Hodge Theory and Degenerations of Projective Manifolds

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What this is

Our goal for Math 690 (Spring 2024) is to obtain a working understanding of the relationship between degenerations of projective manifolds (to singular projective varieties) and mixed Hodge structures. We will cover: classical Hodge theory of Kähler manifolds; variations of Hodge structures; Torelli theorems; mixed Hodge structures; degenerations of algebraic varieties, the Clemens-Schmid exact sequence and applications. It is not possible to cover all these topics in full generality or in significant depth in one semester; rather I aim to give an overview, illuminated by special cases and examples. The guiding reference is the survey [KK98], and is supplemented by [Ara12, CMSP17, CEZGT14, Gri84, GH94, PS08, Voi07].

We work over $\mathbb{C}$.

The appendix of these notes summarizes some standard results from complex algebraic geometry, and is likely to be a useful resource throughout the course. As the semester progresses I will flesh out the body of the notes with: (i) brief summary of some of the material discussed in class, including homework exercises; and (ii) the degenerations that we will study at the end of the course (time allowing).

Caveat emptor

These notes are subject to revision and updates. I appreciate learning of any typos or errors.

Remark on our approach to sheaf cohomology

Good references for sheaves and sheaf cohomology, well-suited to the perspective of this course, include [Ara12, GH94]. (The first is available electronically from Duke Libraries.) We will implicitly use the fact that sheaf cohomology is isomorphic to Čech cohomology $H^q(X, \mathcal{S}) \simeq \check{H}^q(\{U_\alpha\}, \mathcal{S})$ when $\{U_i\}$ is a Leray cover of X with respect to $\mathcal{S}$. (That is, $H^q(U_I, \mathcal{S}) = 0$ for all $q, k \geq 1$.) For example, if X is a smooth manifold, it admits a *good cover*; this is a locally finite cover by open balls with the property that all nonempty intersections are also homeomorphic to open balls. The Poincaré lemma implies that a good cover is a Leray cover for the sheaf of smooth k-forms, and the sheaf of smooth (p,q)-forms. If X is an algebraic variety and $\mathcal{S}$ is quasi-coherent (this includes sheaves of sections of vector bundles), then Serre vanishing (§A.4.4) implies that any open cover by affine varieties is Leray.

Notation

Let $\Delta = \{s \in \mathbb{C} \text{ s.t. } |s| < 1\}$ denote the *unit disc in the complex plane*, and $\Delta^* = \{s \in \mathbb{C} \text{ s.t. } 0 < |s| < 1\}$ the *punctured disc*.

Given a vector space H over a field $\mathbb{F}$, the *dual vector space* will be denoted $H^\vee = \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F})$.

We will write $X \subset \mathbb{P}$ to indicate that X is a *projective variety.*

Contents

Chapter 1

Complex Algebraic Geometry

1.1 Algebraic varieties and complex analytic spaces

1.1.1 Algebraic varieties

Definition 1.1.1. An *algebraic set* $V \subset \mathbb{C}^n$ is the zero locus of a collection $\{f_\alpha\}_{\alpha \in A} \subset \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ of polynomials.

Exercise 1.1.2. Let $V \subset \mathbb{C}^n$ be an algebraic set. Show that the Hilbert basis theorem (§A.3.1) implies that the ideal

$$I(V) \;=\; \{f \in \mathbb{C}[z] \text{ s.t. } f|_V = 0\}$$

is finitely generated.

Exercise 1.1.3. The *Zariski topology* on $\mathbb{C}^n$ is defined by declaring algebraic sets to be the closed sets.

(a) Show that this does indeed specify a well-defined topology.

(b) Is the Zariski topology Hausdorff? (If yes, give a proof; if no, give a counter-example.)

(c) Is the Zariski topology separable? (If yes, give a proof; if no, give a counter-example.)

Exercise 1.1.4. If $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ are algebraic sets, then so is $V_1 \times V_2$.

Definition 1.1.5. The ring of *regular functions* is $\mathbb{C}[V] = \mathbb{C}[z]/I(V)$. We say $f : V_1 \to V_2$ is a *regular map* (or *morphism*) if $f = (f_1, \ldots, f_m)$ with $f_j \in \mathbb{C}[V_1]$.

Exercise 1.1.6. (a) Show that the inclusion map $i : V \hookrightarrow \mathbb{C}^n$ is regular.

(b) Show that the projection $V_1 \times V_2 \twoheadrightarrow V_1$ is regular.

Exercise 1.1.7 (∗)**.** A regular map is equivalent to a morphism $f^* : \mathbb{C}[V_2] \to \mathbb{C}[V_1]$ s.t. $f^*(z_j) = f_j$.

Definition 1.1.8. V is *irreducible* if $I(V)$ is prime.

Exercise 1.1.9. (a) If V is irreducible and $V = V_1 \cup V_2$, then $V = V_i$ for some $i = 1, 2$.

(b) Every V is a finite union of irreducible V_j.

(c) If V is irreducible, then $\mathbb{C}[V]$ is an integral domain.

Definition 1.1.10. Let V be an irreducible algebraic set.[1] The quotient field of $\mathbb{C}[V]$ is the field $\mathbb{C}(V)$ of *rational functions*. The *dimension of* V is the transcendence degree of this field over $\mathbb{C}$. (The cardinality of a maximal set of algebraically independent elements.) The *local ring at* $a \in V$ is the ring $\mathcal{O}_{V,a} = \{f/g \in \mathbb{C}(V) \text{ s.t. } g(a) \neq 0\}$ of rational functions that are regular at a. The *maximal ideal* is the subring $\mathfrak{m}_{V,a} = \{h \in \mathcal{O}_{V,a} \text{ s.t. } h(a) = 0\}$ of functions vanishing at a. The tangent space at a is $T_{V,a} = (\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^2)^\vee$. The local rings are the stalks of the *structure sheaf* $\mathcal{O}_V(U) = \{f/g \in \mathbb{C}(V) \text{ s.t. } g(a) \neq 0 \ \forall\, a \in U\}$.

Exercise 1.1.11 (∗)**.** Hilbert's Nullstellensatz's (§A.3.1) implies $\mathcal{O}_V(V) = \mathbb{C}[V]$.

[1] The definitions that follow can be modified, to account for the presence of zero divisors in $\mathbb{C}[V]$, when V is not irreducible.

Exercise 1.1.12. (a) Given a regular map $f : V_1 \to V_2$, show that prescribes $f_*(v)(\mu) = v(f^*\mu)$, with $v \in T_{V_1,a}$ and $\mu \in T_{V_2,f(a)}$, a well-defined linear map $f_* : T_{V_1,a} \to T_{V_2,f(a)}$.

(b) If $i : V \hookrightarrow \mathbb{C}^n$ is the inclusion, show that $i_* : T_{V,a} \to T_{\mathbb{C}^n,a}$ is injective.

(c) If $f : V_1 \to V_2$ is injective, is f_* necessarily injective?

Exercise 1.1.13. Assume V is irreducible. Suppose that $I(V) = (f_1, \dots, f_k)$. Let $f = (f_1, \dots, f_k) : \mathbb{C}^n \to \mathbb{C}^k$.

(a) Define $\partial_{j,a} = \left.\dfrac{\partial}{\partial z_j}\right|_{z=a}$. Prove that $T_{\mathbb{C}^n,a} = \operatorname{span}_{\mathbb{C}} \{\partial_{j,a}\}_{j=1}^n$.

(b) Given $a \in V$, prove that $T_{V,a} = \ker\{\mathrm{d}_a f : \mathbb{C}^n \to \mathbb{C}^k\}$.

(c) Define $c = \max_{a\in V}\{\operatorname{rank} \mathrm{d}_a f\}$. Prove that $U = \{a \in V \text{ s.t. } \operatorname{rank} \mathrm{d}_a f = c\}$ is Zariski open, and that $\dim V = \dim T_{V,a} = n - c$ for all $a \in U$.

The points of U are the *nonsingular* (or *smooth*) *points* of V. The implicit function theorem implies that U is a complex manifold.

Definition 1.1.14. An *affine variety* is a pair $(V, \mathcal{O}_V)$ consisting of an algebraic set V and its structure sheaf $\mathcal{O}_V$.

Example 1.1.15. Any Zariski open subset $U = \{a \in V \text{ s.t. } f_1(a), \dots, f_k(a) \neq 0\}$ is also an affine variety. (Here $f_j \in \mathbb{C}[V]$.) We realize U as an algebraic set in $\mathbb{C}^{n+k}$ cut out by $I(V)$ and the $w_j\, f_j(z) - 1 \in \mathbb{C}[z_1, \dots, z_n\,,\, w_1, \dots, w_k]$.

Exercise 1.1.16. If V is irreducible, then U is irreducible and $\mathbb{C}(U) \simeq \mathbb{C}(V)$.

Definition 1.1.17. A ringed space $(X, \mathcal{O}_X)$ is an *algebraic variety* if there exists a finite cover $X = \cup V_i$ by open dense $V_i \subset X$ so that the $(V_i, \mathcal{O}_{X|V_i})$ are isomorphic to affine varieties and X is *separable*: the image of the diagonal map $\Delta = (\mathrm{id}, \mathrm{id}) : X \to X \times X$ is closed. The field of *rational functions on* X is the set $\mathbb{C}(X)$ of equivalence classes $[f, U]$ with $U \subset X$ open affine and $f \in \mathbb{C}(U)$. We have $(f_1, U_1) \sim (f_2, U_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$.

Exercise 1.1.18. Prove that any subset $Y \subset X$ that is either open or closed is an algebraic subvariety.

Exercise 1.1.19 ($*$)**.** Show that complex projective n-space $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is a nonsingular algebraic variety, and that the closed subsets are the $X = \{f_1, \ldots, f_k = 0\}$, with $f_j \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ homogeneous polynomials.

Definition 1.1.20. The field of *rational functions on* X is the set $\mathbb{C}(X)$ of equivalence classes $[f, U]$ with $U \subset X$ open affine and $f \in \mathbb{C}(U)$. We have $(f_1, U_1) \sim (f_2, U_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$.

More generally, a *rational mapping* $\Phi : X \to Y$ is an equivalence class of pairs (ϕ, U) where $U \subset X$ is open and $\phi_U : U \to Y$ is a morphism. We say $(\phi_1, U_1) \sim (\phi_2, U_2)$ if $\phi_1 = \phi_2$ on $U_1 \cap U_2$. Every equivalence class $\Phi = [\phi, U]$ contains a unique representative $(\tilde{\phi}, \tilde{U})$ with $\tilde{U}$ maximal; this is the *domain of definition*. If $\tilde{\phi}(\tilde{U})$ is dense in Y, then we have $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$. The map is *birational* if ϕ^* is an isomorphism.

Definition 1.1.21. Let $Y \subset X$ be smooth, $\dim X = n$ and $\dim Y = n - m$. The *blow-up of* X *along* Y is the map $\pi : X' \to X$ defined as follows:

1. Cover X by open affine $U_1, \ldots, U_k$. Set $Y_j = Y \cap U_j$. Let $u_{j,1}, \ldots u_{j,m} \in \mathbb{C}[U_j]$ be polynomials generating $I(Y_j)$. Let $(t_1 : \ldots : t_m)$ be homogeneous coordinates on $\mathbb{P}^{m-1}$, and define

$$U_j' = \{u_{j,a}\, t_b = u_{j,b}\, t_a \mid 1 \le a, b \le m\} \subset U_j \times \mathbb{P}^{m-1}.$$

   (Equivalently, U_j' can be characterized as the closure of the graph of $(u_{j,1} : \cdots : u_{j,m}) : U_j \backslash Y_j \to \mathbb{P}^{m-1}$ in $U_j \times \mathbb{P}^{m-1}$.) Then $\pi_j : U_j' \to U_j$ is an isomorphism away from Y_j, and $\pi_j^{-1}(x) = \mathbb{P}^{m-1}$ for all $x \in Y_j$.

2. Prove that the definition of U_j' does not depend on the choice of defining polynomials $u_{j,a}$ (HW). This implies that the $U_j' \subset U_j \times \mathbb{P}^{m-1}$ may be glued together into $X' \subset X \times \mathbb{P}^{m-1}$.

Exercise 1.1.22 (∗)**.** Let $\pi : X' \to X$ be the blow-up of a smooth variety X along a smooth subvariety $C \subset X$.

(a) Prove that X' is an algebraic variety and that π is a birational morphism.

(b) Let $N_C = T_{V|C}/T_C$ denote the normal bundle. Show that $\pi^{-1}(C) \subset X'$ may be naturally identified with the projectivized normal bundle $\mathbb{P}(N_C) \to C$.

Theorem 1.1.23 (Hironaka 1964)**.** *Let $\phi : X \to Y$ be a rational mapping of nonsingular algebraic varieties. There exists a composition $\sigma : X' \to X$ of (a sequence of) blow-ups and a morphism $\phi' : X' \to Y$ so that $\phi' = \phi \circ \sigma$*

$$\begin{array}{ccc} X' & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\phi'} & \\ X & \xrightarrow[\phi]{} & Y \end{array}$$

1.1.2 Complex analytic spaces

Here we work with the (usual) analytic topology on $\mathbb{C}^n$, and analytic functions. Review the notations and results of §A.3.2 (especially Definition A.3.3).

Definition 1.1.24. We say $V \subset \mathbb{C}^n$ is an *analytic set* if every $a \in V$ admits a neighborhood $a \in U \subset \mathbb{C}^n$ and $f_1, \ldots, f_\ell \in \mathcal{O}_n(U)$ so that $V \cap U = \{f_1, \ldots, f_\ell = 0\}$. The ideals $I_a(V) = \{f \in \mathcal{O}_{n,a} \text{ s.t. } f = 0 \text{ on a nbd } a \in U \subset V\}$ define the stalks $\mathcal{O}_{V,a} = \mathcal{O}_{n,a}/I_a(V)$ of the *structure sheaf* $\mathcal{O}_V$. Given an open $U \subset V$, the ring $\mathcal{O}_V(U)$ consists of all functions $f : U \to \mathbb{C}$ with the property that for every $a \in U$, there exists $\varepsilon > 0$ and $g \in \mathcal{O}_n(\Delta^n_{a,\varepsilon})$ so that $f = g$ on $U \cap \Delta^n_{a,\varepsilon}$. A continuous map $\phi : V_1 \to V_2$ is *holomorphic* if for all $a \in V_1$ and $f \in \mathcal{O}_{V_2,\phi(a)}$, the functions $f \circ \phi : V_1 \to \mathbb{C}$ is analytic; that is, we have a ring morphism $f^* : \mathcal{O}_{V_2,f(a)} \to \mathcal{O}_{V_1,a}$.

Definition 1.1.25. We say V is *irreducible* if $V = V_1 \cap V_2$ implies $V = V_j$ for some $j = 1, 2$. And V is *irreducible at a* if $I_a(V)$ is prime.

Exercise 1.1.26. Show that $V = \{y^2 = x^2 + x^3\} \subset \mathbb{C}^2$ is irreducible, but reducible at the singular point $(0,0)$.

Definition 1.1.27. Assume V is irreducible. Then $a \in V$ is a *regular* (or *smooth*) *point* if $\dim T_{V,a} = \min_{z \in V}\{\dim T_{V,z}\}$. (As in HW 1.1.13, the regular points form a dense open subset.) Points that are not regular are *singular*. And $\dim V = \dim T_{V,a}$ with a regular.

Definition 1.1.28. If V is irreducible at $a \in V$, then the quotient field of the ring $\mathcal{O}_{V,a}$ are *meromorphic fractions.* In general, we say f/g is a *meromorphic fraction* if g is not a zero divisor of $\mathcal{O}_{V,a}$. A *meromorphic function* $f \in M(V)$ is a collection $f = \{(U_i, f_i/g_i)\}$ s.t. $\{U_i\}$ is an open cover of X, $f_i, g_i \in \mathcal{O}_V(U_i)$, with g_i not a zero divisor, and $f_i\, g_j = f_j\, g_i$ on $U_i \cap U_j$.

Definition 1.1.29. The pair $(V, \mathcal{O}_V)$ is an *analytic variety*. A *complex analytic space* is a ringed Hausdorff space $(X, \mathcal{O}_X)$ equipped with an open cover $X = \cup V_i$ so that each $(V_i, \mathcal{O}_{X|V_i})$ is isomorphic to an analytic variety. The space X is a *complex manifold* (or *nonsingular complex analytic space*) if every $x \in X$ is regular.

1.1.3 Algebraic varieties versus complex analytic spaces

Every algebraic set V naturally admits the structure of an analytic set. Every regular function $f \in \mathcal{O}_V(U)$ is a holomorphic function with respect to this structure; and every rational $h \in \mathbb{C}(V)$ is a meromorphic function. If $(V, \mathcal{O}_V)$ is an affine variety, we let $(V^{\text{an}}, \mathcal{O}_{V^{\text{an}}})$ denote the associated analytic variety. This association is an example of an *analytification functor*.

When is a complex manifold algebraic?

In general, the analytification functor $\{\text{algebraic varieties}\} \to \{\text{complex analytic spaces}\}$ is not surjective, and it is a very interesting question to understand when a complex analytic space Y may be realized as $Y = X^{\mathrm{an}}$ for some algebraic variety X. A classical result of this type is

Theorem 1.1.30 (Riemann)**.** *Every Riemann surface Y has enough meromorphic functions to realize it as a projective algebraic curve $Y \subset \mathbb{P}^n$.*

More generally, finite coverings $X \to Y$ of Riemann surfaces are classified (as topological spaces) by permutation representations of the fundamental group of $Y \backslash \{\textit{ramification pts}\}$. It's not difficult to see that these covers are complex analytic maps. Moreover, we have

Theorem 1.1.31 (Riemann existence)**.** *These finite coverings are coverings of algebraic curves: they come from finite extensions of the function field $\mathbb{C}(Y)$.*

Let $\Lambda \simeq \mathbb{Z}^{2g}$ be a lattice in $\mathbb{C}^g$. Then $Y = \mathbb{C}^g/\Lambda$ is a *complex torus*. As a smooth manifold Y is diffeomorphic to $(S^1)^{2g}$. We call Y an *abelian variety* when it can also be realized as a projective algebraic variety. Every complex torus of dimension $g = 1$ can be realized as a projective variety (HW 1.1.33). However, most complex tori do not admit an algebraic structure. The test for algebraicity (which may be interpreted as a special case of Kodaira's embedding theorem (§A.3.11)) is

Theorem 1.1.32 (Lefschetz [Mum08])**.** *The complex torus $Y = \mathbb{C}^g/\Lambda$ is an abelian variety if and only if $\mathbb{C}^g$ admits a positive definite hermitian form $h = g - \mathbf{i}\omega$ whose imaginary part $-\omega$ takes integral values on Λ.*

Exercise 1.1.33. How that every one dimensional complex torus $Y = \mathbb{C}/\Lambda$ is an abelian variety.

The most famous comparison result in algebraic geometry is

Theorem 1.1.34 (Chow 1949)**.** *Every analytic subvariety $Y \subset \mathbb{P}^n$ is algebraic.*

A necessary condition for Y to be algebraic is that it have "enough" meromorphic functions...

Theorem 1.1.35 (Siegel 1955)**.** *Let Y be a compact complex manifold. Then the field $M(Y)$ of meromorphic functions is finitely generated over $\mathbb{C}$ and $\operatorname{trdeg}_{\mathbb{C}} M(Y) \leq \dim Y$.*

If $\operatorname{trdeg}_{\mathbb{C}} M(Y) < \dim Y$, then Y can not be realized as an algebraic variety. For complex surfaces, this necessary condition is also sufficient…

Theorem 1.1.36 (Kodaira 1954)**.** *If Y is a compact complex manifold of dimension two, admitting two algebraically independent meromorphic functions, then Y is projective algebraic variety.*

In general, the necessary condition is not sufficient, but it is close…

Theorem 1.1.37 (Moishezon 1966)**.** *If Y is a compact complex manifold and $\dim Y = \operatorname{trdeg}_{\mathbb{C}} M(Y)$, then there exists a projective algebraic variety Y' and a bi-meromorphic, holomorphic map $\pi : Y' \to Y$ constructed as a composition of (a sequence of) blow-ups.*

Serre's GAGA

As discussed above, every algebraic variety $(X, \mathcal{O}_X)$ may be naturally realized as n complex analytic space $(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$. More precisely, the identity map $\text{id} : X^{\text{an}} \to X$ is a ringed space morphism: the map is continuous (the inverse is *not*), and we have $\text{id}^* : \mathcal{O}_X \to \mathcal{O}_{X^{\text{an}}}$. We have $\mathcal{O}_{X,x} \subset \mathcal{O}_{X^{\text{an}},x}$. And in general containment is strict; for example, $e^x \in \mathcal{O}_{\mathbb{C}^{\text{an}}} \backslash \mathcal{O}_{\mathbb{C}}$.

Exercise 1.1.38. If $f : X \to Y$ is a regular map, then $f^{\text{an}} = \text{id}^{-1} \circ f \circ \text{id} : X^{\text{an}} \to Y^{\text{an}}$ is holomorphic.

Theorem 1.1.39 (Serre 1956)**.** *Let $(X, \mathcal{O}_X)$ be an algebraic variety.*

(i) *X is connected if and only if X^{an} is connected.*

(ii) *X is irreducible if and only if X^{an} is irreducible.*

(iii) $\dim X = \dim X^{\mathrm{an}}$.

(iv) X^{an} *is compact if and only if* X *is* complete *(or* proper*): for every variety* Y *the map* $X \times Y \to Y$ *sends closed sets to closed sets.*

Informally, Serre's GAGA says that the category of coherent algebraic sheaves on a complex projective variety X is equivalent to the category of coherent analytic sheaves on X^{an}. As we will see, the formal statement (Theorem 1.1.45) allows us to construct algebraic objects using analytic tools (which is pretty awesome).

Definition 1.1.40. A sheaf $\mathcal{S}$ of $\mathcal{O}_Y$–modules on a ringed space $(Y, \mathcal{O}_Y)$ is *coherent* if

(i) The sheaf is of *finite type*: for all $y \in Y$ there exists a neighborhood U and a surjective morphism $\mathcal{O}_Y^n(U) \to \mathcal{S}(U)$. (Locally the sheaf is finite generated.)

(ii) For all open $U \subset Y$ and morphisms $\varphi : \mathcal{O}_Y(U)^n \to \mathcal{S}(U)$ of $\mathcal{O}_Y(U)$-modules, the kernel $\ker \varphi$ is of finite type. (This says there are essentially only finitely many relations among the generators.)

Theorem 1.1.41 ([GR84])**.** (i) *The sheaf* $\mathcal{O}_{X^{\mathrm{an}}}$ *is coherent* (Oka 1950).

(ii) *The sheaf* $\mathcal{O}_X$ *is coherent* (Serre 1955).

