Math 105L In Class 2 Practice

Contents

Limits and Continuity

1. Describe the three ways in which a function can fail to be continuous. Draw an example in each case.

2. We said that $f(x)$ is *continuous* at $x = a$ if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.

   We also said that $f(x)$ is continuous at $x = a$ if we both *know what to expect at* $x = a$ and *get what we expect at x=a.*

   Of the two equalities in the first definition, which says we know what to expect, and which says we get it? Explain.

3. Consider the function $f(x) = \frac{x^2-1}{x+1}$.

   (a) Explain why $f(x)$ is not continuous at $x = -1$.

   (b) Consider

   $$g(x) = \begin{cases} f(x) & x \neq -1 \\ k & x = -1 \end{cases}$$

   Is there a value of k that makes $g(x)$ continuous? If so, find it. If not, explain why.

4. Consider the function

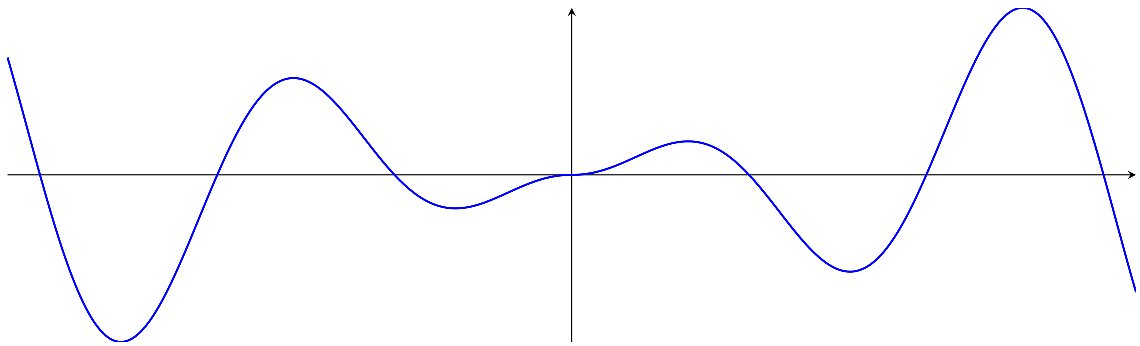
   $$f(x) = \begin{cases} -1 & x > 0 \\ k & x = 0 \\ 1 & x < 0 \end{cases}$$

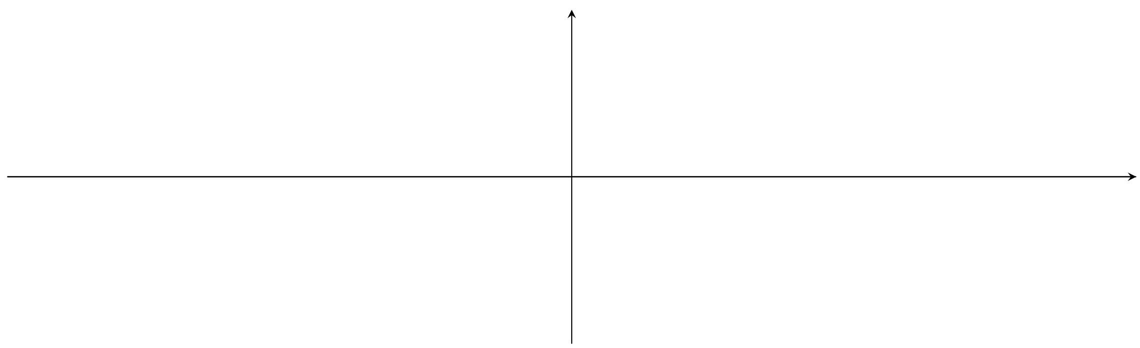
   Is there a value of k that makes $f(x)$ continuous? If so, find it. If not, explain why.

5. Briefly explain why it makes sense that $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$.

Derivative Function

1. Sketch the graph of the derivative of the following function on the axes below.





2. If $f(x)$ is positive from $x = 0$ to $x = 3$, and negative from $x = 3$ to $x = 6$, what is the sign of its derivative from $x = 3$ to $x = 6$? Explain.

