- 11/3 Continuous probability spaces: motivation from arc-sine law; Uncountable sets; general definition of probability space / experiment
- 11/8 Examples of density functions; Continuous random variables
- 11/10 **Midterm 2**. Based on lectures through 11/1 (material up to and including the arcsine law for the random walk)
- 11/15 Expectation for continuous sample spaces; random vectors and independence
- 11/17 Conditional distributions and densities; Poisson processes: definition and construction
- 11/22 Poisson processes: Thinning and superposition
- 11/29 Poisson processes: Further examples. Markov chains: definition and examples
- 12/1 Markov chains: transience and recurrence, decomposition of state space; stationary distributions
- 12/6 Markov chains: detailed balance conditions; convergence to the stationary distribution, and ergodic theorem;
- 12/8 Further examples and review