Example 1.1.42. Let Y be either a complex manifold or a non-singular algebraic variety, and $E \to Y$ a (holomorphic or algebraic) vector bundle. The theorems of Oka and Serre implies that the sheaf of sections $\mathcal{E}_X^0(E)$ is coherent.

Remark 1.1.43. Intuitively, coherent sheaves may be seen as a generalization of vector bundles: they are the smallest abelian category containing vector bundles. For further discussion of coherent sheaves from a perspective well-suited to this course, see [Ara12].

Theorem 1.1.44 (Oka–Cartan [GR84])**.** *The ideal sheaf of an analytic set in a complex space is coherent* (H. Cartan 1950).

Given an sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules over X, there is a natural sheaf

$$\mathcal{S}^{\mathrm{an}} = \mathrm{id}^{-1}\mathcal{S} \otimes_{\mathrm{id}^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\mathrm{an}}}$$

of $\mathcal{O}_{X^{\mathrm{an}}}$-modules over X^{an} (given by the *analytification functor*).

Theorem 1.1.45 (Serre's GAGA 1956)**.** *Let $(X, \mathcal{O}_X)$ be an algebraic variety.*[2] *Let $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ be the associated complex analytic space.*

(i) *The identity* $\mathrm{id} : X^{\mathrm{an}} \to X$ *is a morphism of ringed spaces.*

(ii) *Given a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, there exists a morphism $f^{\mathrm{an}} : (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (Y^{\mathrm{an}}, \mathcal{O}_{Y^{\mathrm{an}}})$ satisfying $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f^{\mathrm{an}}$. If f is an open immersion, then so is f^{an}.*

(iii) *For every sheaf $\mathcal{S}$ over $(X, \mathcal{O}_X)$, there is a sheaf $\mathcal{S}^{\mathrm{an}} = \mathrm{id}^{-1}\mathcal{S} \otimes_{\mathrm{id}^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$ over $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ and a map of sheaves $\mathrm{id}^* : \mathcal{S} \to \mathrm{id}_*\mathcal{S}^{\mathrm{an}}$ over $(X, \mathcal{O}_X)$. The correspondence $\mathcal{S} \mapsto \mathcal{S}^{\mathrm{an}}$ is an exact functor $\mathrm{Sh}(X, \mathcal{O}_X) \to \mathrm{Sh}(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$*

(iv) *Given a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and a coherent sheaf $\mathcal{S}$ over $(X, \mathcal{O}_X)$, the map $(f_*\mathcal{S})^{\mathrm{an}} \to f_*^{\mathrm{an}}\mathcal{S}^{\mathrm{an}}$ is injective.*

If f is proper, the map is an isomorphism and we have isomorphisms $(R^i f_*\mathcal{S})^{\mathrm{an}} \simeq R^i f_*^{\mathrm{an}}\mathcal{S}^{\mathrm{an}}$ of all higher direct image sheaves.

Now assume that $(X, \mathcal{O}_X)$ is projective algebraic. (In particular, X is proper/complete and X^{an} is compact.) *Let $\mathcal{S}, \mathcal{G}$ be coherent sheaves over $(X, \mathcal{O}_X)$.*

(a) *The natural morphism $H^q(X, \mathcal{S}) \to H^q(X^{\mathrm{an}}, \mathcal{S}^{\mathrm{an}})$ is an isomorphism.*

(b) *The natural morphism* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathrm{id}^*\mathcal{S}, \mathrm{id}^*\mathcal{G})$ *is an isomorphism.*

(c) *Every coherent sheaf over $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is isomorphic to $\mathrm{id}^*\mathcal{S}$ for a unique coherent sheaf over $(X, \mathcal{O}_X)$.*

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[2]GAGA holds more generally for schemes of finite type over $\mathbb{C}$. In this setting the identity map is replaced by an inclusion $X^{\mathrm{an}} \hookrightarrow X$, where X^{an} is the set of closed points.

1.2 Complex manifolds

1.2.1 Hermitian vector bundles

Let X be a complex manifold, and $\pi : E \to X$ a holomorphic vector bundle of rank r. Let $\mathcal{E}^0_X(E)$ denote the sheaf of smooth sections of E; let $\mathcal{E}^k_X(E) = \mathcal{E}^k_X \otimes \mathcal{E}^0_X(E)$ denote the sheaf of smooth k-forms taking value in E; and let $\mathcal{E}^{p,q}_X(E) = \mathcal{E}^{p,q}_X \otimes \mathcal{E}^0_X(E)$ denote the sheaf of smooth (p, q)-forms taking value in E.

Definition 1.2.1. A (*smooth*) *framing of E over $U \subset X$* is a collection of sections $e_1, \ldots, e_r \in \mathcal{E}^0_X(E)(U)$ of E over U so that $\{e_1(x), \ldots, e_r(x)\}$ is a basis of E_x for all $x \in U$. The framing is *holomorphic* if each section e_a is holomorphic.

Exercise 1.2.2. Prove that E is trivial over U if and only if E admits a holomorphic framing over U.

Exercise 1.2.3. Show that $\overline{\partial} : \mathcal{E}^{p,q}_X \to \mathcal{E}^{p,q+1}_X$ induces a well-defined operator $\overline{\partial} : \mathcal{E}^{p,q}_X(E) \to \mathcal{E}^{p,q+1}_X(E)$. (Your proof should use the fact that E is *holomorphic*. It is *not* true, in general, that the exterior derivative d induces an operator $\mathrm{d} : \mathcal{E}^k_X(E) \to \mathcal{E}^{k+1}_X(E)$.)

Definition 1.2.4. We say E is *hermitian* if each fibre $E_x = \pi^{-1}(x)$ is equipped with a hermitian scalar product h_x which depends smoothly (not necessarily holomorphically) on E. We call this structure a *hermitian metric on E*. The local framing is *unitary* if $h(e_a, e_b) = \delta_{ab}$.

Exercise 1.2.5. Show that E admits a hermitian metric. [*Hint.* Partition of unity.]

Remark 1.2.6. Given a holomorphic framing we may apply the Gram-Schmidt orthogonalization process to obtain a unitary frame (possibly after shrinking U).

Definition 1.2.7. A *connection on E* is a mapping

$$D : \mathcal{E}^0_X(E) \;\to\; \mathcal{E}^1_X(E)$$

satisfying the Leibniz rule

$$D(f\,\alpha) \;=\; \mathrm{d}f \otimes \alpha \;+\; f\,D\alpha \tag{1.2.8}$$

for all smooth functions $f \in \mathcal{E}^0_X$ and smooth sections $\alpha \in \mathcal{E}^0_X(E)$.

Definition 1.2.9. Fix a (smooth) framing $\{e_1, \dots, e_r\}$ of E over U. The *local connection 1-forms* $\theta^a_b \in \mathcal{E}^1_X(U)$ (*with respect to the local framing*) are defined by $De_a = \theta^b_a \otimes e_b$. Together the Leibniz rule (1.2.8) and the local connection 1-forms determine the connection D on U.

Definition 1.2.10. The *Chern connection* is the unique connection on $E \to X$ satisfying the following conditions:

(i) The natural map $\mathcal{E}^0_X(E) \xrightarrow{D} \mathcal{E}^1_X(E) \to \mathcal{E}^{0,1}_X(E)$ is given by $\overline{\partial}$.

(ii) The hermitian metric is parallel: $\mathrm{d}(\alpha, \beta) = (D\alpha, \beta) + (\alpha, D\beta)$ for all sections α, β.

Exercise 1.2.11. Show that the hermitian metric is parallel if and only if the local connection 1-forms with respect to a local unitary frame satisfy $0 = \theta^a_b + \overline{\theta^b_a}$.

Lemma 1.2.12. *The Chern connection 1-forms with respect to a local holomorphic framing* $\{e_1, \dots, e_r\}$ *over* $U \subset X$ *are given by* $\theta^c_a = (\partial h_{ab})(h^{-1})^{bc} \in \mathcal{E}^{1,0}_X(U)$, *where* $h_{ab} = h(e_a, e_b)$.

Definition 1.2.13. The connection induces a mapping $D : \mathcal{E}^k_X(E) \to \mathcal{E}^{k+1}_X(E)$ by setting $D(\phi \otimes \alpha) = \mathrm{d}\phi \otimes \alpha + (-1)^k \phi \otimes D\alpha$ for all smooth k-forms $\phi \in \mathcal{E}^k_X$ and sections $\alpha \in \mathcal{E}^0_X$. The *curvature* of the connection is the induced map $D^2 : \mathcal{E}^0_X(E) \to \mathcal{E}^2_X(E)$.

Exercise 1.2.14. Show that the curvature $D^2 : \mathcal{E}^0_X(E) \to \mathcal{E}^2_X(E)$ is a $\mathcal{E}^0_X$-linear operator. That is, the map is induced by a bundle mapping $E \to \bigwedge^2 (TX)^\vee \otimes E$.[3]

[3] Here $(TX)^\vee$ is the real cotangent bundle, the cotangent bundle with respect to underlying smooth manifold structure, not the holomorphic co-tangent bundle $T^\vee_X$ of §1.1.2.

Exercise 1.2.15. Given a local framing $\{e_j\} \in \mathcal{E}^0_X(E)(U)$, define $\Theta^i_j \in \mathcal{E}^2_X(U)$ by $D^2 e_j = \Theta^i_j \otimes e_i$. These forms are the coefficients *curvature matrix* $\Theta = (\Theta^i_j)$ *of* D *with respect to* $\{e_j\}$.

(a) If $\{e'_j\}\mathcal{E}^0_X(E)(U)$ is a second framing, then $e'_j = g^i_j\, e_i$ for some $g^i_j \in \mathcal{E}^0_X(U)$. Show that $\Theta' = g\Theta g^{-1}$.

(b) Show that *Cartan's structure equation* $\Theta = \mathrm{d}\theta - \theta \wedge \theta$ holds.

1.2.2 Kähler manifolds

Let X be a complex manifold with complex structure $J : TX \to TX$. Here TX is the real tangent bundle, to be distinguished from the holomorphic tangent bundle T_X of §1.1.2. The two are related by the decomposition of the complexification $TX \otimes \mathbb{C} = T_X \otimes \overline{T_X}$ into J–eigenbundles; the holomorphic tangent bundle T_X is the $+\mathbf{i}$–eigenbundle. In a mild abuse of notation, we will regard $T^\vee_X$ as a subspace of $T^\vee X \otimes \mathbb{C}$.

Definition 1.2.16. We say X is *hermitian* if the holomorphic tangent bundle T_X is equipped with a hermitian metric h. This metric is naturally regarded as a tensor $h \in T^\vee_X \otimes \overline{T^\vee_X} \subset T^\vee X \otimes \mathbb{C}$; locally we write $h = h_{ab}\mathrm{d}z^a \otimes \mathrm{d}\overline{z}^b$.

Remark 1.2.17. The take-away from HW 1.2.18 and 1.2.19 is that a hermitian manifold (X, h) is equivalent to the data of a Riemannian manifold (X, g) equipped with an (integrable) complex structure J that is an isometry of the metric g; and together the Riemannian metric and complex structure define a positive $(1,1)$-form $\omega(u, v) = -g(u, Jv)$.

Exercise 1.2.18. Let h be a hermitian metric on X. Define $g = \mathrm{Re}\, h = \frac{1}{2}(h + \overline{h})$ and $\omega = -\mathrm{Im}\, h = \frac{\mathbf{i}}{2}(h - \overline{h})$.

(a) Show that g is a Riemannian metric on X.

(b) Show that J is an isometry of g; that is, $g(u, v) = g(Ju, Jv)$.

(c) Show that $\omega(u, v) = -g(u, Jv)$.

Exercise 1.2.19. Fix a Riemannian metric g on X, and assume that J is an isometry of the metric; that is, $g(u, v) = g(Ju, Jv)$. Define $\omega(u, v) = -g(u, Jv)$.

(a) Show that ω is a real $(1, 1)$-form.

(b) Show that ω is *positive*. That is, ω is a real $(1, 1)$-form satisfying $\omega(v, Jv) > 0$ for all nonzero $v \in TX$; equivalently, $-\mathbf{i}\omega(u, \overline{u}) > 0$ for all nonzero $u \in T_X$.

(c) Show that $h = g - \mathbf{i}\omega$ is a hermitian metric.

Definition 1.2.20. The hermitian manifold (X, h) is *Kähler* if $\mathrm{d}\omega = 0$. In this case we say that ω is the *Kähler form* ω.

Exercise 1.2.21 (Fubini–Study metric)**.** Let $(u_0 : \cdots : u_n)$ be homogeneous coordinates on $\mathbb{P}^n$. Let $U_j = \{u_j \neq 0\} \subset \mathbb{P}^n$.

(a) Show that the $\omega_j = \mathbf{i}\partial\overline{\partial} \log \sum_{k=0}^{n} |u_k/u_j|^2 \in \mathcal{E}^{1,1}_{\mathbb{P}^n}(U_j)$ agree on intersections, and so define a closed $(1, 1)$–form ω on $\mathbb{P}^n$.

(b) Show that ω is a Kähler form. (The associated metric is the *Fubini-Study metric* on $\mathbb{P}^n$.)

Exercise 1.2.22. Show that every complex submanifold $Y \subset X$ of a Kähler manifold is also Kähler.

Exercise 1.2.23. Show that (X, h) is hermitian if and only if J is parallel with respect to the Levi-Civita connection of g. That is, Kähler manifolds are Riemannian manifolds (M, g) with $\dim_{\mathbb{R}} M = 2n$ and holonomy group contained in $\mathrm{U}(n)$.

1.3 Line bundles and divisors

Let X be a complex manifold, $\mathcal{O}_X^\times$ the sheaf of no-where vanishing holomorphic functions (the units in $\mathcal{O}_X$), and $\mathcal{M}_X^\times$ the sheaf of meromorphic functions that are not identically zero (the units in $\mathcal{M}_X$).

Given an open cover $\{U_\alpha\}$ of X we set $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Definition 1.3.1. A (*holomorphic*) *line bundle* is a submersion $\pi : L \to X$ of complex manifolds s.t. each fibre $L_x = \pi^{-1}$ is canonically a one-dimensional vector space over $\mathbb{C}$, and there exists an open cover $\{U_\alpha\}$ of X and *local trivializations* given by biholomorphisms

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathbb{C} \\ \downarrow{\scriptstyle \pi} & \swarrow & \\ U_\alpha & & \end{array}$$

so that $\varphi_\alpha : L_x \to \{x\} \times \mathbb{C}$ is an isomorphism of $\mathbb{C}$ vector spaces for all $x \in U_\alpha$. This definition implies that we have *transition functions* $g_{\alpha\beta} \in \mathcal{O}_X^\times(U_{\alpha\beta})$ so that

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x, v) \;=\; (x, g_{\alpha\beta}(x)v)$$

for all $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$. Note that $\{g_{\alpha\beta}\} \in H^1(X, \mathcal{O}_X^\times)$.

Exercise 1.3.2. (a) Given a collection $\{g_{\alpha\beta} \in \mathcal{O}_X^\times(U_{\alpha\beta})\}$ such that $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$, show that there exists a line bundle with these transition functions.

(b) Show that the line bundle is trivial if there exists $h_\alpha \in \mathcal{O}_X^\times(U_\alpha)$ so that $g_{\alpha\beta} = h_\beta / h_\alpha$.

Remark 1.3.3. It follows that the *Picard group* of line bundles on X is $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$.

Exercise 1.3.4. Show that every line bundle admits a (smooth) hermitian metric h. [*Hint.* partition of unity.]

Definition 1.3.5. The *Chern form*

$$c_1(L, h) \;=\; -\frac{\mathbf{i}}{2\pi}\partial\overline{\partial}\log h \;\in\; \mathcal{E}^{1,1}(X)$$

is determined as follows. Over U_α we have a holomorphic framing $s_\alpha(x) = \varphi_\alpha^{-1}(x, 1)$ of L, and $h_\alpha(x) = h(s_\alpha(x)) > 0$ is smooth. Define $\partial\overline{\partial}\log h_\alpha = \partial\overline{\partial}\log h_\beta$ on $U_{\alpha\beta}$ if suffices to observe that $s_\alpha(x) = \varphi_\beta^{-1}(x, g_{\alpha\beta}(x)1) = g_{\alpha\beta}(x)\, s_\beta(x)$, so that $h_\alpha = |g_{\alpha\beta}|^2 h_\beta$.

Exercise 1.3.6. The *tautological line bundle* over $\mathbb{P}^n$ is

$$\mathcal{O}_{\mathbb{P}^n}(-1) \;=\; \{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \text{ s.t. } v \in \ell\}\,.$$

If $u = (u_0 : \cdots : u_n)$ are homogeneous coordinates, then $s_j = \frac{1}{u_j} u$ is a holomorphic framing of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $U_j = \{u_j \neq 0\} \subset \mathbb{P}^n$.

(a) Show that $|s_j|^2 = |\frac{1}{u_j} u|^2$ is a globally well-defined hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$.

(b) What is the relationship between the Chern form $c_1(\mathcal{O}_{\mathbb{P}^n}(-1), h)$ and the Kähler form of the Fubini-Study metric (HW 1.2.21)?

Exercise 1.3.7. Show that the de Rham cohomology class $[c_1(L, h)] \in H^2_{\mathrm{d}}(X, \mathbb{R})$ is independent of h. Let $c_1(L) \in H^2_{\mathrm{d}}(X, \mathbb{R})$ denote this *Chern class*.

Exercise 1.3.8. Suppose that ω is a real closed $(1,1)$–form representing the first Chern class $c_1(L)$. Use the $\partial\overline{\partial}$-lemma (HW 2.2.10) to show that there exists a metric h on L so that $\omega = c_1(L, h)$.

Remark 1.3.9. It is a consequence consequence of the Hodge decomposition (§2.2) on a compact Kähler manifold is that the set of Chern classes coincides with $H^2(X, \mathbb{Z})$.

Definition 1.3.10. A *hypersurface* is an irreducible complex analytic space $V \subset X$ with $\dim V = \dim X - 1$.

Exercise 1.3.11. (a) Prove that $I_x(V) = (f)$ for some $f \in \mathcal{O}_{X,x}$. [*Hint.* §A.3.2.]

(b) Prove that $I_{x'}(V) = (f)$ for every x' sufficiently close to x.

(c) Conclude that there exists an open cover $\{U_\alpha\}$ of X and $f \in \mathcal{O}_X(U_\alpha)$ so that $V \cap U_\alpha = \{f_\alpha = 0\}$. These are the *local defining equations of* V.

(d) Suppose that $x \in U_{\alpha\beta}$ and $g \in \mathcal{O}_{X,x}$ is not identically zero. Define $0 \leq k_\alpha, k_\beta \in \mathbb{Z}$ and $h_\alpha, h_\beta \in \mathcal{O}^\times_{X,x}$ by specifying $g = h_\alpha, f_\alpha^{k_\alpha} = h_\beta\, f_\beta^{k_\beta}$. Prove that $k_\alpha = k_\beta$. Conclude that $\mathrm{ord}_V(g) = k_\alpha$, the *order of vanishing of* g *along* V, is well-defined.

If $f \in \mathcal{M}^\times_{X,x}$ is not identically zero, then $f = g/h$, with neither of $g, h \in \mathcal{O}_{X,x}$ identically zero, and we define $\mathrm{ord}_V(f) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h)$.

Definition 1.3.12. A *divisor* is any (locally finite) formal linear combination $D = \sum n_i V_i$ with $n_i \in \mathbb{Z}$. The group of all divisors is

$$\mathrm{Div}(X) \;=\; H^0(X, \mathcal{M}_X^\times/\mathcal{O}_X^\times)\,.$$

The divisor is *effective* (written $D \geq 0$) if $n_i \geq 0$. Every (nonzero) meromorphic function f on X determines a divisor

$$(f) \;=\; \sum_V \mathrm{ord}_V(f)\, V\,.$$

These are the *principal divisors* $\mathrm{Div}^0(X) = \{(f) \text{ s.t. } f \in \mathcal{M}_X^\times(X)\}$. The divisor (f) is effective if and only if f is holomorphic. We say two divisors are *linearly equivalent* (written $D_1 \sim D_2$) if $D_1 - D_2 \in \mathrm{Div}^0(X)$. The *divisor class group* is the associated group of equivalence classes

$$\mathrm{Cl}(X) \;=\; \mathrm{Div}(X)/\mathrm{Div}^0(X)\,.$$

Remark 1.3.13. Fix a hypersurface $V \subset X$. The linear functional $\phi \mapsto \int_V \phi$ on $H^{2n-2}(X,\mathbb{Z})$ determines a homology class $(V) \in H_{2n-2}(X,\mathbb{Z})$. The Poincaré dual $\pi_V \in H^2(X,\mathbb{C})$ is the *fundamental class of* V. It may be shown that $\pi_V = c_1([V])$, cf. [GH94].

Exercise 1.3.14. Show that the line bundle associated to the divisor $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is $[\mathbb{P}^{n-1}] = \mathcal{O}_{\mathbb{P}^n}(1) \stackrel{\mathrm{dfn}}{=\!=} \mathcal{O}_{\mathbb{P}^n}(-1)^\vee$.

Exercise 1.3.15. (a) Show that the line bundle $[D]$ is trivial if and only if $D \in \mathrm{Div}^0(X)$.

(b) Show that $[D_1 + D_2] = [D_1] \otimes [D_2]$.

Conclude that $[\cdot] : \mathrm{Cl}(X) \to \mathrm{Pic}(X)$ is a well-defined morphism.

Exercise 1.3.16. (a) Let s be a meromorphic section of L. Prove that $[(s)] = L$.

(b) Show that $[D]$ has a meromorphic section s with divisor $(s) = D$.

(c) Show that the space $H^0(X, [D])$ of holomorphic sections of $[D]$ may be identified with the space of meromorphic functions f on X such that $(f) + D \geq 0$.

Remark 1.3.17. If X is projective, then every line bundle $L \to X$ admits a nonzero meromorphic section. In this case, every line bundle can be realized as the line bundle $L = [D] = [(s)]$ associated to a divisor. So, when X is projective, $[\cdot] : \mathrm{Cl}(X) \to \mathrm{Pic}(X)$ is an isomorphism. However, Kleiman has exhibited a complete, non-projective 3-dimensional, irreducible scheme that is equipped with a line bundle having no nonzero rational section [Har70, Example 1.3].

Exercise 1.3.18. Fix two effective divisors D_1 and D_2, and set $D = D_1 - D_2$.

Chapter 2

Hodge theory

2.1 Hodge theory on complex manifolds

Recommended reference: [Ara12, GH94, Huy05]; also [Gre94, Lectures 1 & 2] for a nice overview.

Assume X is a *compact*, complex manifold.

