Math 631: Real analysis[1] (Fall 2024)

Course syllabus *(Last updated: 6 Nov)*

Instructor: Nicholas Cook (he/him/his)

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Office hours: W 1–2, F 11–11:45, or by appointment.
Requests to meet should ideally come at least 2 weekdays in advance as my schedule fills up.
Note that I generally do not address work email on evenings or weekends.

Class meetings: TuTh 10:05–11:20, Physics 119
First meeting: 27 Aug
Last meeting: 5 Dec
∗∗The Tuesday 17 Sept lecture will be over Zoom.

Course description: This course covers fundamental objects and methods in modern real analysis at the introductory graduate level. The three columns are:

- (*i*) Measure theory and integration (initially focusing on the Lebesgue integral on $\mathbb{R}^d$)
- (*ii*) Abstract Hilbert spaces
- (*iii*) Compactness

We apply these to develop basic results in Fourier analysis and probability theory. A detailed list of topics follows. (Some topics may be skipped in lectures and covered in homework exercises. Topics that may be skipped are marked with ∗.)

(1) Lebesgue measure and integration, Littlewood's principles, Fubini's theorem
(2) Abstract measure theory, convergence theorems, signed measures and Hahn decomposition, Radon–Nikodym theorem, Lebesgue decomposition
(3) Hardy–Littlewood maximal function, Lebesgue differentiation theorem, approximate identities, functions of bounded variation, absolute continuity
(4) Construction of measures: Carathéodory extension theorem, Hahn–Kolmogorov theorem, construction of product measure, ∗Lebesgue–Stieltjes integral
(5) L^p spaces
(6) Hilbert spaces, orthogonal complements, representation of linear functionals, orthonormal bases, ∗von Neumann's ergodic theorem
(7) Fourier series and the Fourier transform on $\mathbb{R}^d$, Parseval's identity
(8) Compactness: Tychonoff's theorem, Arzelà–Ascoli theorem, ∗Banach–Alaoglu theorem
(9) Foundations of probability theory: Probability spaces, Kolmogorov extension theorem, independence, expectation, conditional expectation, weak convergence, the Central Limit Theorem.

Note: This course serves as a prerequisite for Math 641, which is a *second* course in graduate level measure-theoretic probability theory. The alternative prerequisite for 641 is STA711, which is a standard first course in measure-theoretic probability.

Prerequisites: Mathematics 531 and 532 or equivalent.

Primary texts:

- Tao's *An introduction to measure theory* [Tao11] (available on author's webpage)
  - See also this page for errata.
- Stein and Shakarchi's *Real Analysis* [SS05] (electronic version available on Duke Libraries website)
- Folland's *Real Analysis, Modern techniques and their applications* [Fol99]
- Notes will be provided for probability theory

Additional references:

- Tao's *An epsilon of room, I: Real analysis* [Tao10] (available on author's webpage)
- Stein and Shakarchi's *Functional Analysis* [SS11]
- Lieb and Loss's *Analysis* (2nd ed.) [LL01]
- Probability theory:
  - [Tao12, Section 1.1] (available on the author's webpage)
  - Williams's *Probability with Martingales* [Wil91]

[1] The official title in the course catalogue is "Measure and integration", but the course is somewhat more broad than measure theory.

 – Durrett's *Probability: Theory and Examples* [Dur19]

Grading: Grades will be based on:

(40%) Biweekly homeworks.
- The lowest score will be dropped.
- Homework should be typed up in LaTeX. (It is acceptable to include images of hand-drawn figures, as long as they are clear.)
- Submissions will be done via Gradescope.
- You are welcome (and encouraged!) to collaborate on the homework, but you must list your collaborators at the top of your submission.
- For exercises from Tao, you may freely apply results that appear earlier in the text as "black boxes", including earlier exercises (even if they weren't assigned). You may not cite results that appear after the exercise!

(20%) In-class midterm, Thursday the 10th of October

(40%) Final exam, Friday the 13th of December, 2–5pm

Other class policies and etiquette.
- To promote a distraction-free environment, please limit the use of electronic devices to note-taking.
- You do **not** have permission to take photographs of the classroom or board.

Qualifying credit for the math PhD:. This course can be counted toward the Qualifying requirement for the math PhD program. Qualification in this course will be awarded for an A or A- performance on the final exam; in borderline cases a more holistic assessment may be made based on overall course performance.

A qualifying exam will also be administered at the start of the semester (date TBD). An A performance on this exam will give you credit for 631 and you will not need to take the course. Please contact me and the director of graduate studies Stephanos Venakides by the 2nd week of August if you would like to take the qualifying exam.

References

[Dur19] Rick Durrett. *Probability—theory and examples*, volume 49 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fifth edition, 2019. URL: `https://doi.org/10.1017/9781108591034`, `doi:10.1017/9781108591034`.

[Fol99] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[LL01] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001. URL: `https://doi.org/10.1090/gsm/014`, `doi:10.1090/gsm/014`.

[SS05] Elias M. Stein and Rami Shakarchi. *Real analysis*, volume 3 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2005. Measure theory, integration, and Hilbert spaces.

[SS11] Elias M. Stein and Rami Shakarchi. *Functional analysis*, volume 4 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2011. Introduction to further topics in analysis.

[Tao10] Terence Tao. *An epsilon of room, I: real analysis*, volume 117 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Pages from year three of a mathematical blog. URL: `https://doi.org/10.1090/gsm/117`, `doi:10.1090/gsm/117`.

[Tao11] Terence Tao. *An introduction to measure theory*, volume 126 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011. URL: `https://doi.org/10.1090/gsm/126`, `doi:10.1090/gsm/126`.

[Tao12] Terence Tao. *Topics in random matrix theory*, volume 132 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991. URL: `https://doi.org/10.1017/CBO9780511813658`, `doi:10.1017/CBO9780511813658`.