Localization and quantum ergodicity for Schrödinger operators on large graphs

Math 790-90 (graduate minicourse), Fall 2023

Class meetings: Nov 6 – Dec 6, MW 10:05–11:20, Physics 205

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Office hours (Physics 029A): Tu 11:30–12:30; WeFr 1–2, or by appointment.

Course description. For many large or infinite-dimensional random (or random-like) operators, the eigenvectors/eigenfunctions display one of two opposite behaviors:

(1) *Localization:* the vector is essentially supported on a bounded set of coordinates or a bounded region of space.

(2) *Delocalization:* the ℓ^2-mass of the vector is uniformly distributed over all coordinates.

We consider *Schrödinger operators* $H : \ell^2(X) \to \ell^2(X)$ on large or infinite graphs $G = (X, E)$, taking the form $H = -\Delta + V$, where Δ is the discrete Laplacian $\Delta f(x) = \sum_{y\sim x}(f(y) - f(x))$, with the sum running over the neighbors of x in the graph, and the *potential* V is a multiplication operator (diagonal matrix). Such H can be viewed as the Hamiltonian for a quantum mechanical system with state space X.

An important case is where G is the infinite d-dimensional lattice, with vertices $X = \mathbb{Z}^d$ and nearest-neighbor edges, and the diagonal entries of V are iid random variables. This is the original model of the physicist P.W. Anderson [8], who was interested in insulating/conducting properties of crystals with impurities, which translate to localization/delocalization of eigenvectors of H. Anderson predicted a sharp transition between localization and delocalization for $d \geq 2$ as the energy level or the strength of the disorder V varies [8], though this remains mostly conjectural. A similar transition has been conjectured for random band matrices [18].

In a different direction motivated by questions in quantum chaos, works of Brooks–Lindenstrauss [14] and Anantharaman–Le Masson [6] have shown that, in two different senses, eigenvectors are asymptotically delocalized when $G = G_n$ is a sequence of large regular graphs converging locally (in the Benjamini–Schramm sense) to the infinite tree. The result of [6] can be viewed as a discrete analogue of Shnirelman's celebrated Quantum Ergodicity Theorem [25] establishing delocalization for a density-one sequence of eigenfunctions for the Laplace–Beltrami operator on suitable compact manifolds.[1]

Outline of topics. The first lecture will be an overview (background and motivation, statement of main results, preliminaries on graph spectra). Mostly drawn from the first chapters of the books [5, 1], Sarnak's survey on quantum chaos [22], and Bourgade's survey on random band matrices [11].

Following that the minicourse will be in two parts:

I. (Delocalization) Lectures ≈2–5 will focus on locally tree-like regular graphs, mainly following Anantharaman's book [5], which is available here (free download on Duke network).

[1] Namely, smooth compact manifolds without boundary for which the geodesic flow is ergodic with respect to the Liouville measure.

- The Brooks–Lindenstrauss delocalization result [14] (we'll follow [19] and [5, Ch. 4]);
- The quantum ergodicity theorem of Anantharaman–Le Masson [6] (following [5, Ch. 4] and/or [7])
- Time permitting / recommended further reading:
  - Construction of localized eigenvectors by Alon–Ganguly–Srivastava [2];
  - Irregular graphs and the non-backtracking walk operator [5, Ch. 6];
  - Entropic argument of Backhausz–Szegedy for convergence of eigenvectors to Gaussian waves [9], [5, Ch. 7].

II. (Localization) Lectures ≈6–9 will focus on the lattice $\mathbb{Z}^d$, covering the multiscale analysis approach of Fröhlich–Spencer [17] to prove localization in any dimension at sufficiently low energy levels. This includes:
- Case of random potential with bounded density (as covered in the notes [20]);
- Elements of the Bourgain–Kenig argument for Bernoulli potential [13];
- Time permitting: Recent developments [16, 21] for cases $d = 2, 3$ based on unique continuation principles for discrete harmonic functions [15].

Prerequisites. The lectures should be accessible to mathematics graduate students with background in real analysis (Math 631) and some prior exposure to probability at the undergraduate level. Prior study of spectral theory (at the level of Reed–Simon) would be helpful, especially for understanding results in the literature, but I'll try to minimize its role by reducing to quantitative finite-dimensional problems (where the real ideas happen) as quickly as possible.

Requirements for credit. To receive credit for the course, students have the option to either

A. Write up solutions to six problems that will be posted with lecture notes – three from part I of the course and three from part II – to be turned in at the end of the semester.

B. Select a paper to read outside of class and meet with me one-on-one to discuss it at the end of the semester. A list of suggestions follows below, or you can suggest your own.

Suggestions for independent reading.

(1) Quantum chaos and related topics:
- The original proof of Anantharaman–Le Masson based on semiclassical analysis on locally tree-like graphs, as covered in [5, Ch. 4].
- The proof based on the non-backtracking operator, as covered in [4] and [5, Ch. 6].
- Proof of the random waves conjecture for random regular graphs by Backhaus–Szegedy [5, Ch. 7].
- Construction of localized eigenvectors on expanders of high girth by Alon–Ganguly–Srivastava [2] and Alon–Wei [3]. (These results mean some extra assumption is needed in order to strengthen the Anantharaman–Le Masson theorem to a quantum *unique* ergodicity statement.)
- Existence of continuous spectrum for random graphs [10]

(2) Random Schrödinger operators on the lattice
- Unique continuation principle for planar discrete harmonic functions [15] (see also [16, Section 3] for an extension to random Schrödinger operators)
- The Wegner estimate for Bernoulli potential: [13, Section 5], [16, Section 5]. Perhaps start with the short proof sketch in [12, Section 4].
- Aizenman–Molchanov's fractional moment method approach: [24, Sections 4–5].
- Fractional moment method applied to random band matrices: [23]

References

[1] M. Aizenman and S. Warzel. *Random operators*, volume 168 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015. Disorder effects on quantum spectra and dynamics.

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[5] N. Anantharaman. *Quantum ergodicity and delocalization of Schrödinger eigenfunctions*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, [2022] ©2022.

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