2.1.1 de Rham cohomology

Harmonic representatives of de Rham cohomology

Definition 2.1.1. A *choice of hermitian metric* on X determines

(a) a hermitian product $(\alpha, \beta) \mapsto \int_X (\alpha, \beta)\, \mathrm{d}vol$ on the space $H^0(X, \mathcal{E}_X)$ of global sections of differential forms $\mathcal{E}_X = \oplus\, \mathcal{E}_X^k$;

(b) a *Hodge $*$ operator* $\mathcal{E}_X^{p,q} \to \mathcal{E}_X^{n-p,n-q}$ by $(\alpha, \beta)\mathrm{d}vol = \alpha \wedge *\beta$, and satisfying $*^2 = (-1)^{p+q}\,\mathrm{Id}$; and

(c) an adjoint
$$\mathrm{d}^* = -*\,\mathrm{d}* : H^0(X, \mathcal{E}_X^{k+1}) \ \to\ H^0(X, \mathcal{E}_X^k)$$
to the exterior derivative $\mathrm{d} : H^0(X, \mathcal{E}_X^k) \to \mathcal{E}_X^{k+1}$, also satisfying $(\mathrm{d}^*)^2 = 0$.

The *Laplacian* is the self-adjoint $\Delta_{\mathrm{d}} = (\mathrm{d} + \mathrm{d}^*)^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. The kernel

$$\mathcal{H} \;=\; \ker\{\Delta_{\mathrm{d}} : H^0(X, \mathcal{E}_X) \;\to\; H^0(X, \mathcal{E}_X)\}$$

is the space of *harmonic forms*. (Note that the definition of the harmonic forms depends on the choice of Hermitian metric.)

Theorem 2.1.2 (Hodge[1])**.** *We have a decomposition of the k-forms*

$$H^0(X, \mathcal{E}_X^k) \;=\; \mathcal{H}^k \oplus \mathrm{d}H^0(X, \mathcal{E}_X^{k-1}) \oplus \mathrm{d}^* H^0(X, \mathcal{E}_X^{k+1}),$$

and the closed forms are $\ker \mathrm{d} = \mathcal{H}^k \oplus \mathrm{d}H^0(X, \mathcal{E}_X^{k-1})$*. In particular, each de Rham cohomology class admits a unique harmonic representative, and $H^k_{\mathrm{d}}(X, \mathbb{C})$ is canonically isomorphic to $\mathcal{H}^k$.* This implies that $H^k_{\mathrm{d}}(X, \mathbb{C})$ is finite dimensional, as $\mathcal{H}^k$ is the solution space of an elliptic differential operator.

de Rham's theorem

Let $\mathcal{E}_X^k$ be the sheaf of smooth, $\mathbb{C}$-valued, differential k-forms. The space of globally defined forms is naturally identified with the 0-th sheaf cohomology group $\mathcal{E}_X^k(X) = H^0(X, \mathcal{E}_X^k)$. These sheaves are fine, because X admits partitions of unity. Consequently the sheaf cohomology groups in positive degree vanish: $H^q(X, \mathcal{E}_X^k) = 0$. Then de Rham's theorem asserts that the singular cohomology with complex coefficients (left-hand side) is given by the de Rham cohomology (right-hand side)

$$H^k(X, \mathbb{C}) \;=\; H^k_{\mathrm{d}}(X, \mathbb{C}) \overset{\mathrm{dfn}}{=\!=} \frac{\ker\{\mathrm{d} : H^0(X, \mathcal{E}_X^k) \to H^0(X, \mathcal{E}_X^{k+1})\}}{\mathrm{im}\{\mathrm{d} : H^0(X, \mathcal{E}_X^{k-1}) \to H^0(X, \mathcal{E}_X^k)\}}.$$

2.1.2 Dolbeault cohomology

Harmonic representatives

Definition 2.1.3. Define an adjoint

$$\overline{\partial}^* = - * \overline{\partial} * : H^0(X, \mathcal{E}_X^{p,q+1}) \;\to\; H^0(X, \mathcal{E}_X^{p,q})$$

[1] Hodge's initial arguments were completed in Kodaira and others in the 1940s.

to the Dolbeault differential $\overline{\partial} : H^0(X, \mathcal{E}_X^{p,q}) \to H^0(X, \mathcal{E}_X^{p,q+1})$, a *Laplacian* $\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ and the corresponding space of *harmonic forms*

$$\mathcal{H}^{p,q} \xlongequal{\text{dfn}} \ker\{\Delta_{\overline{\partial}} : H^0(X, \mathcal{E}_X^{p,q}) \to H^0(X, \mathcal{E}_X^{p,q})\} .$$

Theorem 2.1.4. *We have a decomposition of the* (p, q)*-forms*

$$H^0(X, \mathcal{E}_X^{p,q}) = \mathcal{H}^{p,q} \oplus \overline{\partial} H^0(X, \mathcal{E}_X^{p,q-1}) \oplus \overline{\partial}^* H^0(X, \mathcal{E}_X^{p,q+1}) ,$$

and the $\overline{\partial}$*-closed forms are* $\ker \overline{\partial} = \mathcal{H}^{p,q} \oplus \overline{\partial} H^0(X, \mathcal{E}_X^{p,q-1})$. As a corollary, $H^{p,q}_{\overline{\partial}}(X, \mathbb{C})$ is finite dimensional.

Remark 2.1.5. The Hodge $*$ operator commutes with $\Delta_{\overline{\partial}}$, and so induces the *Kodaira–Serre isomorphism*

$$* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q} .$$

Dolbeault's theorem

The complex structure on X induces a decomposition $\mathcal{E}_X^k = \oplus_{p+q=k}\, \mathcal{E}_X^{p,q}$. The sheaves $\mathcal{E}_X^{p,q}$ are also fine. The exterior derivative $\mathrm{d} : \mathcal{E}_X^k \to \mathcal{E}_X^{k+1}$ decomposes as $\mathrm{d} = \partial + \overline{\partial}$ with $\partial : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p+1,q}$ and $\overline{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$. Let $\Omega_X^p = \ker\{\overline{\partial} : \mathcal{E}_X^{p,0} \to \mathcal{E}_X^{p,1}\}$ denote the *sheaves of holomorphic p-forms*. The Dolbeault theorem asserts that the sheaf cohomology (left-hand side) is given by Dolbeault cohomology (right-hand side)

$$H^q(X, \Omega_X^p) = H^{p,q}_{\overline{\partial}}(X, \mathbb{C}) \xlongequal{\text{dfn}} \frac{\ker\{\overline{\partial} : H^0(X, \mathcal{E}_X^{p,q}) \to H^0(X, \mathcal{E}_X^{p,q+1})\}}{\operatorname{im}\{\overline{\partial} : H^0(X, \mathcal{E}_X^{p,q-1}) \to H^0(X, \mathcal{E}_X^{p,q})\}} .$$

For a proof see Example 5.4.21 and Example A.4.13.

Definition 2.1.6. Given a holomorphic vector bundle $E \to X$, let $\mathcal{O}_X(E)$ denote the *sheaf of holomorphic sections*, and let $\mathcal{E}_X^{p,q}(E)$ denote the *sheaf of smooth, E-valued,* (p, q)*-forms on* X.

Exercise 2.1.7. Show that:

(a) The operator $\overline{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$ naturally induces a well-defined operator $\overline{\partial}_E : \mathcal{E}_X^{p,q}(E) \to \mathcal{E}_X^{p,q+1}(E)$ satisfying $\overline{\partial}_E^2 = 0$. Consequently we have a well-defined Dolbeault cohomology groups

$$H_{\overline{\partial}}^{p,q}(X,E) \stackrel{\text{dfn}}{=\!=} \frac{\ker\{\overline{\partial} : H^0(X,\mathcal{E}_X^{p,q}(E)) \to H^0(X,\mathcal{E}_X^{p,q+1}(E))\}}{\text{im}\{\overline{\partial} : H^0(X,\mathcal{E}_X^{p,q-1}(E)) \to H^0(X,\mathcal{E}_X^{p,q}(E))\}}$$

(b) The kernel of $\overline{\partial} : \mathcal{E}_X^0(E) \to \mathcal{E}_X^{0,1}(E)$ is $\mathcal{O}_X(E)$.

Remark 2.1.8. The Dolbeault theorem generalizes to vector-bundle valued forms: the sheaf cohomology (left-hand side) is given by

$$H^q(X, \Omega^p \otimes \mathcal{O}(E)) \;=\; H_{\overline{\partial}}^{p,q}(X,E)\,. \tag{2.1.9}$$

2.2 Hodge theory on Kähler manifolds

In general, the two operators Δ_{d} and $\Delta_{\overline{\partial}}$ are completely unrelated. However, if the compact, complex manifold X is also *Kähler* (§1.2.2), then

$$\Delta \;=\; 2\Delta_{\overline{\partial}} \;=\; 2\Delta_{\partial}\,. \tag{2.2.1}$$

As a corollary

$$\mathcal{H}^k \;=\; \bigoplus_{p+q=k} \mathcal{H}^{p,q} \quad\text{and}\quad \overline{\mathcal{H}^{p,q}} \;=\; \mathcal{H}^{q,p}\,,$$

and we have the *Hodge decomposition*

$$H_{\mathrm{d}}^k(X,\mathbb{C}) \;=\; \bigoplus_{p+q=k} H^{p,q}(X)\,,$$

with $H^{p,q}(X)$ the de Rham cohomology classes that can be represented by (p,q)–forms.

Remark 2.2.2. The cohomology groups $H_{\mathrm{d}}^k(X,\mathbb{C})$ are topological invariants of X. The Hodge decomposition depends on the complex structure. While the definition of harmonic forms depends on the metric, the Hodge decomposition does not.

Example 2.2.3 (Hodge numbers of projective space)**.** We have

$$H^k(\mathbb{P}^n,\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}\,, & k\equiv 0 \text{ mod } 2\,,\\ 0\,, & k\equiv 1 \text{ mod } 2\,. \end{cases}$$

To see this note that $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}$ is a CW-complex with exactly one cell in each degree $2k$ for each $0\le k\le n$, and no cells in odd degree. The attaching maps are zero, and the claim follows. It follows that the Hodge numbers are

$$h^{p,q}(\mathbb{P}^n) \;=\; \delta^{pq}\,,$$

for all $0\le p,q\le n$, and zero otherwise.

Example 2.2.4 (Hodge numbers of complex tori)**.** Fix a complex torus $X=\mathbb{C}^g/\Lambda$; here $\mathbb{Z}^{2g}\simeq\Lambda\subset\mathbb{C}^g$ is a lattice. If $(z_1,\ldots,z_g)$ are complex coordinates on $\mathbb{C}^g$, then $h=\sum \mathrm{d}z_a\wedge\mathrm{d}\overline{z}_a$ is a hermitian metric on X. Writing $z_a = x_a+\mathbf{i}y_a$, the associated Riemannian metric and positive $(1,1)$–form (of HW 1.2.18) are $g=\mathrm{Re}\,h=\sum(\mathrm{d}x_a\otimes\mathrm{d}x_a+\mathrm{d}y_a\otimes\mathrm{d}y_a)$ and $\omega=-\mathrm{Im}\,h = 2\sum\mathrm{d}x_a\wedge\mathrm{d}y_a$.

It is clear that $\mathrm{d}\omega=0$, so that X is a Kähler manifold.

The holomorphic 1-forms on X are $H^{1,0}(X)=\mathrm{span}\{\mathrm{d}z_1\ldots,\mathrm{d}z_g\}$. The Hodge decomposition on $H^1(X,\mathbb{Z})$ determines the Hodge decomposition on $H^k(X,\mathbb{Z}) = \bigwedge^k H^1(X,\mathbb{Z})$:

$$H^{p,q}(X) \;=\; (\textstyle\bigwedge^p H^{1,0}(X))\otimes(\bigwedge^q H^{0,1}(X)) \quad\text{and}\quad h^{p,q} \;=\; \binom{g}{p}\binom{g}{q} \;=\; \binom{g}{p}^2.$$

Exercise 2.2.5. Let X be a compact Kähler manifold. Show that the holomorphic forms are harmonic.

Remark 2.2.6. The equation (2.2.1) is one of the so-called *Kähler identities*. An important consequence of the identities is that $\omega\wedge\eta$ is harmonic whenever η is. In particular, $0\neq\omega^k$ is harmonic for all $1\le k\le\dim X$. An other useful Kähler identity is

$$0 \;=\; \partial^*\,\overline{\partial}+\overline{\partial}\,\partial^*\,.$$

Exercise 2.2.7. Let X be a compact Kähler manifold. Prove that the odd Betti numbers are even, and the even Betti numbers are positive.

Remark 2.2.8. Hopf manifolds violate both these constraints on the Betti numbers, and this is how one sees that they are non-Kähler complex manifolds.

Exercise 2.2.9. Let $i : Y \hookrightarrow X$ be a complex submanifold. Use the fact that $i^*\omega$ is a Kähler form on Y to show that Y is not null-homologous in X.

Exercise 2.2.10 ($\partial\overline{\partial}$-lemma)**.** Let X be a compact Kähler manifold and η a closed (p, q)-form. Prove that the following are equivalent:

(a) η is d-exact.

(b) η is ∂-exact.

(c) η is $\overline{\partial}$-exact.

(d) $\eta = \partial\overline{\partial}\rho$. And if η is real, then ρ may be chosen so that $\mathbf{i}\rho$ is also real.

Remark 2.2.11. The Hodge theory of the $\overline{\partial}$ operator extends to Hermitian vector bundles $E \to X$. There is a well-defined $\overline{\partial}_E$-Laplacian $\Delta_{\overline{\partial}_E}$ on $\mathcal{E}^k_X(E)$ and a notion of $\overline{\partial}_E$-*harmonic* sections $\mathcal{H}^{p,q}(E) = \ker \Delta_{\overline{\partial}_E} \subset H^0(X, \mathcal{E}^{p,q}_X(E))$ yielding $\mathcal{H}^{p,q}(E) = H^{p,q}_{\overline{\partial}}(X, E)$. Consequences of this Hodge theory include Kodaira–Serre duality (§A.3.12) and the Kodaira vanishing theorem (§A.3.10).

2.2.1 Example: Hodge numbers of a projective hypersurface

Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d. The Hodge numbers $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ may be computed from the Lefschetz hyperplane theorem, Kodaira–Serre duality, and the Riemann–Roch–Hirzebruch theorem [Hir66]. (This discussion follows the on-line notes of L.I. Nicolaescu.)

First, Example 2.2.3 and the Lefschetz hyperplane theorem (§A.3.7) imply $h^{p,q}(X) = \delta_{p,q}$ for all $p + q < n$. Then Kodaira–Serre duality (§A.3.12) implies $h^{p,q}(X) = \delta_{pq}$

for all $p+q>n$. It remains to compute $h^{p,q}(X)$ for $p+q=n$. In this section we will sketch how this may be done with the Riemann–Roch–Hirzebruch formula. The approach of yields a generating function, from which extracting the Hodge numbers is laborious. In §5.4.5 we will discuss a more computationally amenable approach via the Jacobian ring.

Note that

$$\chi(X,\Omega^p_X) \;=\; \sum_{q\geq 0}(-1)^q\, h^{p,q} \;=\; \begin{cases} (-1)^p \;+\; (-1)^{n-p}\, h^{p,n-p}\,, & n\neq 2p\,,\\ (-1)^p\, h^{p,p}\,, & n=2p\,. \end{cases} \tag{2.2.12}$$

So we need to compute

$$\chi_y(X) \;\stackrel{\text{dfn}}{=\!=}\; \sum_{p\geq 0} y^p\,\chi(X,\Omega^p_X) \;=\; \sum_{p,q\geq 0}(-1)^q\, h^{p,q}\, y^p\,.$$

Set

$$\mathbf{ch}_y(T^\vee_X) = \sum_{p\geq 0} y^p\,\mathbf{ch}(\Omega^p_X)\,.$$

Then the Riemann–Roch–Hirzebruch formula (§A.3.15) yields

$$\chi_y(X) \;=\; \sum_{p\geq 0} y^p\,\chi(X,\Omega^p_X) \;=\; \langle \mathbf{td}(X)\,\mathbf{ch}_y(T^\vee_X)\,,\,[X]\rangle\,. \tag{2.2.13}$$

We compute $\mathbf{td}(X)$ as follows. Let $H=\mathcal{O}_{\mathbb{P}^{n+1}}(1)$, and set $h=c_1(H)=c_1(\mathbb{P}^n)$. Then $\mathbf{ch}(dH)=e^{dh}$. The adjunction formula (§A.3.6) $dH|_X = N_X$ and SES

$$0 \;\to\; T_X \;\to\; T_{\mathbb{P}^{n+1}}|_X \;\to\; dH|_X \;\to\; 0$$

and (A.3.15) yield

$$\mathbf{td}(X)\,\mathbf{td}(dH|_X) \;=\; \mathbf{td}(T_{\mathbb{P}^{n+1}}|_X)\,.$$

To compute $\mathbf{td}(T_{\mathbb{P}^{n+1}})$, let $\underline{\mathbb{C}}^{n+2}\to\mathbb{P}^{n+1}$ be the trivial line bundle, and define $Q=\underline{\mathbb{C}}^{n+2}/H^\vee$. Then $T_{\mathbb{P}^{n+1}} = H\to Q$, and we have a SES

$$0 \;\to\; \underline{\mathbb{C}} \;\to\; H^{n+2} \;\to\; T_{\mathbb{P}^{n+1}} \;\to\; 0\,. \tag{2.2.14}$$

Then (A.3.15) implies

$$\mathbf{td}(T_{\mathbb{P}^{n+1}}) \;=\; \mathbf{td}(H)^{n+2} \;=\; \left(\frac{h}{1-e^{-h}}\right)^{n+2},$$

so that

$$\mathbf{td}(X) \;=\; \left(\frac{h}{1-e^{-h}}\right)^{n+2}\bigg|_X \,\frac{1-e^{-dh}}{dh}\bigg|_X . \qquad (2.2.15)$$

We compute $\mathbf{ch}_y(T_X^\vee)$ as follows. The SES

$$0 \;\to\; T_X \;\to\; T_{\mathbb{P}^{n+1}}|_X \;\to\; N_X \;\simeq\; dH|_X \;\to\; 0$$

and (A.3.15) yield

$$\mathbf{ch}_y(T_X^\vee) \;=\; \mathbf{ch}_y(T^\vee_{\mathbb{P}^{n+1}})|_X\, \mathbf{ch}_y(-dH)^{-1}|_X\,.$$

Dualizing the SES (2.2.14), and again applying (A.3.15), we have

$$\mathbf{ch}_y(T^\vee_{\mathbb{P}^{n+2}}) \;=\; \frac{(1+ye^{-h})^{n+2}}{1+y}\,.$$

So

$$\mathbf{ch}_y(T_X^\vee) \;=\; \frac{(1+ye^{-h})^{n+2}}{(1+y)(1+ye^{-dh})}\bigg|_X . \qquad (2.2.16)$$

All together (2.2.13), (2.2.15) and (2.2.16) yield

$$\chi_y(X) \;=\; \left\langle \left(\frac{h}{1-e^{-h}}\right)^{n+2} \frac{1-e^{-dh}}{dh}\, \frac{(1+ye^{-h})^{n+2}}{(1+y)(1+ye^{-dh})}\,,\, [X] \right\rangle$$

Since dh is Poincaré dual to $[X]$, we may rewrite this as

$$\chi_y(X) \;=\; \left\langle h^{n+2}\left(\frac{1+ye^{-h}}{1-e^{-h}}\right)^{n+2} \frac{(1-e^{-dh})}{(1+y)(1+ye^{-dh})}\,,\, [\mathbb{P}^{n+1}] \right\rangle$$

It follows that $\chi_y(X)$ is the coefficient of z^{-1} in the Laurent expansion of

$$f(z) \;=\; \left(\frac{1+ye^{-z}}{1-e^{-z}}\right)^{n+2} \frac{(1-e^{-dz})}{(1+y)(1+ye^{-dz})}\,.$$

By the residue formula, this is

$$\chi_y(X) \;=\; \frac{1}{2\pi \mathbf{i}} \int_{|z|=\varepsilon} f(z) \mathrm{d}z$$

The change of variables $\zeta = 1 - e^{-z}$ yields $e^{-dz} = (1-\zeta)^d$ and $-\mathrm{d}z = \mathrm{d}\log(1-\zeta) = -(1-\zeta)^{-1}\mathrm{d}\zeta$. And

$$\chi_y(X) \;=\; \frac{1}{2\pi \mathbf{i}} \int_{C_\varepsilon} \frac{(1+y(1-\zeta))^{n+2}}{\zeta^{n+2}\,(1-\zeta)} \frac{(1-(1-\zeta)^d)}{(1+y)(1+y(1-\zeta)^d)} \,.\mathrm{d}\zeta$$

The integrand

$$g(\zeta) \;=\; \frac{(1-(1-\zeta)^d)}{\zeta^{n+2}\,(1-\zeta)} \frac{(1+y(1-\zeta))^{n+2}}{(1+y)(1+y(1-\zeta)^d)}$$

has a pole of order $n+1$ at $\zeta = 0$. Set

$$h(\zeta) \;=\; \zeta^{n+1}\, g(\zeta) \;=\; \frac{(1-(1-\zeta)^d)}{\zeta\,(1-\zeta)} \frac{(1+y(1-\zeta))^{n+2}}{(1+y)(1+y(1-\zeta)^d)}\,,$$

so that

$$\chi_y(X) \;=\; \frac{h^{(n)}(0)}{n!}\,. \tag{2.2.17}$$

Example 2.2.18 (Planar curve)**.** Let $n = 1$, so that $X \subset \mathbb{P}^2$ is a planar curve of degree d. Then (2.2.12) and (2.2.17) yield

$$\chi_y(X) \;=\; (1-g) \;+\; (g-1)\,y \;=\; \tfrac{1}{2}\,d(d+3)(y-1)\,,$$

and we recover the degree–genus formula (§A.1.2)

$$g \;=\; \tfrac{1}{2}(d-1)(d-2)\,.$$

Exercise 2.2.19. Let $n = 2$, so that $X \subset \mathbb{P}^3$ is a surface of degree d. Show that the Hodge numbers are

$$\begin{aligned} h^{2,0}(X) \;=\; h^{0,2}(X) &\;=\; \tfrac{1}{6}(d-1)(d-2)(d-3) \\ h^{1,1}(X) &\;=\; \tfrac{1}{3}d(2d^2-6d+7)\,. \end{aligned}$$

2.2.2 Example: Hodge numbers of a complete intersection curve

Let $C \subset \mathbb{P}^3$ be the complete intersection of two curves of degrees d_1 and d_2. The adjunction formula (§A.3.6) yields $N_C = (d_1\, H \oplus d_2\, H)|_C$. The SES

$$0 \;\to\; T_C \;\to\; T_{\mathbb{P}^3}|_C \;\to\; (d_1\, H \oplus d_2\, H)|_C \;\to\; 0$$

and (A.3.15) yield

$$1 \,+\, c_1(T_C) \;=\; \mathbf{ch}(T_C) \;=\; \mathbf{ch}(T_{\mathbb{P}^3})|_C \,-\, \mathbf{ch}(d_1\, H)|_C \,-\, \mathbf{ch}(d_2\, H)|_C\,.$$

The SES (2.2.14), with $n = 2$ yields $\mathbf{ch}(T_{\mathbb{P}^3}) = 4e^h - 1$, so that

$$1 \,+\, c_1(T_C) \;=\; (4e^h - 1 - e^{d_1 h} - e^{d_2 h})|_C\,.$$

Then $e(C) = c_1(T_C) = (4 - d_1 - d_2)h|_C$, and

$$\chi(C) \;=\; \langle e(C)\,,\, [C]\rangle \;=\; \langle (4 - d_1 - d_2)h|_C\,,\, [C]\rangle\,.$$

Since $[C]$ is Poincaré dual to $d_1 d_2 h^2$, we may rewrite this as

$$\chi(C) \;=\; \langle (4 - d_1 - d_2)d_1 d_2 h^3\,,\, [\mathbb{P}^3]\rangle \;=\; (4 - d_1 - d_2)d_1 d_2\,.$$

We deduce

$$g(C) \;=\; 1 \,+\, \tfrac{1}{2}(d_1 + d_2 - 4)d_1 d_2\,.$$

This may be reinterpreted as a special case of the genus formula (§A.2.5) for a curve on a surface.