3. Suppose that $F(p)$ is a function giving the number of thousands of fish in a county with p ponds in it. Explain what the following quantities mean:

   (a) $F(20) = 100$.

   (b) $F'(20) = 6$.

4. Let $A(t)$ be the depth below sea level of submarine t hours after its launch, in feet.

   (a) Explain why the statement $A(5) = -1000$ makes no sense. Include units in your answer.

   (b) Suppose that $A'(5) = -500$. Discuss the motion of the submarine at this time. Include units.

5. If $f(x) = x^2 + 2x$, compute $f'(x)$ *from the definition of the derivative.*

Second Derivative

1. If $f''(x) > 0$, then which of the following is true? Explain why each is true, false, or unknowable.

   (a) The function $f(x)$ is positive.

   (b) The tangent lines to $f(x)$ are increasing.

   (c) The slopes of the tangent lines to $f(x)$ are increasing.

2. Sketch graphs of smooth continuous functions with domain all real numbers matching each one of the following descriptions, or explain why it's impossible.

   (a) $f(x) > 0$, $f'(x) < 0$, $f''(x) < 0$, and $\lim_{x\to\infty} = 5$.

   (b) $f(x)$ has two inflection points, $f'(x) < 0$ for $x < 0$, $f'(x) \geq 0$ for $x > 0$.

   (c) $f(x)$ has no inflection points, $f'(x) = 0$ at exactly one point, $f(x)$ has two zeros.

3. True or False? Explain your answers.

   (a) If a function is concave up, then $f''(x) > 0$.

   (b) If a function is concave down, then $f''(x) \leq 0$.

   (c) When a function turns around, it has an inflection point.

   (d) It is possible for a function to be decreasing, but have increasing slopes.

4. Fill in the blanks: If a function is increasing, then its ____________________ is positive. If the slopes of a function are decreasing, then its ____________________ is negative.

5. The position of an object above the ground at time t is given by a function $s(t)$, where $s'(t) < 0$ and $s''(t) > 0$. Is the object rising or falling? Is it speeding up or slowing down? Explain.

Product and Quotient Rules

1. Differentiate $f(x) = x^2(3x^3 + 1)$.
2. Differentiate $g(x) = (x^3 - 2x)(x + 5)$.
3. Differentiate $h(x) = \sqrt{x}\,(2x^2 - 1)$. (Assume $x > 0$.)
4. Differentiate $p(x) = (x - 1)^2(x^2 + 2)$.
5. Differentiate $q(x) = e^x(x^2 - 3x + 1)$.
6. Differentiate $r(x) = \dfrac{x^2 + 1}{x}$.
7. Differentiate $s(x) = \dfrac{3x - 1}{x^2 + 1}$.
8. Differentiate $t(x) = \dfrac{x^3 + 2}{x - 1}$.
9. Differentiate $u(x) = \dfrac{\sqrt{x}}{x + 2}$.
10. Differentiate $v(x) = (x^2 + 1)(2x^3 - x)$.
11. Find the tangent line to $f(x) = x^2(3x + 1)$ at $x = 1$.
12. Find the tangent line to $g(x) = (x^3 - 2x)(x + 2)$ at $x = 0$.
13. Find the tangent line to $h(x) = \sqrt{x}(2x^2 - 1)$ at $x = 1$.
14. Find the tangent line to $p(x) = (x - 1)^2(x^2 + 1)$ at $x = 2$.
15. Find the tangent line to $q(x) = e^x(x^2 - 2x)$ at $x = 0$.
16. Find the tangent line to $r(x) = \frac{x^2+1}{x}$ at $x = 1$.
17. Find the tangent line to $s(x) = \frac{3x-1}{x^2+1}$ at $x = 1$.
18. Find the tangent line to $t(x) = \frac{x^3+1}{x-1}$ at $x = 2$.
19. Find the tangent line to $u(x) = \frac{\sqrt{x}}{x+1}$ at $x = 4$.
20. Find the tangent line to $v(x) = (x^2 + 1)(2x^2 - x)$ at $x = -1$.

