Introduction to Random Matrix Theory

Math 690-40 (Topics in probability), Fall 2025

Instructor: Nicholas Cook (he/him/his)

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Office hours (Phy242): Tu 9:45–10:45, We 10:30–11:30, or by appointment.

Class meetings: TuTh 8:30–9:45, Physics 119.
First meeting: Aug 26th. Last meeting: Dec 2nd (Dec 4th may be used as a makeup day if needed)
∗I will be traveling on Sept 11 and there will be no class. Dec 2nd is a makeup day.

Course description. This course aims for a broad introduction to Random Matrix Theory, covering topics of interest to mathematicians as well as physicists, statisticians and computer scientists. Topics will be selected according to instructor and students' current interests, but will at least cover some core results like Wigner's semicircle law and the Marchenko–Pastur law, the Baik–Ben Arous–Peché (BBP) transition for spiked models, Gaussian ensembles and determinantal point processes, and universality for local spectral statistics.

Further possibilities include:

- Elements of free probability theory
- Dyson Brownian motion
- Beta ensembles, Schwinger–Dyson loop equations
- Matrix concentration inequalities
- Large deviation principles for the spectrum, HCIZ integrals
- Orthogonal polynomials and Riemann–Hilbert problems
- Community detection in random graphs: stochastic block model, planted clique model
- **Connections with physics:** genus expansion, quantum chaos/eigenstate thermalization hypothesis, SUSY methods
- **Connections with number theory:** Montgomery–Dyson pair correlation conjecture for Riemann zeta zeros

(Some of these may be explored as reading projects.)

Prerequisites. The course aims for a broad audience at the graduate level in math, statistics and physics. While a course in measure-theoretic probability is necessary to understand all of the material, my aim is that students with background in probability, linear algebra, real analysis and complex variables at the undergraduate level will understand the main points of every lecture. This will often be achieved by treating concrete, special cases of general results, covering the fundamental ideas while eliminating high-tech measure-theoretic formalism. For example, we will fully develop the theory of determinantal point processes (DPPs) over *finite sets* (which already covers beautiful examples such as uniform random spanning trees) before describing the theory of DPPs on Euclidean space without full proofs.

Grading. Grades will be based on the following:

(1) Two sets of exercises, due Oct 31st and Dec 12th. For each set, students will select some number (tentatively: five) from a list of recommended problems that will be updated as we go through the course. The list will include options to run numerical experiments on eigenvalues/eigenvectors for various types of random matrices.
Collaboration is encouraged. Students should write up their own solutions and acknowledge all collaborators.

(2) A final group presentation on a paper. In order to ensure a good attentive audience for each presentation, part of the grade may be based on scribing notes for another group's presentation.

References.

- *Topics in random matrix theory*, Terence Tao
- *An Introduction to Random Matrices*, Anderson, Guionnet and Zeitouni.
- *The Oxford Handbook of Random Matrix Theory*, Akemann, Baik and Di Francesco (ed.)
  − includes a large number of review articles on applications, particularly in physics
- *Free Probability and Random Matrices*, Mingo and Speicher
- *High-dimensional Probability*, Roman Vershynin

Some options for reading projects:

- Applications to quantum information theory [CDB+24, CB]
- Random models for the Riemann ζ function [KS00, BK13].

- Annealed complexity of spin glasses [ABAČ13]
- Girko's circular law [BC12] and the stability of large ecosystems and neural networks [AT15, AGB+15, RA06]
- Testing for geometry in random graphs [BDER16, LMSY22]
- Braess's paradox for the spectral gap of random graphs [ERS17]
- Universality at the spectral edge for random band matrices [Sod10]
- Spectrum of random inner-product kernel matrices [CS13, LY]
- Wilson loop area law for $SO(N)$ lattice gauge theories [Cha19, BG18]
- Nonlinear spiked Wigner models [GKK+], and other nonlinear models from 2-layer neural networks [LLC18]
- Dyson Brownian motion
- Random matrices for wireless communications [TV04, HLN08]
- ....

References

[ABAČ13] Antonio Auffinger, Gérard Ben Arous, and Jiří Černý. Random matrices and complexity of spin glasses. *Comm. Pure Appl. Math.*, 66(2):165–201, 2013. doi:10.1002/cpa.21422.

[AGB+15] S. Allesina, J. Grilli, G. Barabás, S. Tang, J. Aljadeff, and A. Maritan. Predicting the stability of large structured food webs. *Nature communications*, 6(7842), 2015.

[AT15] S. Allesina and S. Tang. The stability–complexity relationship at age 40: a random matrix perspective. *Population Ecology*, 57(1):63–75, 2015.

[BC12] Charles Bordenave and Djalil Chafaï. Around the circular law. *Probab. Surv.*, 9:1–89, 2012. URL: http://dx.doi.org/10.1214/11-PS183, doi:10.1214/11-PS183.

[BDER16] Sébastien Bubeck, Jian Ding, Ronen Eldan, and Miklós Z. Rácz. Testing for high-dimensional geometry in random graphs. *Random Structures Algorithms*, 49(3):503–532, 2016. doi:10.1002/rsa.20633.

[BG18] Riddhipratim Basu and Shirshendu Ganguly. SO(N) lattice gauge theory, planar and beyond. *Comm. Pure Appl. Math.*, 71(10):2016–2064, 2018. doi:10.1002/cpa.21788.

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[CB] Chi-Fang Chen and Fernando G. S. L. Brandão. Fast thermalization from the eigenstate thermalization hypothesis. arXiv:2112.07646.

[CDB+24] Chi-Fang Chen, Alexander M. Dalzell, Mario Berta, Fernando G. S. L. Brandão, and Joel A. Tropp. Sparse random hamiltonians are quantumly easy. *Phys. Rev. X*, 14:011014, Feb 2024. URL: https://link.aps.org/doi/10.1103/PhysRevX.14.011014, doi:10.1103/PhysRevX.14.011014.

[Cha19] Sourav Chatterjee. Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. *Comm. Math. Phys.*, 366(1):203–268, 2019. doi:10.1007/s00220-019-03353-3.

[CS13] Xiuyuan Cheng and Amit Singer. The spectrum of random inner-product kernel matrices. *Random Matrices Theory Appl.*, 2(4):1350010, 47, 2013. doi:10.1142/S201032631350010X.

[ERS17] Ronen Eldan, Miklós Z. Rácz, and Tselil Schramm. Braess's paradox for the spectral gap in random graphs and delocalization of eigenvectors. *Random Structures Algorithms*, 50(4):584–611, 2017. doi:10.1002/rsa.20696.

[GKK+] Alice Guionnet, Justin Ko, Florent Krzakala, Pierre Mergny, and Lenka Zdeborová. Spectral phase transitions in non-linear Wigner spiked models. arXiv:2310.14055.

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[LLC18] Cosme Louart, Zhenyu Liao, and Romain Couillet. A random matrix approach to neural networks. *Ann. Appl. Probab.*, 28(2):1190–1248, 2018. doi:10.1214/17-AAP1328.

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