Remark 2.2.20. The arguments above may be generalized. Let $X^n \subset \mathbb{P}^{n+k}$ be a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$. Then Hirzebruch's signature formula [Hir66] is

$$\sum_{n=0}^{\infty} \chi_y(X, \mathcal{O}_X(m))\, z^{k+n} \;=\; \frac{(1+zy)^{m-1}}{(1-z)^{m+1}} \prod_{j=1}^{k} \frac{(1+zy)^{d_j} - (1-z)^{d_j}}{(1+zy)^{d_j} + (1-z)^{d_j}} \tag{2.2.21}$$

Taking $m = 0$ we have

$$\sum_{n=0}^{\infty} \chi_y(X, \mathcal{O}_X)\, z^{k+n} \;=\; \frac{1}{(1+zy)(1-z)} \prod_{j=1}^{k} \frac{(1+zy)^{d_j} - (1-z)^{d_j}}{(1+zy)^{d_j} + (1-z)^{d_j}}\,. \tag{2.2.22}$$

2.2.3 Primitive cohomology

Definition 2.2.23. Let (X, ω) be a compact Kähler manifold of dimension n. The *primitive cohomology* is

$$P^{n-k}(X) \;=\; \ker\{\omega^{k+1} : H^{n-k}(X) \to H^{n+k+2}(X)\}\,,$$

and inherits the Hodge decomposition

$$P^m(X) \;=\; \bigoplus_{p+q=m} P^{p,q}(X)\,, \qquad \text{where} \qquad P^{p,q}(X) = P^{p+q}(X) \cap H^{p,q}(X)\,.$$

Theorem 2.2.24 (Hard Lefschetz)**.** *The map $\omega^k : H^{n-k}(X) \to H^{n+k}(X)$ is an isomorphism.*

Corollary 2.2.25 (Lefschetz decomposition)**.** *We have*

$$H^m(X) \;=\; \bigoplus_{0 \le k \le m/2} \omega^k \wedge P^{m-2k}(X)\,.$$

Exercise 2.2.26. Show that the Betti numbers of a compact Kähler manifold satisfy $b_k(X) \ge b_{k-2}(X)$ for all $k \le \dim X$.

Remark 2.2.27 (Geometric interpretation)**.** Let ω be the Fubini-Study $(1,1)$ form on $\mathbb{P}^m$ (HW 1.2.21). It can be shown that $[\omega] \in H^2(\mathbb{P}^m)$ is Poincaré dual to the homology class $[\mathbb{P}^{m-1}] \in H_{2m-2}(\mathbb{P}^m)$, cf. Remark 1.3.13, and Exercises 1.3.6 and 1.3.14.

Let $i : X \hookrightarrow \mathbb{P}^m$ be a nonsingular projective variety of dimension d. Then $\omega_X = i^*\omega$ is Poincaré dual to the homology class $[H] \in H_{2d-2}(X)$ of the hyperplane section $H = X \cap \mathbb{P}^{m-1}$. (Bertini's theorem (§A.3.4) assures us H will be smooth for generic choice of $\mathbb{P}^{m-1}$.) Duality gives the hard Lefschetz theorem the following dual formulation: the operation of intersecting with $\mathbb{P}^{m-k} \subset \mathbb{P}^m$ defines an isomorphism $H_{n+k}(X, \mathbb{C}) \to H_{n-k}(X, \mathbb{C})$.

Poincaré duality identifies the primitive cohomology $P^{n-k}(X)$ with the subgroup of $(n-k)$ cycles that do not intersect H. This is the image of the map $H_{n-k}(X \backslash H) \to H_{n-k}(X)$. Regarding $\mathbb{P}^{m-1} \subset \mathbb{P}^m$ as the "hyperplane at infinity", we call these the *finite cycles.*

Definition 2.2.28. Suppose $n = k + \ell$, with $k, \ell \geq 0$. Define a bilinear pairing $Q : H^k(X) \otimes H^k(X) \to \mathbb{C}$ by

$$Q(\alpha, \beta) \; = \; (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^\ell \, .$$

Exercise 2.2.29. Prove that $Q(\alpha, \beta) = (-1)^k Q(\beta, \alpha)$.

Theorem 2.2.30 (Hodge–Riemann bilinear relations)**.**

$$\begin{aligned} Q(H^{p,q}, H^{r,s}) \; &= \; 0 \, , \quad \textit{if } (p,q) \neq (s,r) \, , \\ \mathbf{i}^{p-q} Q(\alpha, \overline{\alpha}) \; &> \; 0 \textit{ for all } 0 \neq \alpha \in P^{p,q}(X) \subset P^m(X) \, . \end{aligned}$$

Exercise 2.2.31 (Hodge Index Theorem for surfaces)**.** Let X be a Kähler manifold of dimension 2. Show that Q has signature $(1 + 2 \dim H^{2,0}(X) \, , \, \dim H^{1,1}(X) - 1)$.

Exercise 2.2.32 (Hodge filtration)**.** Define a filtration $F^k \subset F^{k-1} \subset \cdots F^1 \subset F^0 = P^m(X)$ by specifying

$$F^k \; \stackrel{\text{dfn}}{=\!=} \; P^{m,0}(X) \, \oplus \, P^{m-1,1}(X) \, \oplus \cdots \oplus \, P^{k,m-k}(X) \, .$$

(a) Prove that $F^k \cap \overline{F^{m-k}} = P^{k,m-k}(X)$.

(b) Show that the first Hodge–Riemann bilinear relation is equivalent to $Q(F^k, F^{m-k+1}) = 0$.

Definition 2.2.33. The Kähler manifold (X, ω) is *Hodge* if $\omega \in H^2(X, \mathbb{Z})$. In this case the primitive cohomology $P(X)$ has the structure of a vector space over $\mathbb{Q}$.

Example 2.2.34. Since $H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$ (Example 2.2.3), the Kähler form associated with the Fubini-Study metric (HW 1.2.21) is proportional to an integral form. So $\mathbb{P}^n$ is a Hodge manifold. Since the property of being Hodge is inherited by complex submanifolds $Y \subset X$, it follows that every projective manifold $X \subset \mathbb{P}$ is Hodge.

2.2.4 Example: abelian varieties

Suppose $(X = \mathbb{C}^g/\Lambda, \omega)$ is an abelian variety (Theorem 1.1.32). Fix a basis $\{\lambda_1, \ldots, \lambda_{2g}\} \subset \Lambda \simeq H_1(X, \mathbb{Z})$. Let $(u_1, \ldots, u_{2g}) : \mathbb{C}^g \to \mathbb{R}^{2g}$ be the dual $\mathbb{R}$-coordinates of $H^k(X, \mathbb{Z})$. Then $\{\mathrm{d}u_j\}_{j=1}^{2g} \subset H^1(X, \mathbb{Z})$ is dual to the $\{\lambda_j\}_{j=1}^{2g}$. Define a skew-symmetric matrix $R = (r_{ij}) \in \mathrm{GL}_{2g}(\mathbb{Z})$ by $\omega = \frac{1}{2} \sum r_{ij} \mathrm{d}u_i \wedge \mathrm{d}u_j$.

Exercise 2.2.35 (Smith normal form)**.** (a) Show that there exists a choice of basis $\{\lambda_j\}_{j=1}^{2g}$ so that

$$R = \begin{bmatrix} 0 & -\Delta \\ \Delta & 0 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_g \end{bmatrix},$$

with $0 < \delta_i \in \mathbb{Z}$ and $\delta_a | \delta_{a+1}$.

(b) Show that $(\delta_1, \ldots, \delta_g)$ is an invariant of ω; that is, does not depend on the choice of normalizing basis.

Assume this normalization is in effect.

The polarized Hodge structure

Note that $\{\delta_1^{-1} \lambda_1, \ldots, \delta_g^{-1} \lambda_g\}$ is a complex basis of $\mathbb{C}^g$. And

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_g \\ \lambda_{g+1} \\ \vdots \\ \lambda_{2g} \end{bmatrix} = \begin{bmatrix} \Delta \\ P \end{bmatrix} \begin{bmatrix} \delta_1^{-1} \lambda_1 \\ \vdots \\ \delta_g^{-1} \lambda_g \end{bmatrix},$$

with $P = (p_{ab}) \in \mathrm{GL}_g\mathbb{C}$.

Let $(z_1, \ldots, z_g)$ be the complex coordinates dual to the basis $\{\delta_a^{-1} \lambda_a\}$. Using $\mathrm{d}z_a = \delta_a \, \mathrm{d}u_a + \sum p_{ab} \, \mathrm{d}u_{b+g}$, one may check that the polarization on $H^1(X, \mathbb{Z})$ satisfies

$$Q(\mathrm{d}z_a, \mathrm{d}z_b) \;=\; |\delta|(p_{ba} - p_{ab}) \quad \text{and} \quad Q(\mathrm{d}z_a, \mathrm{d}\overline{z}_b) \;=\; |\delta|(\overline{p}_{ab} - p_{ab}) \,,$$

where $|\delta| = \delta_1 \cdots \delta_g$. The Hodge–Riemann bilinear relations (Theorem 2.2.30) then force P to be symmetric with $\mathrm{Im}\, P$ positive definite, cf. Example 2.3.16.

Remark 2.2.36. Conversely, any symmetric $\tilde{P}$, with $\mathrm{Im}\, \tilde{P}$ positive definite, defines a Hodge decomposition on $H_{\mathbb{Z}} = H^1(X, \mathbb{Z})$ with $\tilde{H}^{1,0} = \mathrm{span}_{\mathbb{C}}\{\delta_a \, \mathrm{d}u_a + \sum \tilde{p}_{ab} \, \mathrm{d}u_{b+g}\}$.

Theta divisor

Definition 2.2.37. We call $(\delta_a)_{a=1}^g$ the *polarization type* of (X, ω), and say that X is *principally polarized* if all $\delta_a = 1$.

Theorem 2.2.38 ([Mum08])**.** *The polarization type* (Definition 2.2.37) *uniquely determines the polarizing/ample line bundle L on X, up to translation, and* $\dim H^0(X, L) = \delta_1 \cdots \delta_g$.

Definition 2.2.39. If X is principally polarized, then the *theta divisor* $\mathbb{P}\, H^0(X, L)$ is uniquely determined up to translation.

2.3 Hodge structures

Fix a lattice $H_{\mathbb{Z}} \simeq \mathbb{Z}^r$ of rank r. Given a field $\mathbb{Q} \subset \mathbf{k} \subset \mathbb{C}$, let $H_{\mathbf{k}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbf{k}$ be the associated vector space of dimension r. Fix an integer $n \in \mathbb{Z}$ and a non-degenerate bilinear form

$$Q : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \;\to\; \mathbb{Q}$$

that is (skew-)symmetric

$$Q(u, v) \;=\; (-1)^n \, Q(v, u) \,, \qquad \text{for all} \quad u, v \in H_{\mathbb{Q}} \,.$$

Let $\mathrm{Aut}(H) \simeq \mathrm{GL}_r$ be the group of invertible linear maps $H \to H$, and let $\mathrm{End}(H) \simeq \mathfrak{gl}_r$ be the Lie algebra of linear maps $H \to H$. Let

$$\mathbf{G} \;=\; \mathrm{Aut}(H,Q) \;=\; \{g \in \mathrm{Aut}(H) \mid Q(gu,gv) = Q(u,v)\,,\ \forall\, u,v \in H\} \qquad (2.3.1)$$

be the $\mathbb{Q}$-algebraic subgroup of automorphisms preserving Q, and let

$$\mathfrak{g} \;=\; \mathrm{End}(H,Q) \;=\; \{\xi \in \mathrm{End}(H) \mid Q(\xi u,v) + Q(u,\xi v) = 0\,,\ \forall\, u,v \in H\} \quad (2.3.2)$$

be its Lie algebra.

Definition 2.3.3. A (pure, rational) *Hodge structure of weight* $n \in \mathbb{Z}$ on the lattice $H_\mathbb{Z}$ is given by either of the following two equivalent objects: A *Hodge decomposition*

$$H_\mathbb{C} \;\stackrel{\mathrm{dfn}}{=\!=}\; H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \;=\; \bigoplus_{p+q=n} H^{p,q}\,, \qquad \text{such that} \quad \overline{H^{p,q}} = H^{q,p}\,. \qquad (2.3.4)$$

A (finite, decreasing) *Hodge filtration*

$$0 \subsetneq F^m \subset F^{m-1} \subset \cdots \subset F^{n-m+1} \subsetneq F^{n-m} \;=\; H_\mathbb{C}\,, \qquad (2.3.5)$$

with $\ell = 2m - n \geq 0$ the *level* of the Hodge structure, and such that

$$H_\mathbb{C} \;=\; F^k \oplus \overline{F^{n+1-k}}\,.$$

The equivalence of the two definitions is given by

$$F^k \;=\; \bigoplus_{p \geq k} H^{p,n-p} \quad \text{and} \quad H^{p,q} \;=\; F^p \cap \overline{F^q}\,.$$

The *Hodge numbers* $\mathbf{h} = (h^{p,q})$ and $\mathbf{f} = (f^p)$ are

$$h^{p,q} \;=\; \dim_\mathbb{C} H^{p,q} \quad \text{and} \quad f^p \;=\; \dim_\mathbb{C} F^p\,.$$

The Hodge structure is *effective* if $h^{p,q} \neq 0$ implies both $p,q \geq 0$. In this case $n \geq 0$, and the Hodge filtration is usually expressed as $0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = H_\mathbb{C}$.

Example 2.3.6. The *Tate Hodge structure* the pure, weight $n = -2$ Hodge structure on the lattice $\mathbb{Z}(1) \stackrel{\text{dfn}}{=\!=} 2\pi\mathbf{i}\mathbb{Z} \hookrightarrow \mathbb{C}$. Likewise, $\mathbb{Z}(m)$ is the pure, weight $n = -2m$ Hodge structure on the lattice $\mathbb{Z}(m) \stackrel{\text{dfn}}{=\!=} (2\pi\mathbf{i})^m\mathbb{Z} \hookrightarrow \mathbb{C}$.

Example 2.3.7 (effective, weight one)**.** An effective, weight $n = 1$ Hodge structure is given by a subspace $H^{1,0} = F^1 \subset H_{\mathbb{C}}$ such that $H_{\mathbb{C}} = H^{1,0} \oplus \overline{H^{1,0}}$. We will denote the Hodge numbers $\mathbf{h} = (h^{1,0}, h^{0,1}) = (g, g)$. The Hodge filtration is $F^1 = H^{1,0}$. For a geometric example, see Example 2.2.4.

Example 2.3.8 (effective, weight two)**.** An effective, weight $n = 2$ Hodge structure is given by subspaces $H^{2,0} \oplus H^{1,1} \subset H_{\mathbb{C}}$ so that $\overline{H^{1,1}} = H^{1,1}$ and $H_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus \overline{H^{2,0}}$. We will denote the Hodge numbers $\mathbf{h} = (h^{2,0}, h^{1,1}, h^{0,2}) = (a, b, a)$. The Hodge filtration is $F^2 = H^{2,0}$ and $F^1 = H^{2,0} \oplus H^{1,1}$.

Example 2.3.9 (effective, weight three)**.** An effective, weight $n = 3$ Hodge structure is given by subspaces $H^{3,0} \oplus H^{2,1} \subset H_{\mathbb{C}}$ so that $H_{\mathbb{C}} = H^{3,0} \oplus H^{2,1} \oplus \overline{H^{2,1} \oplus H^{3,0}}$. We will denote the Hodge numbers $\mathbf{h} = (h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (a, b, b, a)$. The Hodge filtration is $F^3 = H^{3,0}$, $F^2 = H^{3,0} \oplus H^{2,1}$, and $F^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2}$.

Remark 2.3.10. Note that (2.3.4) implies that $\dim H = 2g$ is even when n is odd.

Example 2.3.11 (compact Kähler manifolds)**.** The n-th cohomology group $H = H^n(X, \mathbb{Q})$ of a compact Kähler manifold admits an effective Hodge structure of weight n (§2.2). Here $H^{p,q} = H^{p,q}(X) \subset H^n(X, \mathbb{C})$ are the de Rham cohomology classes that can be represented by (p, q)–forms.

Remark 2.3.12. There are interesting and important Hodge structures that are not effective. An important example is the induced weight zero Hodge structure on the Lie algebra of the automorphism group (HW 2.3.24). We will see others when we study mixed Hodge structures (Chapter 5). Regardless, every Hodge structure can be converted to an effective Hodge structure via a Tate twist (Remark 2.3.21).

Definition 2.3.13. The Hodge structure (Definition 2.3.3) is *Q-polarized* if the *Hodge–Riemann bilinear relations* hold:

$$Q(H^{p,q}, H^{r,s}) \;=\; 0 \qquad \text{if} \qquad (p,q) \neq (s,r)\,, \tag{2.3.14}$$

$$\mathbf{i}^{p-q}\, Q(v,\bar{v}) \;>0 \quad \text{for all} \quad 0 \neq v \in H^{p,q}\,. \tag{2.3.15}$$

Example 2.3.16 (effective, weight one, polarized)**.** The first Hodge–Riemann bilinear relation is $Q(F^1, F^1) = 0$. Note that F^1 is maximal with this property: $(F^1)^\perp = F^1$. The second Hodge–Riemann bilinear relation is $\mathbf{i}Q(v,\bar{v}) > 0$ for all $0 \neq v \in H^{1,0}$.

These PHS are realized geometrically by algebraic curves and abelian varieties (Example 2.2.4).

Example 2.3.17 (effective, weight two, polarized)**.** The first Hodge–Riemann bilinear relation is $Q(F^2, F^1) = 0$. In this case we have $(F^2)^\perp = F^1$. The second Hodge–Riemann bilinear relation asserts that $-Q(u,u) > 0$ for all $0 \neq u \in H^{2,0}$ and $Q(v,v) > 0$ for all $0 \neq v \in H^{1,1}$.

Example 2.3.18 (effective, weight three, polarized)**.** The first Hodge–Riemann bilinear relation is $Q(F^2, F^2) = 0$. Again, F^2 is maximal with this property: $(F^2)^\perp = F^2$. The second Hodge–Riemann bilinear relation is $-\mathbf{i}Q(u,\bar{u}) > 0$ for all $0 \neq u \in H^{3,0}$, and $\mathbf{i}Q(v,\bar{v}) > 0$ for all $0 \neq v \in H^{2,1}$.

Example 2.3.19 (smooth projective varieties)**.** Let $X \subset \mathbb{P}^N$ be a projective manifold of dimension d with hyperplane class $\omega \in H^2(X,\mathbb{Z})$. Given $n \leq d$, and keeping in mind that X is Hodge (Definition 2.2.33), the primitive cohomology (Definition 2.2.23)

$$H \;=\; \{\alpha \in H^n(X,\mathbb{Q}) \mid \omega^{d-n+1} \wedge \alpha = 0\}$$

inherits the weight n Hodge decomposition

$$H_{\mathbb{C}} \;=\; \bigoplus_{p+q=n} H^{p,q}(X) \cap H_{\mathbb{C}}$$

from $H^n(X,\mathbb{Q})$. The Hodge–Riemann bilinear relations (Theorem 2.2.30) for X assert that this Hodge structure is polarized by

$$Q(\alpha,\beta) \;=\; (-1)^{n(n-1)/2}\int_X \alpha\wedge\beta\wedge\omega^{d-n}\,.$$

Exercise 2.3.20. Fix two lattices $H_{1,\mathbb{Z}}$ and $H_{2,\mathbb{Z}}$. Set $H_\mathbb{Z} = H_{1,\mathbb{Z}}\otimes H_{2,\mathbb{Z}}$. Given Q_j-polarized Hodge decompositions $H_{j,\mathbb{C}} = \oplus H_j^{p,q}$ of weight n_j, show that

$$H^{p,q} \;=\; \bigoplus_{\substack{p_1+p_2=p\\ q_1+q_2=q}} H_1^{p_1,q_1}\otimes H_2^{p_2,q_2}$$

defines a weight $n=n_1+n_2$ Hodge decomposition $H_\mathbb{C}=\oplus H^{p,q}$ that is polarized by $Q=Q_1\otimes Q_2$.

Remark 2.3.21. Recall the Tate Hodge structure (Example 2.3.6). Note that $\mathbb{Z}(m) = \mathbb{Z}(1)^{\otimes m}$ and $\mathbb{Z}(-m) = \mathbb{Z}(-1)^{\otimes m}$ for all $m\geq 0$. Moreover, given a weight n Hodge structure on $H_\mathbb{Z}$, the induced Hodge structure on $H(m)_\mathbb{Z} = H_\mathbb{Z}\otimes\mathbb{Z}(m)$ has weight $n-2m$, and is given by $H(m)^{p,q} = H^{p+m,q+m}$. For $m\gg 0$, $H(-m)$ will be effective.

Exercise 2.3.22. Show that the real automorphism group $G_\mathbb{R} \stackrel{\text{dfn}}{=\!=} \mathbf{G}(\mathbb{R}) = \mathrm{Aut}(H_\mathbb{R},Q)$ is isomorphic to:

- $\mathrm{Sp}(2g,\mathbb{R})$, where $2g=\dim H$, when n is odd;
- $\mathrm{O}(b,2a)$, where

  $$b \;=\; \sum_k h^{m+2k,m-2k} \quad\text{and}\quad 2a \;=\; \sum_k h^{m+1+2k,m-1-2k}\,,$$

  when $n=2m$ is even.