Compound Interest

1. Which of the following is good approximation of $Ae^{0.04t}$? Explain your answer.

   (a) $A\left(1+\frac{1}{0.04}\right)^{1,000,000}$

   (b) $A\left(1+\frac{0.04}{1,000,000}\right)^{1,000,000}$

   (c) $A\left(1+\frac{1,000,0000}{0.04}\right)^{1,000,000}$

   (d) $A\left(1+\frac{1}{0.04}\right)^{0.04}$

2. "The more often you compound, the more money you make, so it's worth insisting on an initial investment of \$1,000 being compounded every second rather than every minute."

   Critique this statement.

3. How much money would you have if you invested \$1000 at 50% annual interest, compounded twice a year? Don't use a formula here: show computations from scratch.

4. Write an equation whose solution gives the amount of money you'd need to invest to have \$100,000 in 20 years, assuming you put it in a bank account with continuously compounded 6% interest.

5. Suppose that you have the opportunity either get \$10,000 in ten years, or \$5,000 now, given that you can put the money in a bank account with 8% interest, compounded continuously. Which would you prefer? Explain.

Differentiability

1. True or false? Justify your answer.
   (a) If a function is continuous at $x = a$, then it is also differentiable there.
   (b) Suppose that a function has a vertical tangent line at a point $x = a$. Then since it has a tangent line there, it's differentiable there.
   (c) If a function is differentiable at $x = a$, it is continuous there.

2. A student was asked to check if the function

$$f(x) = \begin{cases} x^2 + 1 & x \leq 0 \\ x^3 & x > 0 \end{cases}$$

is differentiable at $x = 0$. What is wrong with the following answer?

"We differentiate away from $x = 0$ to get

$$f'(x) = \begin{cases} 2x & x < 0 \\ 3x^2 & x > 0 \end{cases}$$

Then

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} 3x^2 = 0$$

and

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} 2x = 0$$

The two limits are the same, so $f'(x)$ is continuous at $x = 0$, making $f(x)$ differentiable at $x = 0$."

3. Use the limit definition of the derivative (no shortcut!) to show that the function

$$f(x) = \begin{cases} 2x + 7 & x > 1 \\ x^2 + 8 & x \leq 1 \end{cases}$$

is differentiable.

4. Is there a value a so that the function

$$f(x) = \begin{cases} x^3 + ax & x > 1 \\ ax^2 + 1 & x \leq 1 \end{cases}$$

is differentiable at $x = 1$? If so, find such a value.

5. A function can fail to be differentiable in three ways. Name them and give an example of each.

Exponential Functions

1. Which of the following functions is exponential?

   (a) $f(x) = 3x + 2$
   (b) $f(x) = x^3$
   (c) $f(x) = 2^x$
   (d) $f(x) = e^{x-1}$
   (e) $f(x) = 7^{x^2}$
   (f) $f(x) = |x| + 7$
   (g) $f(x) = x^x$
   (h) $f(x) = e^x + x$

2. Suppose that $f(x) = P \cdot a^x$, with $a > 0$ and $a \neq 1$. Which of the following are true regardless of the values of P and a (subject to $a > 0$, $a \neq 1$)? For those that are not always true, give values of P and a that make them true.

   (a) $f(x)$ is increasing.
   (b) $f(x)$ is decreasing.
   (c) $f(x)$ is concave up.
   (d) $f(x)$ is concave down.

3. Write the function $22e^{0.2t}$ in the form P_0a^t.

4. Suppose that a radioactive substance has a half life of 4 days. If we start with 20 moles, find a formula of the form P_0a^t for the number of moles left at time t days, and use it to find how much is left after 6 days. Show all your work – no shortcut formulas.

5. Suppose that $a > 0$. What coordinate do the graphs of all functions of the form $f(x) = a^x$ pass through?

Logs

1. Find the quadrupling time of a bank account with 6% continuously compounded interest.

2. (a) Find the inverse of $f(x) = 3e^{2x} - 1$.
   (b) Find the inverse of $g(x) = \ln(5x - 2)$.
   (c) Find the inverse of $h(x) = 2^{x+3} + 4$.
   (d) Find the inverse of $p(x) = \frac{1}{4}\ln(x) + 7$.
   (e) Find the inverse of $q(x) = \ln\big(2 - e^{-x}\big)$.

