High-dimensional probability

Math 690-40 (Topics in probability), Spring 2024

Instructor: Nicholas Cook (he/him/his)
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Office hours: M2–3, W1–2, or by appointment.
Requests to meet should ideally come at least 48hrs in advance as my schedule fills up.
Note that I generally do not check email on evenings or weekends.

Class meetings: TuTh 8:30–9:45, Physics 259.
First meeting: Jan 11th. Last meeting: April 16th.

Course description. High-dimensional probability has only coalesced as a distinct discipline with its own textbook relatively recently, bringing together mathematical results and techniques that were developed over the past several decades in many areas, such as geometric functional analysis, harmonic analysis, statistical physics and combinatorics. Tools from high-dimensional probability have found increasing application in other areas of probability and math, as well as computer science, signal processing, high dimensional statistics and learning theory. This course aims to cover the fundamental tools and some important applications.

Target audience. Mathematics graduate students, as well as graduate students from domains of application (computer science, statistics, electrical engineering, physics, . . .) who are interested in mathematical foundations.

Prerequisites. Math 631 or Stats 711 (measure theory and probability) are recommended.

Rough outline of topics. Some topics may be explored in homework exercises.
($*$ indicates optional topics that may be covered if time permits, or which could be explored in reading projects.)

(1) Concentration of measure I: martingale and isoperimetry methods
- Martingale method
- Isoperimetry method, Talagrand's inequality
- Application: Johnson–Lindenstrauss Lemma for dimension reduction
- Applications to random graphs and combinatorial optimization
- Sub-Gaussian vectors
- Largest and smallest singular values of random rectangular matrices
- Application to compressed sensing: RIP matrices
- $*$ Spectral gap for random graphs

(2) Anti-concentration and normal approximation
- Berry–Esseen theorem
- Essential least common denominator
- The smallest singular value for square random matrices
- $*$ Application: Smoothed analysis of algorithms
- $*$ Joint cumulants and Edgeworth expansion
- $*$ Concentration and anti-concentration for low-degree polynomials.

(3) Suprema of random processes
- Gaussian comparison inequalities
- Chaining
- VC dimension
- $*$ Generic chaining

(4) Further topics in random matrix theory
- Resolvent method
- Spiked covariance matrices and BBP transition
- $*$ Universality phenomenon in random matrix theory

(5) Concentration of measure II: further topics
- Entropy method
- $*$ Transportation method
- $*$ Stein's method
- Matrix concentration inequalities
- $*$ Superconcentration

(6) Threshold phenomena for functions on the hypercube

- Fourier analysis of Boolean functions, hypercontractivity
  ∗ Semigroup methods
- Talagrand's L^1-L^2 inequality
  ∗ Applications to random graphs, percolation, social choice functions

References. (With links to online versions.)

- *High-dimensional Probability*, Roman Vershynin (textbook)
- *Probability in High Dimensions*, Ramon van Handel (lecture notes)
- *Concentration Inequalities*, Boucheron, Lugosi and Massart [BLM13] (textbook, free on Duke libraries website)
- *The Concentration of Measure Phenomenon*, Michel Ledoux [Led01] (textbook)
- *Topics in random matrix theory*, Terence Tao (textbook).
- *A Brief Introduction to Fourier Analysis on the Boolean Cube*, Roland de Wolf
- *Superconcentration and Related Topics*, Sourav Chatterjee [Cha14] (downloadable on Duke network via Springer-Link)
- Recommended reading on effective adoption of asymptotic notation: https://terrytao.wordpress.com/2023/09/30/bounding-sums-or-integrals-of-non-negative-quantities/

Grading. Grades will be based on:

(70%) 4 problem sets, due Feb 8th, March 7th, April 4th and May 2nd.
- Your grade will be based on your best 5 writeups from each set (so for full credit you don't need to do more than 5 per set).
- You are welcome (and encouraged) to collaborate with fellow students. If you do so, you must write your solutions up in your own words, and acknowledge your collaborators at the top of your writeup.

(30%) An in-class final presentation on a reading project.
- In teams of ≈ 3 (depending on class enrollment), you will select a paper from the list below. Team and paper assignments will be determined by March 21st.
- Talks will be scheduled for the last 3 meetings (April 9th, 11th and 16th).
- *Attendance to others' talks will factor into the grade for final presentations. Please let me know ahead of time if you unable to attend these three dates.*
- The presentation should cover the main results and important context, applications, and ideas of the proofs. (It's key to determine what's really "important" enough to present in the allotted time, and how to communicate it clearly and efficiently! I'm happy to give my opinions and feedback as you prepare.)

Papers for reading projects. For some of these it may be appropriate to select a subset of chapters / results. There are clickable doi links to the papers in the references.

Please choose from the following list:

(1) Concentration for quadratic forms and applications [RV13]
(2) Stein's method for concentration of measure: [Cha07]; see also Chatterjee's PhD thesis: arXiv:math/0507526.
(3) Bounding the operator norm of structured random matrices [BvH16]
(4) Restricted Isometry Property for sub-sampled Fourier matrices: [RV08, Bou14, HR17] (these build on each other)
(5) Gaussianity of high dimensional projections of random vectors [Mec12a, Mec12b].
(6) Semigroup methods for concentration, hypercontrativity, suprema and transportation/information inequalities: Ledoux's *Four Talagrand inequalities under the same umbrella* (You can pick two of the four to present in more detail.)

Some additional papers you might consider:

- Concentration of the chromatric number: [AK97, Sco08, AN05]
- Testing for geometry in random graphs: [BDER16, LMSY22]
- Braess's paradox for the spectral gap of random graphs: [ERS17]
- Small ball probabilities for high-dimensional distributions [RV15]
- Universality for the smallest singular value of random matrices [TV10]
- Spectral stability from Gaussian noise (toward applications in numerical linear algebra) [BKMS19]
- The circular law for iid matrices [BC12]
- Invertibility and the single ring theorem [RV14]
- Stein's method for matrix concentration
- The spectrum of random inner-product kernel matrices [CS13]
- Sharp thresholds for graph properties and the k-SAT problem [Fri99]
- Matrix Spencer conjecture: [BJM]; see also series of three posts by Meka at the Windows on Theory blog

- Grothendieck's inequality and applications: [AN04]
- ...

References

[AK97] Noga Alon and Michael Krivelevich. The concentration of the chromatic number of random graphs. *Combinatorica*, 17(3):303–313, 1997. doi:10.1007/BF01215914.

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[BJM] Nikhil Bansal, Haotian Jiang, and Raghu Meka. Resolving matrix spencer conjecture up to poly-logarithmic rank. https://arxiv.org/abs/2208.11286.

[BKMS19] Jess Banks, Archit Kulkarni, Satyaki Mukherjee, and Nikhil Srivastava. Gaussian regularization of the pseudospectrum and davies' conjecture, 2019. arXiv:1906.11819.

[BLM13] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities*. Oxford University Press, Oxford, 2013. A nonasymptotic theory of independence, With a foreword by Michel Ledoux. doi:10.1093/acprof:oso/9780199535255.001.0001.

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