Exercise 2.3.23. Show that

$$\mathcal{H}(u,v) \;=\; \mathbf{i}^n\,Q(u,\bar v)$$

defines a nondegenerate Hermitian form on $H_\mathbb{C}$ of signature

- (g,g), where $2g = \dim H$, when n is odd;
- $(b, 2a)$, when n is even.

Exercise 2.3.24 (Induced Hodge structure on the endomorphism algebra)**.** Fix a Q-polarized Hodge structure on $H_\mathbb{Z}$ of weight n. Let $H_\mathbb{C} = \oplus H^{r,s}$ denote the Hodge decomposition. Recall the Lie algebra $\mathfrak{g}$ defined in (2.3.2). Show that

$$\mathfrak{g}^{p,-p} \ = \ \{\xi \in \mathfrak{g}_\mathbb{C} \text{ s.t. } \xi(H^{r,s}) \subset H^{r+p,s-p} \, , \ \forall \, r,s\}$$

defines a weight zero Hodge structure on $\mathfrak{g}$.

Exercise 2.3.25. Show that the induced Hodge structure on $\mathfrak{g}$ (HW 2.3.24) is polarized by $-\kappa$, where κ is the Killing form.

2.4 Complex tori constructed from Hodge structures

Many important complex tori (Example 2.2.4) are constructed from Hodge structures.

2.4.1 Albanese variety

The dual to the Picard variety (§A.3.13) is the *Albanese variety*

$$\mathrm{Alb}(X) \ = \ \frac{H^0(X,\Omega^1_X)^\vee}{H_1(X,\mathbb{Z})} \ = \ \frac{H^{1,0}(X)^\vee}{H_1(X,\mathbb{Z})} \, .$$

It has the universal property that any morphism from X to an abelian variety factors uniquely through the *Albanese map*

$$\alpha : X \ \to \ \mathrm{Alb}(X) \, , \quad \alpha(x)(\eta) \ = \ \int_{x_o}^x \eta \, .$$

Exercise 2.4.1. Fix a basis $\omega_1, \ldots, \omega_g$ of $H^0(X, \Omega^1_X)$, and show that the Albanese map may be identified with the map

$$\alpha(x) \ = \ \left[\int_{x_o}^x \omega_1 \, , \ \ldots \, , \ \int_{x_o}^x \omega_g \right] \, .$$

2.4.2 Jacobian variety of a curve

In the case that X is a nonsingular algebraic curve of genus g, the Albanese variety is known as the Jacobian $\mathrm{Jac}(X)$ of the curve.

Exercise 2.4.2. Show that $\mathrm{Jac}(X)$ is a principally polarized abelian variety (Definition 2.2.37). [*Hint.* The Hodge–Riemann bilinear relations imply that $h(u,v) = -\mathbf{i}Q(u,\overline{v})$ is a positive definite hermitian form on $H^{1,0}(X)$.]

2.4.3 Griffiths tori

Fix a Q-polarized Hodge structure φ on $H_{\mathbb{Z}}$ of odd weight $n = 2p-1$ (as in Definitions 2.3.3 and 2.3.13), and with Hodge decomposition

$$H_{\mathbb{C}} \;=\; \underbrace{H^{n,0}_{\varphi} \oplus \cdots \oplus H^{p,p-1}_{\varphi}}_{\overline{L_{\varphi}}} \;\oplus\; \underbrace{H^{p-1,p}_{\varphi} \oplus \cdots \oplus H^{0,n}_{\varphi}}_{L_{\varphi}}\,.$$

Exercise 2.4.3. (a) Show that the image of $H_{\mathbb{Z}} \hookrightarrow H_{\mathbb{C}} = L_{\varphi} \oplus \overline{L_{\varphi}} \twoheadrightarrow L_{\varphi}$ is a lattice.

The *intermediate Jacobian* is the associated complex torus $J(\varphi) = L_{\varphi}/H_{\mathbb{Z}}$.

(b) Show that $h(u,v) = -\mathbf{i}Q(u,\overline{v})$ defines a non-degenerate bilinear form on L of signature (s_+, s_-) with $s_+ = h^{p-1,p} + h^{p-3,p+2} + h^{p-5,p+4} + \cdots$.

(c) Show that the imaginary part of $h = g - \mathbf{i}\omega$ satisfies $\omega \in H^{1,1}(J(\varphi)) \cap H^2(J(\varphi),\mathbb{Z})$.

We call ω a *pseudo-polarization*. The intermediate Jacobian is an abelian variety when $s_+\,s_- = 0$. (Eg. whenever $L_{\varphi} = H^{p-1,p}_{\varphi}$.)

Example 2.4.4. Suppose that X is a nonsingular projective variety of dimension d. Then the intermediate Jacobian $J^{2d-1}(X) = H^{d-1,d}(X)/H^{2d-1}(X,\mathbb{Z})$ is an abelian variety ($s_- = 0$). Serre duality implies that $J^{2d-1}(X)$ can be identified with the Albanese variety $\mathrm{Alb}(X) = H^0(X,\Omega^1_X)^{\vee}/H_1(X,\mathbb{Z})$ of §2.4.1.

Example 2.4.5. Suppose that X is a nonsingular projective variety of dimension d. If $d = 2p-1$ is odd, then $J^d(X) = (H^{p-1,p}(X) \oplus \cdots \oplus H^{0,d}(X))/H^d(X,\mathbb{Z})$. The intersection form on d-cycles is unimodular (this is essentially equivalent to Poincaré duality [GH94]). So, if $s_- = 0$, the Jacobian will be principally polarized.

Exercise 2.4.6. Suppose that $s_+ s_- = 0$, so that $J(\varphi)$ is an abelian variety. Assume that the polarization Q is *unimodular* on $H_\mathbb{Z}$: the matrix representation of with respect to an integral basis has determinant ± 1. Show that the polarization on $J(\varphi)$ is principal.

Theorem 2.4.7 (Griffiths)**.** *Two Hodge structures $\varphi, \varphi' \in D$ belong to the same $G_\mathbb{Z}$ orbit if and only if $J(\varphi) \simeq J(\varphi')$ as pseudo-polarized tori.*

Remark 2.4.8. Replacing L_φ in the construction above with $H^{p-1,1}_\varphi \oplus H^{p-3,3} \oplus \cdots \oplus H^{0,n}$, we obtain the Weil torus $I(\varphi)$, which is always polarized. However, the Weil tori $I(\varphi)$ do not vary holomorphically with $\varphi \in D$, while the Griffiths tori $J(\varphi)$ do. (The two tori may be interpreted as different complex structures on the real torus $H_\mathbb{R}/H_\mathbb{Z}$).

2.5 Hodge structures: a third definition*

Recommended references: [GGK12, Pat16].

We have seen that a Hodge structure may be defined by either a Hodge decomposition (2.3.4), or by a Hodge filtration (2.3.5). There is a third definition by group homomorphisms.[2] Let $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ be the group of nonzero complex numbers. Define a homomorphism

$$\tilde{\varphi} : \mathbb{C}^\times \rightarrow \mathrm{Aut}(H_\mathbb{R}) \tag{2.5.1}$$

[2]We will not have much use for this third definition in this class (and this section is optional reading). However, of the three this is in many respects the optimal definition (for example, if one wishes to discuss Hodge tensors or Mumford–Tate groups), and so worthwhile including here.

by specifying

$$\tilde{\varphi}(z) \;=\; z^p \bar{z}^q \, v\,, \qquad \text{for all} \quad v \in H^{p,q}\,;$$

that is, we specify that the Hodge decomposition (2.3.4) is an eigenspace decomposition for $\tilde{\varphi}$.

Remark 2.5.2. Observe that (2.5.1) satisfies $\varphi(x) = x^n \, \mathrm{Id}$, for all nonzero real numbers $x \in \mathbb{R}^\times$.

Exercise 2.5.3. (a) Verify that $\tilde{\varphi}$ does indeed take value in $\mathrm{Aut}(H_\mathbb{R})$.

(b) Verify that the restriction $\tilde{\varphi}|_{S^1}$ takes value in $\mathrm{Aut}(H_\mathbb{R}, Q)$ if and only if the Hodge structure satisfies the first Hodge–Riemann bilinear relation (2.3.14). In this case, the Hermitian form $\mathbf{i}^n Q(u, \overline{v})$ is nondegenerate on $H^{p,q}$.

We wish to view $\mathbb{C}^\times$ as real group. At the very least you should think of it a real Lie group. If you are familiar with algebraic groups, then you should think of $\mathbb{C}^\times$ as the real points

$$\mathbf{S}(\mathbb{R}) \;=\; \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \;\middle|\; \begin{array}{l} x, y \in \mathbb{R} \\ x^2 + y^2 \neq 0 \end{array} \right\}$$

of the *Deligne torus*, the $\mathbb{R}$–algebraic group $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, G_{m,\mathbb{C}}$. Likewise, we identify $S^1 \subset \mathbb{C}^\times$ with the maximal compact subgroup

$$\mathbf{U}(\mathbb{R}) \;=\; \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \;\middle|\; \begin{array}{l} x, y \in \mathbb{R} \\ x^2 + y^2 = 1 \end{array} \right\}.$$

Exercise 2.5.4. Conversely suppose that you are given a homomorphism (2.5.1), with the property that $\tilde{\varphi}|_{\mathbb{R}^\times}$ is defined over $\mathbb{Q}$.

(a) Show that $H \;=\; \oplus_{n \in \mathbb{Z}} H_n$ where

$$H_n \;=\; \{ v \in H \mid \tilde{\varphi}(x)(v) = x^n \, \mathrm{Id}\,,\ x \in \mathbb{R}^\times \}\,.$$

(b) Set $H^{p,q} \;=\; \{ v \in H_\mathbb{C} \mid \tilde{\varphi}(z)(v) \;=\; z^p \bar{z}^q \, v\,,\ \forall\ z \in \mathbb{C}^\times \}$. Show that $H_{n,\mathbb{C}} = \oplus_{p+q=n} H^{p,q}$ is a Hodge decomposition of weight n.

Definition 2.5.5. The upshot of the discussion above is that we may define a (real) *Hodge structure* as a homomorphism (2.5.1) of $\mathbb{R}$–algebraic groups. The Hodge structure is *rational* if $\tilde{\varphi}|_{\mathbb{R}^\times}$ is defined over $\mathbb{Q}$; it is *pure* of weight $n \in \mathbb{Z}$ if $\tilde{\varphi}(r) = r^n \, \mathrm{Id}$ for all $r \in \mathbb{R}^\times$; and if the Hodge structure is Q–polarized, then $\varphi = \tilde{\varphi}|_{S^1}$ takes value in $\mathrm{Aut}(H_{\mathbb{R}}, Q)$. We may identify the period domain D with the $\mathrm{Aut}(H_{\mathbb{R}}, Q)$ conjugacy classes of φ, and the isotropy group H is clearly seen to be the centralizer of the circle $\varphi : S^1 \to \mathrm{Aut}(H_{\mathbb{R}}, Q)$.

Chapter 3

Families and period maps

3.1 Monodromy

Recommended reference: [Ara12, Voi07].

Consider a smooth surjective holomorphic mapping $f : \mathcal{X} \to S$ of complex manifolds with compact fibres. In this context "smooth" means $\mathrm{d}f$ has maximal rank everywhere; equivalently, f is a submersion as a map of smooth manifolds. In particular, the fibres $X_s = f^{-1}(s)$ are compact, complex submanifolds of $\mathcal{X}$; and we regard $f : \mathcal{X} \to S$ as a family compact, complex manifolds $\{X_s\}_{s \in S}$ that is parameterized by S.

Suppose that $U \subset S$ is open and contractible. Fix $u_o \in U$. Then *Ehresmann's theorem* asserts that there is a diffeomorphism $\varphi_U : f^{-1}(U) \to U \times X_{u_o}$ so that

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\varphi_U} & U \times X_{u_o} \\ \downarrow{\scriptstyle f} & \swarrow & \\ U & & \end{array}$$

commutes. In particular, the fibres X_s are all diffeomorphic. However, in general they will not be biholomorphic. So $f : \mathcal{X} \to S$ may also be viewed as a family of complex structures on a fixed smooth manifold X_{s_o}.

Consider $f : \mathcal{X} \to S$ as a submersion of smooth manifolds. Since f is a submersion $\mathcal{V} = \ker \mathrm{d}f$ defines a subbundle of the (real) tangent bundle $T\mathcal{X}$. Note that $\mathcal{V}|_{X_s} = TX_s$. Fix a Riemannian metric g on $\mathcal{X}$ and consider the decomposition $TX = \mathcal{V} \oplus \mathcal{H}$ given by $\mathcal{H} = \mathcal{V}^\perp$.

Given a curve $\gamma : [0,1] \to S$ and $x_0 \in X_{\gamma(0)}$ there is a unique lift $\tilde{\gamma}(x_o, \cdot) : [0,1] \to \mathcal{X}$ determined by $\gamma(t) = f \circ \tilde{\gamma}(x_o, t)$, $\tilde{\gamma}(x_o, 0) = x_0$ and $\partial_t \tilde{\gamma}(x_o, t) \in \mathcal{H}_{\tilde{\gamma}(x_o t)}$. This defines $\tilde{\gamma} : X_{\gamma(0)} \times [0,1] \to \mathcal{X}$ with $\tilde{\gamma}(x, 0) = x$ for all $x \in X_{\gamma(0)}$, and $\tilde{\gamma}(x, 1) \in X_{\gamma(1)}$. In particular, we have a map $\tilde{\gamma}(\cdot, 1) : X_{\gamma(0)} \to X_{\gamma(1)}$. This map is a (homeomorphism) diffeomorphism if γ is (piecewise) smooth; and induces a map $\mu(g, \gamma) : H_k(X_{\gamma(0)}, \mathbb{Z}) \to H_k(X_{\gamma(1)}, \mathbb{Z})$.

Exercise 3.1.1. (a) Show that the map $\mu(g, \gamma)$ is independent of our choice of Riemannian metric g on $\mathcal{X}$.

(b) Show that the map $\mu(g, \gamma) = \mu(\gamma)$ depends only on the homotopy class of γ (with fixed endpoints).

Definition 3.1.2. In the case that the curve is closed $s_o = \gamma(0) = \gamma(1)$, this yields the *monodromy representation*

$$\rho : \pi_1(S, s_o) \ \to \ \mathrm{Aut}(H_k(X_{s_o}, \mathbb{Z}))\,. \tag{3.1.3}$$

Example 3.1.4. In the case that $f : \Delta^* \to \Delta^*$ is given by $f(x) = x^k$, and $\gamma(t) = e^{2\pi \mathbf{i} t}\, s_o$ is a generator of $\pi_1(\Delta^*) = \mathbb{Z}$, we have $\Gamma(x, 1) = x e^{2\pi \mathbf{i}/k}$, where $x^k = s_o$. That is, monodromy is analytic continuation of a branch of $f^{-1}(s) = \sqrt[k]{s}$, and the image of the monodromy representation is isomorphic to $\mathbb{Z}/k\mathbb{Z}$.

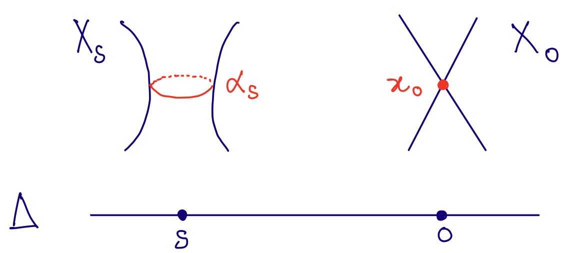
Definition 3.1.5. Suppose that $f : \mathcal{X} \to \Delta$ is a proper surjective morphism, and that the restriction of f to $\mathcal{X}^* = f^{-1}(\Delta^*)$ is smooth (again, this means f is a submersion on $\mathcal{X}^*$). Let $\gamma \in \pi_1(\Delta^*, s_o) \simeq \mathbb{Z}$ be a generator represented by a counter-clockwise loop $t \mapsto e^{2\pi \mathbf{i} t}\, s_o$. The induced monodromy

$$T : H^\bullet(X_{s_o}, \mathbb{Q}) \ \to \ H^\bullet(X_{s_o}, \mathbb{Q}) \tag{3.1.6}$$

is the *Picard–Lefschetz transformation of the family f.*

Definition 3.1.7. If $x_o \in X_0$ is a *node* (a.k.a. simple, isolated singularity), then there exist local coordinates $(z_0, \ldots, z_n)$ on $\mathcal{X}$ and centered at x_0 so that $f(z) = z_0^2 + \cdots z_n^2$. The fibres X_s close to X_0 contain a cycle $\alpha_s \in H_n(X_s, \mathbb{Z})$ that is represented by an n-sphere $i_s : S^n \hookrightarrow X_s$. If we write $s = r^2\, e^{2\mathbf{i}\theta}$, in polar coordinates, then the embedding i_s maps $\zeta \in S^n \mapsto re^{\mathbf{i}\theta}\,\zeta$. We call α_s a *vanishing cycle* because these spheres collapse to a point as $s \to 0$. Cf. Figure 3.1.

Figure 3.1: Vanishing cycle



Exercise 3.1.8. The proof that $\alpha_s \neq 0 \in H_n(X_s, \mathbb{Z})$ is outlined below. Fill-in the details.

Fix $0 < \rho < \varepsilon \ll 1$. Define

$$B \;=\; \left\{ z \text{ s.t. } |z_0|^2 + \cdots + |z_n|^2 \leq \varepsilon\,,\ |z_0^2 + \cdots + z_n^2| \leq \rho \right\}\,.$$

Fix $s = \rho \in \Delta^*$. Write $z = x + \mathbf{i}y$, with $x, y \in \mathbb{R}^{n+1}$.

(a) Show that we may identify

$$X_s \cap B \;=\; \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \quad \text{s.t.} \quad \|x\|^2 + \|y\|^2 \leq \varepsilon\,, \\ \|x\|^2 - \|y\|^2 = \rho\,,\ x \cdot y = 0\}\,.$$

(b) Show that $\|x\| \neq 0$ and $\|y\|^2 \leq \frac{1}{2}(\varepsilon - \rho)$. Prove that $(x, y) \mapsto \left(\dfrac{x}{\|x\|}, \dfrac{2y}{2-\rho} \right)$ defines a homeomorphism

$$X_s \cap B \;\to\; \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \text{ s.t. } \|x\|^2 = 1\,,\ \|y\|^2 \leq 1\,,\ x \cdot y = 1\}\,.$$

(c) Show that $X_s \cap B$ deformation retracts onto the sphere $S^n = \{(x,0) \text{ s.t. } \|x\| = 1\}$.

(d) Conclude that S^n generates $H_n(X_s \cap B, \mathbb{Z}) = \mathbb{Z}$, and $H_k(X_s \cap B, \mathbb{Z}) = 0$ for all $k \neq 0, n$.

It remains to observe that Poincaré duality implies $H^n_{\mathrm{cpt}}(X_s \cap B) \simeq H^n_{\mathrm{d}}(X_s \cap B) \neq 0$. So there exists a closed n-form η on X_s with compact support contained in $X_s \cap B$, so that $\int_{i_s(S^n)} \eta \neq 0$. It follows that $\alpha_s = [i_s(S^n)] \subset H_n(X_s, \mathbb{Z})$ is nonzero.

Theorem 3.1.9 (Picard–Lefschetz formulas)**.** *Assume that the simple singularity $x_o \in X_0$ is the the unique singular point of the family $f : \mathcal{X} \to \Delta$ (the unique point where* $\mathrm{d}f$ *drops rank). Then the Picard–Lefschetz transformation* (3.1.6) *is:*

(i) *the identity $T = \mathrm{id}$ on $H_k(X_{s_o}, \mathbb{Q})$ for all $k \neq n$; and*

(ii) *given by $T(\beta) = \beta + \epsilon\,(\beta, \alpha_{s_o})\,\alpha_{s_o}$, for all $\beta \in H_n(X_{s_o}, \mathbb{Q})$, with*

$$\epsilon \;=\; \begin{cases} 1\,, & \text{if } n \equiv 2, 3 \text{ mod } 4\,, \\ -1\,, & \text{otherwise,} \end{cases}$$

and

$$T(\alpha_{s_o}) \;=\; \begin{cases} \alpha_{s_o}\,, & \text{if } n \equiv 1, 3 \text{ mod } 4\,, \\ -\alpha_{s_o}\,, & \text{otherwise.} \end{cases}$$

Exercise 3.1.10. Fix a nonsingular projective variety $X \subset \mathbb{P}^m$ of dimension n, and $L \simeq \mathbb{P}^{m-2} \subset \mathbb{P}^m$.

(a) Write $\mathbb{P}^m = \mathbb{P}(V)$, with V a complex vector space of dimension $m+1$. Show that $\check{\mathbb{P}}^m = \mathbb{P}(V^\vee)$ parameterizes the set of all hyperplanes $H = \mathbb{P}^{m-1} \subset \mathbb{P}(V)$.

(b) Fix a projective subspace $L \simeq \mathbb{P}^{m-2}$. Show that the set of all hyperplanes $H = \mathbb{P}^{m-1}$ containing L is parameterized by a $\mathbb{P}^1$.

Definition 3.1.11. A *pencil of hypersurfaces on* X is a family $\{X_s = X \cap H_s\}_{s \in \mathbb{P}^1}$. We say the family is *Lefschetz pencil* if:

(i) The intersection $X \cap L$ is nonsingular.

(ii) There is a finite set $\{p_1, \ldots, p_k\} \subset \mathbb{P}^1$, so that $X_s = X \cap H_s$ is nonsingular for all $s \in S = \mathbb{P}^1 \backslash \{p_1, \ldots, p_k\}$.

(iii) For each p_j, the variety X_{p_j} has a single simple singularity $x_j \in L \cap X_{p_j}$.

Theorem 3.1.12. *For a generic choice of $L = \mathbb{P}^{m-2}$ the family $\{X_s\}_{s \in \mathbb{P}^1}$ is Lefschetz pencil.*

Exercise 3.1.13. The set

$$\mathcal{X} \;=\; \{(x, s) \in X \times S \text{ s.t. } x \in H_s\}\,.$$

is a fibre bundle over S. The monodromy representation (3.1.3) is computed as follows. Each p_j determines vanishing cycle $\alpha_j \in H_n(X_{s_o}, \mathbb{Q})$.[1] Fix a curve $\gamma_j \in \pi_1(S, s_o)$ traveling from s_o to a point near p_j, looping once around p_j counter-clockwise, and returning to s_o. Without loss of generality the paths γ_j are pairwise disjoint away from s_o. The induced monodromy action $T_j = \rho(\gamma_j)$ on $H_k(X_{s_o}, \mathbb{Q})$ is trivial if $k \neq n$, and is given by

$$T_j(\beta) \;=\; \beta \,+\, \epsilon\,(\beta, \alpha_j)\,\alpha_j \tag{3.1.14}$$

for all $\beta \in H_n(X_{s_o}, \mathbb{Q})$. Let $\mathrm{Van} = \mathrm{span}_{\mathbb{C}}\{\alpha_1, \ldots, \alpha_k\} \subset H_n(X_{s_o}, \mathbb{Q})$ be the subspace spanned by the vanishing cycles. It can be shown that the intersection pairing is nondegenerate on Van. Let $\Gamma = \rho(\pi(S, s_o))$ be the image of the monodromy representation (3.1.3).