3. (a) Solve $5^x = 12$.
   (b) Solve $10^{2x+1} = 7 \cdot 10^{x-2}$.
   (c) Solve $9^{7-2x} + 6 = e$.
   (d) Solve $\log(2x+3) + \log(x-1) = 1$. (Give exact answer and check if your answers are possible!)
   (e) Solve $\log(4x) - \log(x+1) = \frac{1}{2}$. (Give exact answer and check if your answers are possible.)

4. Let $x = \log A$ and $y = \log B$. Rewrite each of the following in terms of x and y:
   (a) $\log\sqrt{\dfrac{A}{B}}$
   (b) $\log\left(\dfrac{A^3\sqrt{B}}{10}\right)$
   (c) $\log\left(\dfrac{\sqrt[3]{A}\,B^{-2}}{A^{-1}}\right)$
   (d) $\dfrac{\log\big(A^2B\big)}{\log\big(\sqrt{AB}\big)}$

5. Write 9^x in the form 8^{kx}.

Chain Rule

1. Find the derivatives of the following functions:

(a) $y = (2x^3 - x)^5$.

(b) $y = e^{3x^2+5}$.

(c) $y = ((x^2+1)^3 + 2)^4$.

(d) $y = (e^{x^3+3x^2+2x})^2$.

(e) $y = (x^4 + 2x^2 + 3)^{1/3}$.

(f) $y = e^{(4x^3+1)^{1/2}}$.

(g) $y = (5x^4 - 7x^2 + 2)^3$.

(h) $y = ((x^2+1)^4 + 3x)^5$.

(i) $y = e^{(x^2+1)^3}$.

(j) $y = (3x^2 + (x+1)^3)^4$.

2. Supose that $f(x)$ is a differentiable function. Differentiate the following. Most involve the triple chain rule.

(a) $f(x)^9$

(b) $f\big((x^4+2)^5\big)$

(c) $e^{f(x)^2+1}$

(d) $2^{f(x)^3}$

(e) $\big(f(x)^4 + x\big)^8$

(f) $f\big((x^2+1)^7\big)$

(g) $(10^{f(x)} + 1)^5$

(h) $f\big(e^{x^3+2x}\big)$

(i) $(f(x)^5 + 2)^9$

(j) $f\big((x^4 + 3x^2 + 1)^3\big)$

3. Let $f(x)$ and $g(x)$ be differentiable. Find a formula for the derivative of $\frac{f(x)}{[g(x)]^2}$. Then ***use it*** to compute the derivative of $\frac{e^x}{x^4}$.

4. Suppose that the depth of a lake in feet at distance d feet from its center is give by a function $f(d)$. Suppose also that the density of fish (in fish/ft^3) in an area of the lake with depth D feet is given by a function $p(D)$.

(a) Explain what the function $F(d) = p(f(d))$ means.

(b) Explain why the function $f(p(d))$ doesn't make sense.

(c) What are the units of $f'(d)$, $p'(D)$, and $F'(d)$?

(d) Suppose that:

- at a distance of $1,000$ feet from the center of the lake, the depth of the lake is 20 feet, and is getting shallower at a rate of 0.1 ft of depth for each foot further from the center.
- When the depth is 20 ft, for each ft deeper, the density of fish increases by 100 fish/ft^3.

Compute the rate at which the density of fish increases or decreases for every foot you go further out from the center when you are $1,000$ feet from the center. Give units.

Differentiating Exponentials

Note: some of the chain rule problems above cover derivatives of exponentials.

1. Fill in the blanks: Let $a > 0$. Then the derivative of $f(x) = a^x$ is ________ proportional to $f(x)$ with constant of proportionality _____.
2. Suppose that $e^{g(x)} = x$. By differentiating both side of this equation, show that $g'(x) = \frac{1}{x}$.
3. Find the slope of the tangent line to $f(x) = 2^x + e^x + e^2 + 2^e + x^2 + x^e$ at $x = 1$.
4. Find a function of the form $f(x) = a^x$, so that $f'(x) = -f(x)$.
5. What is the exact value of $\lim_{h\to 0} \frac{7^h - 1}{h}$? Explain.