(a) Show that the subspace Van is invariant under the monodromy.

(b) Show that $H_n(X_{s_o}, \mathbb{Q}) = \mathrm{Van} \oplus \mathrm{Inv}$ with $\mathrm{Inv} = \{\beta \in H_n(X_{s_o}, \mathbb{Q}) \text{ s.t. } T(\beta) = \beta\,,\ \forall\, T \in \Gamma\} = \mathrm{Van}^{\perp}$ the cycles invariant under the monodromy representation. (For a generalization, see Remark 5.4.31.)

[1] The cycle α_j is defined only up to the action of the monodromy group. However, the space spanned by the vanishing cycles is well-defined (HW 3.1.13).

(c) Show that the monodromy acts irreducibly on Van (with no nontrivial invariant subspaces).

[*Hint.* The loops γ_j generate $\pi(S, s_o)$. So the T_j generate Γ.]

3.2 The compact dual

The first Hodge–Riemann bilinear relation (2.3.14) asserts that the Hodge filtration (2.3.5) is *Q–isotropic*

$$Q(F^p, F^q) \;=\; 0\,, \qquad \text{for all} \quad p+q = n+1\,.$$

This is precisely the statement that the Hodge filtration is an element of the complex flag manifold

$$\check{D} \;=\; \mathrm{Flag}^Q(\mathbf{f}, H_{\mathbb{C}}) \tag{3.2.1}$$

of Q–isotropic filtrations $F = (F^p)$ of $H_{\mathbb{C}}$ satisfying $\dim F^p = f^p$. The variety $\check{D}$ is the *compact dual* of the period domain D (which will be defined next).

Example 3.2.2 (effective, weight one)**.** The compact dual is $\mathbb{P}^1$, when $g = 1$.

For $g \geq 1$, the compact dual is the Lagrangian grassmannian $\mathrm{LG}(g, \mathbb{C}^{2g})$ of g-dimensional subspaces $F^1 \subset H_{\mathbb{C}} \simeq \mathbb{C}^{2g}$ that are isotropic with respect to a nondegenerate skew-symmetric bilinear form.

Example 3.2.3 (effective, weight two)**.** The compact dual is the Grassmannian $\mathrm{Gr}^Q(a, \mathbb{C}^{2a+b})$ of a–dimensional subspaces $F^2 \subset H_{\mathbb{C}} \simeq \mathbb{C}^{2a+b}$ that are isotropic with respect to the nondegenerate, symmetric bilinear form Q.

Example 3.2.4 (effective, weight three)**.** The compact dual is the isotropic flag manifold $\mathrm{Flag}^Q(a, g; \mathbb{C}^{2g})$, consisting of pairs $F^3 \subset F^2$ with $F^2 \in \mathrm{LG}(g, \mathbb{C}^{2g})$, $\dim_{\mathbb{C}} F^3 = a$, and where $g = a + b$.

Exercise 3.2.5. Show that the complex automorphism group $G_{\mathbb{C}} = \mathbf{G}(\mathbb{C}) = \mathrm{Aut}(H_{\mathbb{C}}, Q)$ acts transitively on $\check{D}$.

3.3 Period domain

Definition 3.3.1. The *period domain* $D = D_{\mathbf{h},Q} \subset \check{D}$ is the set of all Q-polarized Hodge structures on $H_{\mathbb{Z}}$ with Hodge numbers $\mathbf{h}$.

Slogan. The compact dual $\check{D} \subset \mathrm{Flag}(\mathbf{f}, H_{\mathbb{C}})$ parameterizes filtrations satisfying the first Hodge–Riemann bilinear relation, and the period domain $D \subset \check{D}$ parameterizes filtrations satisfying both Hodge–Riemann bilinear relations.

Example 3.3.2 (effective, weight one)**.** When $g = 1$ the period domain is the upper-half plane, and $\mathrm{Aut}(H_{\mathbb{R}}, Q) = \mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$ acts transitively.

For $g \geq 1$, the period domain $D = \mathrm{Sp}(2g, \mathbb{R})/U(g)$ is the Siegel upper-half space of symmetric $g \times g$ matrices with complex entries and positive definite imaginary part. Alternatively D is the set of $E \in \mathrm{LG}(g, \mathbb{C}^{2g})$ with the property that the Hermitian form $\mathbf{i}Q(u, \bar{u})$ restricts to be positive definite on E.

We recover the Hodge decomposition from E by setting $H^{1,0} = E$ and $H^{0,1} = \overline{E}$.

Example 3.3.3 (effective, weight two)**.** The period domain $D = \mathrm{O}(b, 2a)/\mathrm{U}(a) \times \mathrm{O}(b)$ is the subset of elements $E \in \mathrm{Gr}^Q(a, \mathbb{C}^{2a+b})$ on which the Hermitian bilinear form $-Q(u, \bar{v})$ restricts to be positive definite.

We recover the Hodge decomposition from E by setting $H^{2,0} = E$ and $H^{0,2} = \overline{E}$, and $H^{1,1} = (E \oplus \overline{E})^{\perp}$.

Example 3.3.4 (effective, weight three)**.** The period domain $D = \mathrm{Sp}(2g, \mathbb{R})/\mathrm{U}(a) \times \mathrm{U}(b)$ is the subset of filtrations $(F^3 \subset F^2) \in \mathrm{Flag}^Q(a, g; \mathbb{C}^{2g})$ with the property that the Hermitian form $-\mathbf{i}Q(u, \bar{v})$ restricts to be positive definite on F^3, and nondegenerate on F^2 with signature (a, b).

Given a point $\varphi \in D$, the associated Hodge decomposition will be expressed as $H_{\mathbb{C}} = \oplus H^{p,q}_{\varphi}$, and the associated Hodge filtration will be expressed as $F_{\varphi} = (F^p_{\varphi})$.

Exercise 3.3.5. Show that the real automorphism group $G_{\mathbb{R}}$ of (2.3.1) acts transitively on D with compact isotropy L (stabilizer of a point $\varphi \in D$) isomorphic to:

- $\mathrm{U}(h^{n,0}) \times \cdots \times \mathrm{U}(h^{m+1,m})$, if $n = 2m+1$ is odd;
- $\mathrm{U}(h^{n,0}) \times \cdots \times \mathrm{U}(h^{m+1,m-1}) \times \mathrm{O}(h^{m,m})$, if n is even.

Exercise 3.3.6. Show that $D \subset \check{D}$ is open (in the analytic topology). In particular, D inherits the structure of a complex manifold from $\check{D}$, and is a "flag domain" in the sense of [Wol69, FHW06].

Exercise 3.3.7. Fix a Hodge structure $\varphi \in D$. Recall the induced Hodge structure of HW 2.3.24.

(a) Show that the Lie algebra of the stabilizer $L_\varphi = \mathrm{Stab}_{G_\mathbb{R}}(\varphi)$ has complexification $\mathfrak{l}_\mathbb{C} = \mathfrak{g}_\varphi^{0,0}$.

(b) Show that the Lie algebra of the stabilizer $P_\varphi = \mathrm{Stab}_{G_\mathbb{C}}(\varphi)$ of $\varphi \in \check{D}$ is $\mathfrak{p} = \oplus_{p\geq 0}\, \mathfrak{g}_\varphi^{p,-p}$.

3.4 Horizontal subbundle

The compact dual $\check{D} = \mathrm{Flag}^Q(\mathbf{f}, H_\mathbb{C})$ naturally sits inside the flag manifold

$$\mathrm{Flag}(\mathbf{f}, H_\mathbb{C}) \;=\; \left\{ (F^n, \ldots, F^0) \in \prod_{p=n}^{0} \mathrm{Gr}(f^p, H_\mathbb{C}) \text{ s.t. } F^p \subset F^{p-1}\,,\ \forall\, 1 \leq p \leq n \right\}.$$

Exercise 3.4.1. Show that $T_{\mathrm{Flag}(\mathbf{f},H_\mathbb{C}),F} \;=\; \bigoplus_{p=n}^{1} \mathrm{Hom}(F^p, H_\mathbb{C}/F^p)$.

Let $\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^0 = \mathrm{Flag}(\mathbf{f}, H_\mathbb{C}) \times H_\mathbb{C}$ be the tautological filtration of the trivial bundle. Then $T_{\mathrm{Flag}(\mathbf{f},H_\mathbb{C})} \;=\; \bigoplus_{p=n}^{1} \mathrm{Hom}(\mathcal{F}^p, \mathcal{F}^0/\mathcal{F}^p)$.

Definition 3.4.2 (Horizontal subbundle, first definition)**.** The *horizontal subbundle of the flag manifold* is

$$T^{\mathrm{h}}_{\mathrm{Flag}(\mathbf{f},H_\mathbb{C})} \;\overset{\mathrm{dfn}}{=\!=}\; \bigoplus_{p=n}^{1} \mathrm{Hom}(\mathcal{F}^p, \mathcal{F}^{p-1}/\mathcal{F}^p)\,.$$

The *horizontal subbundle of the compact dual* is

$$T^{\mathrm{h}}_{\check D} \ \stackrel{\mathrm{dfn}}{=\!=}\ T_{\check D} \cap T^{\mathrm{h}}_{\mathrm{Flag}(\mathbf{f},H_{\mathbb{C}})}\,.$$

A holomorphic map $f : M \to \check D$ is *horizontal* (or satisfies the *infinitesimal period relation* (IPR)) if $f_*(T_xM) \subset T^{\mathrm{h}}_{\check D, f(x)}$. The IPR is *trivial* if $T^{\mathrm{h}}_{\check D} = T_{\check D}$.

Example 3.4.3. A holomorphic curve $\gamma(t) = (F^p_t) : \Delta \to \check D$ is *horizontal* if and only if for every curve $e : \Delta \to H_{\mathbb{C}}$ with $e(t) \in F^p_t$, for all t, we have $\dot e(t) \in F^{p-1}_t$. The *horizontal subbundle* $T^{\mathrm{h}}_{\check D} \subset T_{\check D}$ is the set of all $\dot\gamma(t)$ with γ horizontal.

For this reason, the IPR is often expressed as

$$\mathrm{d}F^p \ \subset\ F^{p-1}\,.$$

Exercise 3.4.4. Suppose that D is a period domain parameterizing weight $n = 1$ polarized Hodge structures. Prove that the IPR is trivial.

Exercise 3.4.5. Suppose that D is a period domain parameterizing weight $n = 2$ polarized Hodge structures with Hodge numbers $\mathbf{h} = (1, h, 1)$. Prove that the IPR is trivial.

Exercise 3.4.6. Suppose that D is a period domain parameterizing weight $n = 2$ polarized Hodge structures with Hodge numbers $\mathbf{h} = (2, h, 2)$. Prove that $T^{\mathrm{h}}_{\check D} \subset T_{\check D}$ has corank 1. (In fact, $T^{\mathrm{h}}_{\check D}$ is a contact distribution.)

Definition 3.4.7 (Horizontal subbundle, second definition)**.** Recall the notations of HW 2.3.24, 3.2.5 and 3.3.7. A a homogeneous space the compact dual is $\check D = G_{\mathbb{C}} \cdot \varphi = G_{\mathbb{C}}/P_\varphi$. In particular,

$$T_{\check D,\varphi} \ \simeq\ \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_\varphi$$

as vector spaces. As a homogeneous vector bundle, the holomorphic tangent bundles is (dropping the subscript φ)

$$T_{\check D} \ =\ G_{\mathbb{C}} \times_P (\mathfrak{g}_{\mathbb{C}}/\mathfrak{p})\,.$$

The induced action of P on $\mathfrak{g}_\mathbb{C}/\mathfrak{p}$ preserves $F^{-1}(\mathfrak{g}_\mathbb{C})/\mathfrak{p}$; the *horizontal subbundle* is the associated homogeneous subbundle

$$T^{\mathrm{h}}_{\check{D}} \;=\; G_\mathbb{C} \times_P \left(F^{-1}(\mathfrak{g}_\mathbb{C})/\mathfrak{p}\right) .$$

3.5 Period maps

Let S be a complex manifold with universal cover $\tilde{S}$, and $\Gamma \subset G_\mathbb{Z} = \mathrm{Aut}(H_\mathbb{Z}, Q)$. We say that $\Phi : S \to \Gamma\backslash D$ is a *period map* if there is a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Phi} & \Gamma\backslash D \end{array} \tag{3.5.1}$$

with $\tilde{\Phi}$ holomorphic and horizontal.

Example 3.5.2. The identity map $D \to D$ is a period map if and only if the IPR is trivial.

Example 3.5.3. Fix a family $f : \mathcal{X} \to S$ as in §3.1. Recall that each $\gamma \in \pi_1(S; s, s')$ induces an isomorphism $\gamma : H^n(X_s, \mathbb{Z}) \simeq H^n(X_{s'}, \mathbb{Z})$. Assume that $\mathcal{X} \subset \mathbb{P}^m \times S$, and the map f is the restriction to $\mathcal{X}$ of the projection $\mathbb{P}^m \times S \to S$. Then an integral Kähler form $\omega \in H^2(\mathbb{P}^m, \mathbb{Z}) \cap H^{1,1}(\mathbb{P}^m)$ on $\mathbb{P}^m$ restricts to an integral Kähler form $\omega_s \in H^2(X_s, \mathbb{Z}) \cap H^{1,1}(X_s)$ on the fibres. The ω_s are invariant under monodromy, by construction. And $\gamma \in \pi_1(S; s, s')$ maps $H^n_{\mathrm{prim}}(X_s, \mathbb{Z})$ onto $H^n_{\mathrm{prim}}(X_{s'}, \mathbb{Z})$. It can be shown that the Hodge numbers $h^{p,q}_s = \dim_\mathbb{C} H^{p,q}_{\mathrm{prim}}(X_s)$ are locally constant [Gri68].[2]

[2] The key point is that the dimension of the kernel ($= \mathcal{H}^{p,q}$) of an ellipic opertor (the Laplacian on $\mathcal{E}^{p,q}$) depending smoothly on a parameter s is upper-semicontinuous. So, for s' in a small disc about s, we have $\dim H^n(X_{s'}, \mathbb{C}) = \sum_{p+q=n} \dim H^{p,q}(X_{s'}) \le \sum_{p+q=n} \dim H^{p,q}(X_s) = \dim H^n(X_s, \mathbb{C})$. Since $H^n(X_{s'}, \mathbb{C}) \simeq H^n(X_s, \mathbb{C})$, we necessarily have $\dim H^{p,q}(X_{s'}) = \dim H^{p,q}(X_s)$. Cf. [CMSP17, p. 138–139].

Fix $s_o \in S$. Set $X = X_{s_o}$, and $H_{\mathbb{Z}} = H^n_{\mathrm{prim}}(X, \mathbb{Z})/\{\mathrm{torsion}\}$. Set $Q(\alpha, \beta) = (-1)^{n(n-1)/2} \int_X \alpha \wedge \beta \wedge \omega_{s_o}^{d-n}$, where $d = \dim X$. Let D be the period domain parameterizing Q–polarized Hodge structures on $H_{\mathbb{Z}}$ with Hodge numbers $\mathbf{h} = (h^{p,q})$. Each $\gamma \in \pi_1(S; s_o, s)$ defines a Q–polarized Hodge structure $\tilde{\varphi}(s, \gamma)$ on $H_{\mathbb{Z}}$. In this way f induces commutative diagram (3.5.1), with Γ the image of the monodromy representation $\pi_1(S, s_o) \to \mathrm{Aut}(H_{\mathbb{Z}}, Q)$. The lift $\tilde{\Phi}$ is holomorphic and horizontal [Gri68]. In this way f induces a period map $\Phi : S \to \Gamma \backslash D$.

3.6 Derivative of the period map

Fix a family

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathbb{P}^m \times S \\ {\scriptstyle f}\downarrow & \swarrow & \\ S & & \end{array} \tag{3.6.1}$$

as in Example 3.5.3. Fix $s_o \in S$ and set $X = f^{-1}(s_o)$. Our goal here is to compute the derivative of the induced period map $\Phi : S \to \Gamma \backslash D$ at s_o. Since this is a local question we may assume that S is simply connected (for example, a polydisc), and $\Gamma = \{1\}$.

Note that $\mathrm{d}f$ induces an isomorphism $N_{\mathcal{X}/X,x} \to T_{s_o}S$ for all $x \in X$. So every $\xi \in T_{s_o}S$ defines $\tilde{\xi} \in H^0(X, N_{\mathcal{X}/X})$. The SES

$$0 \ \to \ T_X \ \hookrightarrow \ T_{\mathcal{X}}|_X \ \twoheadrightarrow \ N_{\mathcal{X}/X} \ \to \ 0$$

induces $\delta : H^0(X, N_{\mathcal{X}/X}) \to H^1(X, T_X)$. The *Kodaira–Spencer mapping*

$$\rho : T_{S,s_o} \ \to \ H^1(X, T_X)$$

is defined by $\rho(\xi) = \delta(\tilde{\xi})$.

Notice that the cup-product induces

$$H^1(X, T_X) \times H^{p,q}(X) \ \to \ H^{p-1,q+1}(X)\,.$$

So we have a map

$$\varepsilon : H^1(X, T_X) \ \to \ \bigoplus_p \mathrm{Hom}(H^{p,q}, H^{p-1,q+1}) \ \subset \ T_{D,\Phi(s_o)} \,. \tag{3.6.2}$$

The derivative of the period map is [Gri68]

$$\mathrm{d}\Phi_{s_o}(v) \ = \ \varepsilon \circ \rho(v) \,.$$

3.7 Deformations

A *deformation* of X is given by a family (3.6.1), with $X \simeq f^{-1}(s_o)$ for some $s_o \in S$. The following theorem suggests that we regard $H^1(X, T_X)$ as parameterizing the "infinitesimal deformations" of X.

Theorem 3.7.1 (Frölicher–Nijenhuis 1957)**.** *If $H^1(X, T_X) = 0$, then there exists a neighborhood $s_o \in U \subset S$ so that $f^{-1}(u)$ is biholomorphic to X for all $u \in U$.*

We say X is *rigid* when $H^1(X, T_X) = 0$.

Theorem 3.7.2 (Kodaira–Nirenberg–Spencer 1958)**.** *If $H^2(X, T_X) = 0$, then there exists a complete deformation $f : \mathcal{X} \to S$ of X over some polydisc S so that the Kodaira–Spencer map is an isomorphism.*

A few remarks on the theorem:

- *Complete* means that any other deformation $g : \mathcal{Y} \to T$ of $X \simeq g^{-1}(t_o)$ is obtained from $f : \mathcal{X} \to S$ by local base change: there exists a neighborhood $t_o \in U \subset T$ and a map $\phi : U \to S$ so that $\phi(t_o) = s_o$ and the family $g|_U$ is isomorphic to $\mathcal{X} \times_S U$.
- The fact that the Kodaira–Spencer map is an isomorphism implies the deformation f is *versal*: the differential $\mathrm{d}\phi_{t_o}$ is uniquely determined.
- As a complete, versal deformation of X, the family $f : \mathcal{X} \to S$ in Theorem 3.7.2 is the *Kuranishi family* of X.

- A complete family is *universal* if the germ of ϕ at t_o is uniquely determined. The family f in the theorem is universal if $H^0(X, T_X) = 0$. In this case we say that X *satisfies the infinitesimal Torelli theorem* if the differential $\mathrm{d}\Phi_{s_o} : T_{s_o}S \to T_{D,\Phi(s_o)}$ is injective. (It is sometimes easier to check the dual statement that $\oplus\,(H^{p,q} \otimes H^{d-p+1,d-q+1})$ surjects onto $H^{d-1}(X, \Omega^1_X \otimes K_X)$.)

Exercise 3.7.3. Suppose that K_X is trivial. Show that $H^k(X, T_X)^\vee \simeq H^{1,d-k}(X)$, where $d = \dim X$. [*Hint.* §A.3.12.]

Chapter 4

Torelli theorems

4.1 Moduli space of polarized algebraic varieties

To every polarized algebraic variety (X, ω) of dimension d we associate the following *Hodge data*: for each $0 \le n \le d$, the lattice $H_{\mathbb{Z}} = H^n_{\text{prim}}(X, \mathbb{Z})/\{\text{torsion}\}$; the Hodge numbers $\mathbf{h} = (h^{p,q})$, and the polarization $Q(\alpha, \beta) = (-1)^{n(n-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{d-n}$. Let D be the associated period domain.

Let X^{sm} be the underlying smooth manifold. Let $\mathfrak{M}$ be the moduli space of polarized algebraic varieties $(\tilde{X}, \tilde{\omega})$ with the same underlying smooth structure $\tilde{X}^{\text{sm}} = X^{\text{sm}}$, and the same Hodge data as (X, ω). We naturally have $\psi : \mathfrak{M} \to G_{\mathbb{Z}} \backslash D$.

Suppose that $\mathfrak{M}$ admits the structure of an algebraic variety, and that $f : \mathcal{X} \to S$ is an algebraic deformation of (X, ω). If the natural map $\pi : S \to \mathfrak{M}$ is a morphism, then we say $\mathfrak{M}$ is the *coarse moduli space* for (X, ω). By construction we have

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \Gamma \backslash D \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathfrak{M} & \xrightarrow{\Psi} & G_{\mathbb{Z}} \backslash D \, . \end{array}$$

We say the *global Torelli theorem* holds for $\mathfrak{M}$ if Ψ is an embedding of the closed points; we say the *local Torelli theorem* holds for (X, ω) if $\mathrm{d}\Psi$ is an inclusion of tangent spaces $T_{\mathfrak{M},[X]} \to T_{G_\mathbb{Z}\backslash D,\Psi([X])}$; we say *weak global Torelli holds* if there exists a Zariski-dense $\mathfrak{M}' \subset \mathfrak{M}$ so that $\Psi|_{\mathfrak{M}'}$ embeds the closed points.

Analogous terminology may be applied to the family f and the period map Φ.

4.2 Algebraic curves

If $\dim X = 1$, then $H^2(X, T_X) = 0$. The Kodaira–Nirenberg–Spencer Theorem 3.7.2 applies: there exists a complete deformation of X over a polydisc so that the Kodaira–Spencer map $\rho : T_{S,s_o} \to H^1(X, T_X)$ is an isomorphism.

Exercise 4.2.1. (a) Show that $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-4))^\vee = 0$.

(b) Let E be an elliptic curve (a curve of genus one). Show that $H^q(E, T_E) = \mathbb{C}$ for $a = 0, 1$. Note that the complete deformation is not universal.

(c) Let X be a curve of genus $g \geq 2$. Show that $H^1(X, T_X) = H^0(X, K_X^{\otimes 2})$ has dimension $3g - 3$. And $H^0(X, T_X) = 0$, so that the deformation is universal.

[*Hint.* §A.1.6.]

Let $\mathfrak{M}_g$ denote the moduli space of genus g curves X.

Exercise 4.2.2. Let X be a curve of genus 0.

(a) Fix $p \in X$. Use the Riemann–Roch theorem (§A.1.7) to show that there exists a meromorphic function $f : X \to \mathbb{P}^1$ with a single pole of order one at p.

(b) Use the Riemann–Hurwitz theorem (§A.1.4) to conclude that f is a biholomorphism.

We conclude that there is only one algebraic curve of genus $g = 0$: the moduli space $\mathcal{M}_0 = \{\mathbb{P}^1\}$ is a point. This is consistent with HW 4.2.1(a).

4.2.1 Elliptic curves

The moduli space for curves of genus $g = 1$ is $\mathfrak{M}_1 = \mathbb{A}^1 = \mathrm{SL}(2,\mathbb{C})\backslash\mathcal{H}$ the quotient of the upper half-plane by $\mathrm{SL}(2,\mathbb{Z})$. (Cf. HW 4.2.1(b).) The key observations are the following:

(a) A curve X of genus $g = 1$ can be embedded in $\mathbb{P}^1$ as $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$. Two curves are isomorphic if and only if their j *invariants* coincide $j(X) = 2^6 3^2 g_2^3/(g_2^3 - 27\, g_3^2)$. And the map $j : \mathfrak{M}_1 \to \mathbb{C}$ is surjective. Thus $\mathfrak{M}_1 = \mathbb{A}^1$.

(b) Every elliptic curve may be expressed as $E_\tau = \mathbb{C}^2/\Pi_\tau$, with $\Pi = \mathbb{Z} + \tau\,\mathbb{Z}$ and $\tau \in \mathcal{H}$. Two curves are isomorphic if and only if $\tau_2 = (a\tau_1 + b)/(c\tau_1 + d)$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Thus $\mathfrak{M}_1 = \mathrm{SL}(2,\mathbb{C})\backslash\mathcal{H}$.

(c) The two interpretations are related as follows. Define

$$\begin{aligned} g_2(\tau) &= 60 \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} (m\tau + n)^{-4} \\ g_3(\tau) &= 140 \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} (m\tau + n)^{-6} \\ j(\tau) &= \frac{2^6 3^2 g_2(\tau)^3}{g_2(\tau)^3 - 27\, g_3(\tau)^2}\,. \end{aligned}$$

4.2.2 Curves of general type

If $g \geq 2$, then Mumford showed that $\mathfrak{M}_g$ is quasi-projective of dimension $3g - 3$ (cf. HW 4.2.1(c)).

Recall that the Jacobian $\mathrm{Jac}(X)$ is principally polarized (HW 2.4.2), and so carries a theta divisor Θ (§2.2.4).

Theorem 4.2.3 (Global Torelli 1913 [GH94])**.** *Any nonsingular projective curve X may be reconstructed from its polarized Jacobian* $(\mathrm{Jac}(X), \Theta)$.

It is remarkable that the global Torelli theorem holds for curves, because the infinitesimal Torelli theorem fails for hyperelliptic curves of genus $g > 2$. This is related to the observation that, given a Kuranishi deformation $f : \mathcal{X} \to S$ of an hyperelliptic curve X, the neighborhood of $[X]$ in $\mathcal{M}_g$ is analytically isomorphic to the quotient of S by an involution.

The key players in the proof are the canonical map and Albanese map. After reviewing these, we outline the proof.

The canonical map

The complete linear system $|K_X| = \mathbb{P}\, H^0(X, K_X)$ defines the *canonical map*

$$\kappa : X \ \to\ \mathbb{P}\, H^0(X, K_X)^\vee \ =\ \mathbb{P}^{g-1}\,;$$

the point $x \in X$ is mapped to the hyperplane $\{s \in H^0(X, K_X) \text{ s.t. } s(x) = 0\}$. In the following exercises you will show that κ is either an embedding, or a double cover of $\mathbb{P}^1$ branched over $2g + 2$ points. In particular, either K_X is very ample, or X is hyperelliptic.

Exercise 4.2.4. Assume $g \geq 2$.

(a) Suppose that z is a local coordinate on a neighborhood $x_o \in U \subset X$. Fix a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, K_X)$. Show that $\kappa|_U$ can be identified with the map

$$z \ \mapsto\ \left[\frac{\omega_1(z)}{\mathrm{d}z} : \cdots : \frac{\omega_g(z)}{\mathrm{d}z}\right] .$$

(b) Show that $|K_X|$ is base point free. [*Hint.* Exercises A.1.7 and A.1.12.]

Exercise 4.2.5. Assume $g = 2$.

(a) Use $\deg K_X = 2$ (Example A.1.8) to conclude that κ is 2-to-1 onto its image $\kappa(X) = \mathbb{P}^1$. In particular, all genus two curves are hyperelliptic (§A.1.5).

(b) Use the Riemann–Hurwitz formula (§A.1.4) to show that the canonical map is branched at four points.

Exercise 4.2.6. Assume $g \geq 3$. Show that $\phi_{K_X^{\otimes 2}}$ is a closed embedding (cf. HW A.1.12).

Remark 4.2.7. If $\kappa = \phi_{K_X}$ is *not* a closed embedding (equivalently, K_X is ample, but not very ample) then κ is 2:1 onto its image $\kappa(C) \simeq \mathbb{P}^1$ (HW A.1.6).

Albanese map

Review the Albanese map $\alpha : X \to \mathrm{Jac}(X)$ of §§2.4.1-2.4.2. Let $X^{(k)}$ denote the k-th symmetric power of X, and define $\alpha_k : X^{(k)} \to \mathrm{Jac}(x)$ by $(x_1, \ldots, x_k) \mapsto \alpha(x_1) + \cdots + \alpha(x_k)$. The points of $A_k = \alpha_k(X^{(k)})$ parameterize equivalence classes of degree k divisors on X:

Theorem 4.2.8 (Abel [GH94])**.** *The divisor $\sum_{i=1}^k x_i - y_i$ is linearly equivalent to zero if and only if $\alpha_k(x_1, \ldots, x_k) = \alpha_k(y_1, \ldots, y_k)$.*

Exercise 4.2.9. Prove the following:

(a) The differential $\mathrm{d}\alpha_k$ is degenerate at $(x_1, \ldots, x_k)$ if and only if the points $\kappa(x_1), \ldots, \kappa(x_k)$ lie in a $\mathbb{P}^{k-2}$.

(b) The differential $\mathrm{d}\alpha_k$ is nondegenerate at a generic point.

Conclude that $Z = \alpha_{(g-1)}(X^{(g-1)})$ is a divisor in $\mathrm{Jac}(X)$.

Theorem 4.2.10 (Riemann)**.** *The divisors Θ and A_{g-1} coincide up to a translation.*

Proof of Global Torelli for curves of genus $g \geq 2$

Let $T_0 = T_{\mathrm{Jac}(X),0}$ be the tangent space at the identity, and let $\phi : \mathrm{Gr}(g-1, T_{\mathrm{Jac}(X)}) \to \mathrm{Gr}(g-1, T_0) = \mathbb{P}T_0^\vee$ be the *Gauss map.* Let $Y \subset X^{(g-1)}$ be the Zariski open subset where the differential $\mathrm{d}\alpha_{g-1}$ is nondegenerate (HW 4.2.9), and define $\psi : Y \to \mathbb{P}T_0^\vee$ by $y \mapsto \phi(\mathrm{d}\alpha_{g-1}(T_y Y))$.

Exercise 4.2.11. Show that the map $\psi : Y \to \mathbb{P}T_0^\vee$ is finite-to-one, and of degree $\binom{2g-2}{g-1}$. [*Hint.* Example A.1.8.]

Note that we may naturally view the canonical curve $\kappa(X)$ as sitting in $\mathbb{P}T_0 = \mathbb{P}^{g-1}$.

Lemma 4.2.12 ([GH94])**.** *Let $B \subset \mathbb{P}T_0^\vee$ be the closure of the branch locus of ψ. Then $B = C^*$.*

If X is not hyperelliptic (§A.1.5), then the Global Torelli Theorem 4.2.3 follows from Lemma 4.2.12. If X is hyperelliptic, then the proof requires the following modifications: The set B is the closure of the set of of hyperplanes $H \subset \mathbb{P}^{g-1}$ that are either tangent $\kappa(X)$ or pass through a branch point of κ. It follows that we can reconstruct both $\kappa(X)$ and the branch points. Since $\kappa(X) = \mathbb{P}^1$, this suffices to determine X. □

4.3 Infinitesimal Torelli for Calabi–Yau manifolds

Let X be a compact complex manifold of dimension n with trivial canonical bundle $K_X = \Omega_X^n = \det(\Omega_X^1)$.

Exercise 4.3.1. Show that the bundles Ω_X^{n-1} and T_X are isomorphic.

HW 4.3.1 implies that $H^k(X, T_X) \simeq H^k(X, \Omega_X^{n-1})$. In particular, $H^2(X, T_X) = H^2(X, \Omega_X^{n-1}) = H^{n-1,2}(X) = 0$. The Kodaira–Nirenberg–Spencer Theorem 3.7.2 implies that the deformation space of X is unobstructed, and X has a Kuranishi family $f : \mathcal{X} \to S$.

Definition 4.3.2. A *Calabi–Yau manifold* is a compact complex manifold of dimension n with trivial canonical bundle and $H^0(X, \Omega_X^k) = H^{k,0}(X) = 0$ for all $0 < k < n$. A *K3 surface* is a Calabi–Yau manifold of dimension $n = 2$.

Exercise 4.3.3. Show that a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $n+2$ is a Calabi–Yau manifold. [*Hint.* The adjunction formula (§A.3.6) and the Lefschetz hyperplane theorem (§A.3.7).]

Exercise 4.3.4. Let X be a K3 surface. Show that the Hodge numbers of $H^2(X)$ are $\mathbf{h} = (1, 20, 1)$. [*Hint.* Noether's formula (§A.2.1).]

Exercise 4.3.5. Let X be a K3 surface.

(a) Let D be an effective divisor on X. Show that $H^0(X,D) = 2 + \frac{1}{2}D^2$. [*Hint.* Kodaira vanishing (§A.3.10) and the Riemann–Roch formula (§A.2.4).]

(b) Let $C \subset X$ be a(n irreducible and reduced) curve. Show that the arithmetic genus satisfies $p_a(C) = 1 + \frac{1}{2}C^2$. [*Hint.* Genus formula (§A.2.5).]

Proof of infinitesimal Torelli for CYs. For the remainder of §4.3 we assume that X is a Calabi–Yau manifold. Then $H^0(X,T_X) = H^0(X,\Omega_X^{n-1}) = H^{n-1,0}(X) = 0$. So the Kuranishi family of X is a universal deformation (§3.7). Then X satisfies the infinitesimal Torelli theorem if the map (3.6.2) is injective. That is, if $H^1(X,T_X) = H^1(X,\Omega_X^{n-1}) = H^{n-1,1}(X)$ injects into $\oplus \operatorname{Hom}(H^{p,q}(X), H^{p-1,q+1}(X))$.

We claim that the map $H^1(X,T_X) \to \operatorname{Hom}(H^{n,0}(X), H^{n-1,1}(X))$ is an isomorphism. To see this, fix a generator $\eta \in H^{n,0}(X) \simeq \mathbb{C}$. Any $\xi \in H^1(X,T_X)$ may be represented, in Dolbeault cohomology, by a closed $(0,1)$-form taking values in T_X. And $\xi(\eta) \in H^{n-1,1}(X)$ is precisely the image of ξ under the isomorphism $H^1(X,T_X) \simeq H^{n-1,1}(X)$. □

4.4 Infinitesimal Torelli for hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and dimension n. The Lefschetz hyperplane theorem (§A.3.7) implies that

$$H^k(X,\mathbb{Z}) = \begin{cases} \mathbb{Z}\,, & k \equiv 0 \bmod 2 \\ 0\,, & k \equiv 1 \bmod 2\,. \end{cases}$$

for all $k < n$. What about $k = n$? What is the Hodge decomposition $H^n(X,\mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X)$?

4.4.1 Griffiths Jacobian ring

Let $S^d \subset \mathbb{C}[x_0,\ldots,x_{n+1}]$ be the homogeneous polynomials of degree d. Set $S = \mathbb{C}[x_0,\ldots,x_{n+1}]$. Then $S = \oplus_{d\geq 0} S^d$. We have $X = \{s = 0\}$ for some $f \in S^d$. The

Jacobian ideal $J_f = (\partial f/\partial x_j)_{j=0}^{n+1} \subset S$ is the ideal generated by the partial derivatives of f. The *Jacobian ring* is $R_f = S/J_f$.

Let $F^n \subset \cdots \subset F^0 = H^n_{\text{prim}}(X,\mathbb{C})$ be the Hodge filtration. Note that $F^p/F^{p+1} \simeq H^{p,n-p}_{\text{prim}}(X)$. Define

$$t(p) \ \stackrel{\text{dfn}}{=\!=} \ d(n+1-p) - (n+2)\,.$$

Then (§5.4.5)

$$F^p/F^{p+1} \ \simeq \ R_f^{t(p)}\,.$$

In particular, the Hodge numbers are

$$\dim H^{p,q}_{\text{prim}}(X) \ = \ \dim R_f^{t(p)}$$

for all $p+q=n$.

Example 4.4.1. Let $X = \{x_0^d + x_1^d + x_2^d = 0\} \subset \mathbb{P}^2$ be a planar curve of degree d.

- Then $R_s = S/(x_0^{d-1}, x_1^{d-1}, x_2^{d-1})$ implies $R_s^k = S^k$ if $k \le d-2$.
- We have $t(a) = d(2-a) - 3$, so that $t(1) = d-3$.
- It follows that $g = h^{1,0} = \frac{1}{2}(d-1)(d-2)$, and we recover the degree–genus formula (§A.1.2).

Exercise 4.4.2. Compute the Hodge numbers for $n=2$ and $d=3,4$. [*Hint.* Remark 5.4.48.]

Exercise 4.4.3. Compute the Hodge numbers for $n=3$ and $d=3,4,5$. [*Hint.* Remark 5.4.48.]

4.4.2 Moduli of hypersurfaces

The space of degree d hypersurfaces in $\mathbb{P}^{n+1}$ is parameterized by $\mathbb{P}\,S^d$. Let $\mathcal{U} \subset \mathbb{P}\,S^d$ be the locus of nonsingular hypersurfaces. Then $G = \text{PGL}(n+2)$ acts on $\mathcal{U}$. Mumford [Mum65] showed that the moduli space $\mathcal{M} = \mathcal{U}/G$ of nonsingular hypersurfaces

$X \subset \mathbb{P}^{n+1}$ of degree d is a quasi-projective variety. (The nonsingular hypersurfaces are GIT stable.) Suppose $X = \{s = 0\} \in U$. We have

$$T_{U,X} \;=\; T_{\mathbb{P}\,S^d,X} \;\simeq\; (\mathbb{C}\,s)^\vee \otimes (S^d/\mathbb{C}\,s) \quad \text{and} \quad T_{(GX),X} \;\simeq\; (\mathbb{C}\,s)^\vee \otimes (J^d_s/\mathbb{C}\,s)\,.$$

Kodaira–Serre duality implies $H^0(X, T_X)^\vee = H^n(X, \Omega^1_X \otimes K_X) = H^{1,n}(X, K_X)$. If $d \geq 3$ and $n \geq 2$, then $H^0(X, T_X) = 0$, implying the automorphism group $\mathrm{Aut}(X)$ is finite. For generic X the automorphism group is trivial. Then $[X] \in \mathcal{M}$ is a smooth point. It follows that

$$T_{\mathcal{M},[X]} \;\simeq\; R^d_s\,.$$

4.4.3 Infinitesimal Torelli for hypersurfaces

Fix a simply connected neighborhood $[X] \in S \subset \mathcal{M}$. Then we have a period map $\Phi : S \to D$. Griffiths' infinitesimal Torelli theorem asserts that Φ is a local embedding ($\mathrm{d}\Phi_{[X]}$ is injective) if either $d > 2$ and $n \neq 2$, or $d > 3$ and $n = 2$, [Gri69]. The idea of the proof is to:

1. Show that the differential

$$\mathrm{d}\Phi_{[X]} : T_{\mathcal{M},[X]} \;\to\; \bigoplus_{p=n}^{1} \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p)$$

   is induced by multiplication in the Jacobian ring:

$$R^d_s \,\times\, R^{t(p)}_s \;\to\; R^{t(p)+d}_s\,.$$

2. Apply Macaulay's theorem, which asserts that the pairing $R^a_s \times R^b_s \to R^{a+b}_s$ is nondegenerate for all $a + b \leq (n+2)(d-2)$.

4.5 Summary of some other Torelli results

1. The cubic threefold $X \subset \mathbb{P}^4$ has Hodge decomposition $H^3(X) = H^{2,1}(X) \oplus H^{1,2}(X)$. This allows one to use ideas very similar to those in the proof of Theorem 4.2.3 to establish a global Torelli theorem [CG72, Tju71].

2. Global Torelli holds for K3 surfaces [Pvv71]. The period mapping is also surjective in this case [Kul77b].

3. The ideas of [Pvv71] led to a proof of the global Torelli theorem for elliptic pencils [Cha84].

4. For smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d and dimension n: generic Torelli holds, with the following possible exceptions: if $n = 2$ and $d = 3$; d divides $n+2$; $d = 4$ and $4|m$; or $d = 6$ and $n \equiv 1 \bmod 6$ [Don83, DG84]. The proof builds on the approach to the infinitesimal Torelli theorem developed by Griffiths (§4.4). A key ingredient here is Donagi's "symmetrizer lemma", which is equivalent to the vanishing of a Koszul cohomology group.

5. The global Torelli theorem holds for smooth cubic fourfolds $X \subset \mathbb{P}^5$ [Voi86]. (Note this case is *not* covered by Donagi's result.)

6. The infinitesimal Torelli theorem *fails* for certain surfaces of general type [Kyn77, Cat79, Tod80]; the global Torelli theorem also fails for these surfaces [Cat80, Cha80].

7. For more results and counter-examples, see the collection [Gri84].

Chapter 5

Mixed Hodge structures

Recommended references: [Dur83, EZT14, PS08].

5.1 Introduction

Mixed Hodge structures (Definition 5.1.2) are a generalization of (pure) Hodge structures (Definition 2.3.3). Recall that the cohomology of a compact Kähler manifold admits a Hodge decomposition (§2.2). Analogously, we have

Theorem 5.1.1 (Deligne)**.** *Let X be an algebraic variety defined over $\mathbb{C}$. Then the cohomology groups $H^n(X, \mathbb{Q})$ can be equipped with a natural mixed Hodge structure, with the following properties:*

(i) *If $f : X \to Y$ is a morphism of algebraic varieties, then $f^* : H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q})$ is a weight zero morphism of mixed Hodge structure.*

(ii) *The weight filtration is $0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2n-1} \subset W_{2n} = H^n(X, \mathbb{Q})$. If X is complete, then $W_n = H^n(X, \mathbb{Q})$; and if X is smooth, then $W_{n-1} = 0$. If X if smooth and complete, Deligne's mixed Hodge structure coincides with the usual (pure) Hodge structure.*

(iii) *The mixed Hodge structure is compatible with algebraic constructions, including duality, Künneth formulas, et cetera.*

We will discuss Theorem 5.1.1 in two special, but important cases: X is complete, simple normal crossing (§5.3); and X is smooth, but not necessarily complete (§5.4).

Definition 5.1.2. A *mixed Hodge structure* on a finite-dimensional rational vector space $H_{\mathbb{Q}}$ consists of:

- an increasing, rational *weight filtration*

$$0 \subsetneq W_a \subset W_{a+1} \subset \cdots \subset W_{b-1} \subset W_b = H_{\mathbb{Q}}\,, \quad a \le b\,, \quad \text{and}$$

- a decreasing, complex *Hodge filtration*

$$0 \subsetneq F^m \subset F^{m-1} \subset \cdots \subset F^{\ell+1} \subsetneq F^\ell = H_{\mathbb{C}}\,, \quad \ell \le m\,,$$

such that F induces a pure Hodge structure of weight n on $\mathrm{Gr}^W_n = W_n/W_{n-1}$. Here the induced Hodge filtration is

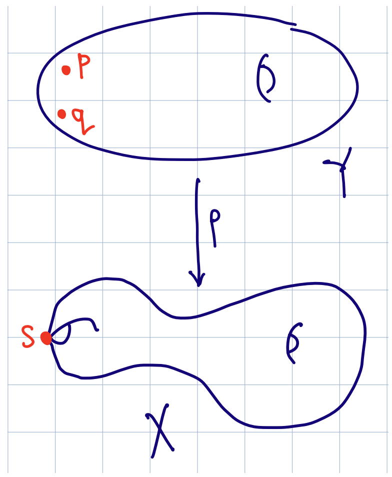
$$F^p(\mathrm{Gr}^W_n) = \frac{F^p \cap W_\ell}{F^p \cap W_{\ell-1}}\,.$$

Example 5.1.3. A pure Hodge structure of weight n is a mixed Hodge structure with trivial weight filtration $0 = W_{n-1} \subset W_n = H_{\mathbb{Q}}$.

5.1.1 Two toy examples

Example 5.1.4 (Complete, singular curve [Dur83])**.** Suppose that $X \subset \mathbb{P}$ is an irreducible curve with (at worst) ordinary double point singularities. Let $S = \{s_1, \ldots, s_k\}$ denote the singular points, and let $\rho : Y \to X$ denote the normalization of X (as in Figure 5.1). Then $\rho^{-1}(s_j) = \{p_j, q_j\}$ consists of two points. Let $T = \rho^{-1}(S)$. Then Y is smooth and the restriction $\rho : Y\backslash T \to X\backslash S$ is an isomorphism. Each of $H^n(Y)$, $H^n(S)$ and $H^n(T)$ admit pure Hodge structures of weight n; our goal is to use these

Figure 5.1: Normalization



to describe the mixed Hodge structure on X. Let $i : S \hookrightarrow X$ and $j : T \hookrightarrow Y$ be the inclusions. The maps

$$\begin{array}{ccccc} & & T & & \\ & {}^{\rho}\swarrow & & \searrow^{j} & \\ S & & & & Y \\ & {}_{i}\searrow & & \swarrow_{\rho} & \\ & & X & & \end{array}$$

define a SES of sheaves

$$0 \longrightarrow \mathbb{Q}_X \xrightarrow{\alpha = i^* \oplus \rho^*} \mathbb{Q}_S \oplus \mathbb{Q}_Y \xrightarrow{\beta = \rho^* - j^*} \mathbb{Q}_T \longrightarrow 0$$

which induces a LES in cohomology

$$\begin{array}{ccccccccccc} 0 \longrightarrow & H^0(X) & \xrightarrow{\alpha_0} & H^0(S) \oplus H^0(Y) & \xrightarrow{\beta_0} & H^0(T) & \xrightarrow{\gamma_0} & H^1(X) & \xrightarrow{\alpha_1} & H^1(Y) & \longrightarrow 0\,. \\ & \parallel & & \parallel & & \parallel & & & & & \\ & \mathbb{Q} & & \mathbb{Q}^{k+1} & & \mathbb{Q}^{2k} & & & & & \end{array}$$

This suggests the weight filtration $0 \subset W_0 \subset W_1 = H^1(X, \mathbb{Q})$ defined by

$$W_0 H^1(X) \stackrel{\text{dfn}}{=\!=} \operatorname{im} \gamma_0 \simeq \operatorname{coker} \beta_0 = \mathbb{Q}^k \quad \text{with} \quad \operatorname{Gr}_1^W H^1(X) = H^1(Y)\,.$$

Example 5.1.5 (Smooth, incomplete curve [Dur83]). Let X be a Riemann surface, and $D = \{p_1, \ldots, p_k\} \subset X$ a finite set of points. Both $H^n(X, \mathbb{Q})$ and $H^n(D, \mathbb{Q})$ have pure polarized Hodge structures of weight n. Our goal is to use these to describe the mixed Hodge structure on $U = X \backslash D$. What follows is a sketch of the approach, and mixed Hodge structures of this type will be discussed in greater generality in §5.4.

Let $j : U \hookrightarrow X$ and $i : D \to X$ be the inclusion maps. The Gysin map $i_! : H^0(D) \to H^2(X)$ and residue map $\mathrm{Res} : H^1(U) \to H^0(D)$ complete to a LES

$$0 \longrightarrow H^1(X) \xrightarrow{j^*} H^1(U) \xrightarrow{\mathrm{Res}} \underset{\mathbb{C}^k}{\underset{\parallel}{H^0(D)}} \xrightarrow{i_!} \underset{\mathbb{C}}{\underset{\parallel}{H^2(X)}} \xrightarrow{j^*} \underset{\mathbb{C}}{\underset{\parallel}{H^2(U)}} \longrightarrow 0 .$$

Passing to the reduced cohomology, we have

$$0 \longrightarrow H^1(X) \xrightarrow{j^*} H^1(U) \xrightarrow{\mathrm{Res}} \widetilde{H}^0(D) = \mathbb{C}^{k-1} \longrightarrow 0 .$$

This suggests weight filtration $0 \subset W_1 \subset W_2 = H^1(U)$ with $W_1 = H^1(X)$ and $\mathrm{Gr}_2^W = \mathrm{coker}\, i^* \simeq \mathrm{im}\,\mathrm{Res} = \widetilde{H}^0(D)$.

5.1.2 Induced mixed Hodge structures

Example 5.1.6. Given a mixed Hodge structure on H, the induced mixed Hodge structure on $H^\vee$ is

$$W_p(H^\vee) \ = \ \mathrm{Ann}(W_{-p-1}(H)) \quad \text{and} \quad F^p(H^\vee) \ = \ \mathrm{Ann}(F^{1-p}(H)) .$$

Example 5.1.7. Given mixed Hodge structures on H_1 and H_2,

$$W_n(H) \ = \ \sum_{p+q \le n} W_a(H_1) \otimes W_b(H_2) \quad \text{and} \quad F^k(H) \ = \ \sum_{p+q \ge k} F^p(H_1) \otimes F^q(H_2)$$

defines a mixed Hodge structure on $H = H_1 \otimes H_2$.

Exercise 5.1.8. Together Examples 5.1.6 and 5.1.7 induce a mixed Hodge structure on $H = \mathrm{Hom}(H_1, H_2) \simeq H_2 \otimes H_1^\vee$. Show that this induced mixed Hodge structure is

$$\begin{aligned} F^p(H) &= \{\phi \in H \text{ s.t. } \phi(F^k(H_1)) \subset F^{k+p}(H_2) \,,\ \forall\, k\} \\ W_p(W) &= \{\phi \in H \text{ s.t. } \phi(W_k(H_1)) \subset W_{k+2p}(H_2) \,,\ \forall\, k\} . \end{aligned}$$

5.2 Morphisms

Definition 5.2.1 (Morphism of pure Hodge structure)**.** A *weight* 2ℓ *morphism of (pure) Hodge structures* is a $\mathbb{Q}$–linear map $\phi : H_1 \to H_2$ such that $\phi(H_1^{p,q}) \subset H_2^{p+\ell,q+\ell}$.

Example 5.2.2 (Pullback)**.** Given map $f : X \to Y$ of compact Kähler manifolds, the pull-back $f^* : H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q})$ is a weight zero morphism of pure Hodge structures.

Definition 5.2.3 (Gysin map)**.** Let $f : M \to N$ be a smooth map of compact oriented manifolds of (real) dimensions m, n. The *Gysin map* $f_! : H^k_{\mathrm{d}}(M) \to H^{k+n-m}_{\mathrm{d}}(N)$ is characterized by

$$\int_N f_!(\alpha) \wedge \beta \; = \; \int_M \alpha \wedge f^*(\beta)\,,$$

and admits the following description. Use Poincaré duality to identify the pullback $f^* : H^\ell_{\mathrm{d}}(N) \to H^\ell_{\mathrm{d}}(M)$ with a map $H^{n-\ell}_{\mathrm{d}}(N)^* \to H^{m-\ell}_{\mathrm{d}}(M)^*$. Now take the dual map and set $k = m - \ell$.

Exercise 5.2.4 (Gysin map)**.** Let $f : X \to Y$ be a morphism of compact Kähler manifolds of (complex) dimension m, n. Show that the Gysin map $f_! : H^k_{\mathrm{d}}(X) \to H^{k+2(n-m)}_{\mathrm{d}}(Y)$ (Definition 5.2.3) is weight $2(n - m)$ morphism of Hodge structures.

Exercise 5.2.5. Let $\phi : H_1 \to H_2$ be a weight 2ℓ morphism of Hodge structures (as in Definition 5.2.1).

(a) Prove that $\phi(F_1^p) \subset F_2^{p+\ell}$.

(b) Show that the morphism is *strict*: $\phi(F_1^p) = \phi(H_1) \cap F_2^{p+\ell}$.

(c) Show that the kernel, cokernel and the image of a morphism of pure Hodge structures are pure Hodge structures.

Definition 5.2.6 (Morphism of mixed Hodge structure)**.** A *weight* 2ℓ *morphism of mixed Hodge structures* is a $\mathbb{Q}$–linear map $\phi : H_1 \to H_2$ that is compatible with the weight and Hodge filtrations:

$$\phi(W_p(H_1)) \;\subset\; W_{p+2\ell}(H_2) \quad \text{and} \quad \phi(F^p(H_1)) \;\subset\; F^{p+\ell}(H_2)\,.$$

Exercise 5.2.7. Show that a weight 2ℓ morphism $\phi : H_1 \to H_2$ of mixed Hodge structures naturally induces a weight 2ℓ morphism $\phi_n : \mathrm{Gr}^W_n(H_1) \to \mathrm{Gr}^W_{n+2\ell}(H_2)$ of pure Hodge structures.

Exercise 5.2.8. Let (H, W, F) be a mixed Hodge structure.

(a) Show that there are naturally induced mixed Hodge structures on $W_n(H)$ and $H/W_n(H)$.

(b) Show that the maps $i : W_n(H) \hookrightarrow H$ and $j : H \twoheadrightarrow W/W_n(H)$ are weight zero morphisms of mixed Hodge structures.

In general, mixed Hodge structures do not have direct sum decompositions like the Hodge decomposition. The best we can say is the following

Lemma 5.2.9 (Deligne splitting [GS75])**.** *Let (W, F) be a mixed Hodge structure on H. Define*

$$I^{p,q}_{W,F} \ \overset{\mathrm{dfn}}{=\!=} \ (F^p \cap W_{p+q}) \cap \left(\overline{F^q} \cap W_{p+q} + \sum_{i \geq 1} \overline{F^{q-i}} \cap W_{p+q-i-1} \right) . \qquad (5.2.10)$$

Then (5.2.10) *is the unique splitting with the properties*

$$H_{\mathbb{C}} \ = \ \bigoplus I^{p,q}_{W,F} \,, \qquad \overline{I^{p,q}_{W,F}} \ \equiv \ I^{q,p}_{W,F} \quad mod \ \bigoplus_{\substack{r<p \\ s<q}} I^{r,s}_{W,F} \ \subset \ W_{p+q-2} \,,$$

$$W_\ell \ = \ \bigoplus_{p+q \leq \ell} I^{p,q}_{W,F} \,, \qquad F^p \ = \ \bigoplus_{r \geq p} I^{r,q}_{W,F} \,.$$

Moreover, the projection $I^{p,q} \hookrightarrow W_m \to \mathrm{Gr}^W_m$ maps $I^{p,q}$ isomorphically onto the Hodge decomposition summand $(\mathrm{Gr}^W_m)^{p,q}$, where $m = p+q$. And if $\phi : H_1 \to H_2$ is any weight 2ℓ morphism of mixed Hodge structure, then $\phi(I^{p,q}_1) \subset I^{p+\ell,q+\ell}_2$.

This lemma has a number of powerful consequences

Exercise 5.2.11. Let $\phi : H_1 \to H_2$ be a weight 2ℓ morphism of mixed Hodge structure. Show that ϕ is *strict*:

$$\phi(W_m H_1) \ = \ (W_{m+2\ell} H_2) \cap \phi(H_1) \quad \text{and} \quad \phi(F^p H_1) \ = \ (F^{p+\ell} H_2) \cap \phi(H_1) \,.$$

Exercise 5.2.12. Let $\phi : H_1 \to H_2$ be a weight 2ℓ morphism of mixed Hodge structure. Then there are induced mixed Hodge structures on $\ker \phi$ and $\operatorname{coker} \phi$.

Exercise 5.2.13. Let $0 \to H_1 \stackrel{\alpha}{\to} H \stackrel{\beta}{\to} H_2 \to 0$ be an exact sequence of morphisms of mixed Hodge structures, with α of weight 2ℓ and β of weight $2m$. Prove that the induced sequence $0 \to \mathrm{Gr}^W_{n-2\ell}(H_1) \xrightarrow{\alpha_n} \mathrm{Gr}^W_n(H) \xrightarrow{\beta_n} \mathrm{Gr}^W_{n+2m}(H_2) \to 0$ is exact.

Hodge diamonds are very useful "hieroglyphics" that encode much of the discrete data in a mixed Hodge structure (just as Dynkin diagrams and Young tableaux encode representation theoretic data).

Definition 5.2.14 (Hodge diamond)**.** Given a mixed Hodge structure (W, F), the *Hodge diamond* is a configuration of points $(p, q) \in \mathbb{Z}^2$, with each node labelled with $\dim_{\mathbb{C}} I^{p,q}_{W,F}$.

5.3 Complete, normal crossing varieties

The goal of this section is to describe Deligne's mixed Hodge structure on the cohomology of a complete, simple normal crossing variety.

Definition 5.3.1. A variety X is *normal crossing* if every point $x \in X$ admits an analytic neighborhood U centered at x such that $U \simeq \{(z_0, \ldots, z_d) \in \Delta^{d+1} \text{ s.t. } z_0\, z_1 \cdots z_k = 0\}$. We say X is *simple* (or *strict*) *normal crossing* if the irreducible components of X are smooth.

Let's begin by considering the simplest nontrivial case...

5.3.1 Example: two irreducible components $X = X_1 \cup X_2$

Let $X = X_1 \cup X_2$ be complete and simple normal crossing. The SES

$$0 \to \mathbb{Q}_X \stackrel{\alpha}{\to} \mathbb{Q}_{X_1} \oplus \mathbb{Q}_{X_2} \stackrel{\beta}{\to} \mathbb{Q}_Y \to 0$$

induces a LES (cohomology coefficients are suppressed to save space)

$$\begin{array}{cccccccc} 0 & \to & H^0(X) & \xrightarrow{\alpha_0} & H^0(X_1) \oplus H^0(X_2) & \xrightarrow{\beta_0} & H^0(Y) & \xrightarrow{\gamma_0} \cdots \\ & & \vdots & & \vdots & & \vdots & \\ \cdots & \xrightarrow{\gamma_{k-2}} & H^{k-1}(X) & \xrightarrow{\alpha_{k-1}} & H^{k-1}(X_1) \oplus H^{k-1}(X_2) & \xrightarrow{\beta_{k-1}} & H^{k-1}(Y) & \xrightarrow{\gamma_{k-1}} \\ & \xrightarrow{\gamma_{k-1}} & H^k(X) & \xrightarrow{\alpha_k} & H^k(X_1) \oplus H^k(X_2) & \xrightarrow{\beta_k} & H^k(Y) & \xrightarrow{\gamma_k} \cdots \end{array}$$

Since X_1, X_2 and $Y = X_1 \cap X_2$ are nonsingular, $H^k(X_1) \oplus H^k(X_2)$ has a HS of weight k, and $H^{k-1}(Y)$ has a HS of weight $k-1$. We will use these two HS to define the weight filtration

$$0 \ = \ W_{k-2} \ \subset \ W_{k-1} \ \subset \ W_k \ = \ H^k(X, \mathbb{Q}) \, .$$

To begin, note that β_k is a weight zero morphism of Hodge structures (Definition 5.2.1). It follows that $\ker \beta_k = \operatorname{im} \alpha_k$ and $\operatorname{im} \beta_k = \ker \gamma_k$ and $\operatorname{coker} \beta_k$ all have induced Hodge structures of weight k (HW 5.2.5). So there is an induced Hodge structure of weight $k-1$ on

$$W_{k-1} \ \stackrel{\text{dfn}}{=\!=} \ \operatorname{im} \gamma_{k-1} \ \simeq \ \operatorname{coker} \beta_{k-1} \, .$$

Likewise,

$$\mathrm{Gr}^W_k \ = \ \frac{W_k}{W_{k-1}} \ = \ \frac{H^k(X,\mathbb{Q})}{\operatorname{im} \gamma_{k-1}} \ = \ \frac{H^k(X,\mathbb{Q})}{\ker \alpha_k} \ \simeq \ \operatorname{im} \alpha_k \ = \ \ker \beta_k$$

has an induced Hodge structure of weight k.

5.3.2 Example: curve $X = \cup X_i$

Suppose that each $X = \cup X_i$ is complete, simple normal crossing, with each X_i a nonsingular curve of genus g_i. Let $X_{ij} = X_i \cap X_j$ The SES

$$0 \ \to \mathbb{Q}_X \ \xrightarrow{\alpha} \ \oplus_i \, \mathbb{Q}_{X_i} \ \xrightarrow{\beta} \ \oplus_{i<j} \, \mathbb{Q}_{X_{ij}} \ \to \ 0$$

induces a LES (cohomology coefficients are suppressed to save space)

$$\begin{array}{cccccccc} 0 & \to & H^0(X) & \xrightarrow{\alpha_0} & \oplus_i \, H^0(X_i) & \xrightarrow{\beta_0} & \oplus_{i<j} \, H^0(X_{ij}) & \xrightarrow{\gamma_0} \\ & \xrightarrow{\gamma_0} & H^1(X) & \xrightarrow{\alpha_1} & \oplus_i \, H^1(X_i) & \xrightarrow{\beta_1} & 0 & \\ 0 & \xrightarrow{\gamma_1} & H^2(X) & \xrightarrow{\alpha_k} & \oplus_i \, H^2(X_i) & \xrightarrow{\beta_k} & 0 \, . & \end{array}$$

As in §5.3.1, the weight filtration on $H^1(X,\mathbb{Q})$ is

$$W_0 \;=\; \mathrm{im}\,\gamma_0 \;\simeq\; \mathrm{coker}\,\beta_0\,, \quad \text{and} \quad W_1 \;=\; H^1(X,\mathbb{Q})\,,$$

with

$$\mathrm{Gr}^W_1 \;=\; \frac{W_1}{W_0} \;=\; \frac{H^1(X,\mathbb{Q})}{\ker\alpha_1} \;\simeq\; \mathrm{im}\,\alpha_1 \;=\; \oplus_i\, H^1(X_i,\mathbb{Q})\,.$$

5.3.3 Deligne's MHS

Assume X is a complete, simple normal crossing variety. The mixed Hodge structure (W,F) on $H^n(X,\mathbb{Q})$ is obtained as follows. Write $X = \cup X_i$, with X_i the irreducible components of X. Set

$$X_I \;=\; \bigcap_{i\in I} X_i\,.$$

Then $\mathrm{codim}\, X_I \;=\; |I|-1$. The X_I are nonsingular, and $H^n(X_I,\mathbb{Q})$ admits a pure Hodge structure of weight n. Set

$$X^{(k)} \;=\; \bigsqcup_{|I|=k} X_I\,. \tag{5.3.2}$$

Let $i_k : X^{(k)} \to X$ be the map defined by the inclusions $X_I \hookrightarrow X$. Given $1 \le a \le k$, let $j_a : X^{(k)} \to X^{(k-1)}$ be the map defined by the inclusions $X_I \hookrightarrow X_{I-\{i_a\}}$. Define a double complex $C^{p,q} = (i_{q+1})_*\mathcal{E}^p_{X^{(q+1)}}$ with differentials $D_1 = \mathrm{d}$ the exterior derivative, and D_2 the signed restriction $\sum_{a=1}^{q+1}(-1)^{p+a} j_a^*$. Deligne's mixed Hodge structure on $H^n(X,\mathbb{Q})$ (Theorem 5.1.1) is exhibited by constructing a spectral sequence from the associated simple/total complex ($\mathcal{C}^m = \oplus_{p+q=m}, D = D_1 + D_2$) that collapses at the second page with

$$\begin{aligned}
\mathrm{Gr}^W_q\, H^n(X,\mathbb{Q}) \;&=\; E^{p,q}_\infty \;=\; E^{p,q}_2 \;=\; H^p(E^{\bullet,q}_1, d_1)\\
&=\; H^p(H^q(X^{(\bullet+1)},\mathbb{Q})\,,\, d_1) \qquad\qquad (5.3.3)\\
&=\; \frac{\ker\{d_1 : H^q(X^{(p+1)}) \to H^q(X^{(p+2)})\}}{\mathrm{im}\,\{d_1 : H^q(X^{(p)}) \to H^q(X^{(p+1)})\}}\,, \qquad p+q=n\,,
\end{aligned}$$

where $E_1^{p,q} = H^q(X^{(p+1)}, \mathbb{Q})$ is equipped with the natural pure Hodge structure of weight q, and the differential $d_1 : H^q(X^{(p+1)}, \mathbb{Q}) \to H^q(X^{(p+2)}, \mathbb{Q})$, which is induced by a signed restriction map, is a morphism of Hodge structures. A number of corollaries follow.

Corollary 5.3.4. (i) *The weight filtration satisfies* $W_n = H^n(X, \mathbb{Q})$.

(ii) *If* $X^{(k+1)} = \emptyset$, *then* $W_{n-k} = 0$.

Corollary 5.3.5. *If* X *and* Y *are normal crossing varieties and* $f : X \to Y$ *is a morphism, then* $f^* : H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q})$ *is a weight zero morphism of mixed Hodge structures.*

It follows from this corollary and strictness of morphisms (HW 5.2.11) that we have...

Corollary 5.3.6. *If* X *is a normal crossing variety,* Y *is smooth and complete, and* $f : X \to Y$ *is a morphism, then* $W_{n-1}H^n(X, \mathbb{Q}) \cap \mathrm{Im}\, f = 0$ *is a weight zero morphism of mixed Hodge structures.*

The Lefschetz hyperplane theorem and Corollary 5.3.6 yield

Corollary 5.3.7. *Let* Z *be a nonsingular projective variety of dimension* $d+1$, *and let* $X \subset Z$ *be an ample divisor (a hyperplane section of* Z*) with normal crossings. Then* $W_{n-1}H^n(X, \mathbb{Q}) = 0$ *when* $n < d$.

Definition 5.3.8. The *dual complex* $\Gamma(X)$ of $X = \cup X_i$ is the polyhedron whose vertices correspond to the irreducible components X_i of X. The vertices $X_{i_0}, \ldots, X_{i_k}$ form a k-simplex if $X_I \neq \emptyset$.

Corollary 5.3.9. *If* Γ *is the dual complex of a complete, normal crossing* X, *then* $W_0(H^k(X)) = H^k(|\Gamma|)$.

Proof. The key observation is that it follows from (5.3.3) and Corollary 5.3.4 that

$$W_0 H^k(X) \;=\; H^k(H^0(X^{(*+1)})\,,\, d_1)\,,$$

keeping in mind that d_1 is induced by a signed restriction map, and the definition of the dual complex. □

5.3.4 Example: surface $X = \cup X_i$

Let $C_{ij} = X_i \cap X_j$, $i < j$, denote the double curves; and P_{ijk}, $i < j < k$, the triple points. Then Corollary 5.3.9 implies $W_0 H^n(X) = H^n(\Gamma)$. The observations that follow are all consequences of (5.3.3) and the E_1 page of the spectral sequence:

$$\bigoplus_i H^4(X_i) \longrightarrow 0$$

$$\bigoplus_i H^3(X_i) \longrightarrow 0$$

$$\bigoplus_i H^2(X_i) \xrightarrow{d_1} \bigoplus_{i<j} H^2(C_{ij}) \xrightarrow{d_1} 0$$

$$\bigoplus_i H^1(X_i) \xrightarrow{d_1} \bigoplus_{i<j} H^1(C_{ij}) \xrightarrow{d_1} 0$$

$$\bigoplus_i H^0(X_i) \xrightarrow{d_1} \bigoplus_{i<j} H^0(C_{ij}) \xrightarrow{d_1} \bigoplus_{i<j<k} H^0(C_{ijk}) \xrightarrow{d_1} 0\,.$$

In degree one Corollary 5.3.4 yields $H^1(X) = W_1 \supset W_0 \supset 0$, and (5.3.3) yields

$$\mathrm{Gr}^W_1 H^1(X) \;=\; \ker\left\{ d_1 : \bigoplus_i H^1(X_i) \;\to\; \bigoplus_{i<j} H^1(C_{ij}) \right\}\,.$$

In degree two we have $H^2(X) = W_2 \supset W_1 \supset W_0 \supset 0$. The weight-graded quotient in degree two is

$$\mathrm{Gr}^W_2 H^2(X) \;=\; \ker\left\{ d_1 : \bigoplus_i H^2(X_i) \;\to\; \bigoplus_{i<j} H^2(C_{ij}) \right\}\,.$$

Moreover, if $\mathrm{Gr}_2^W H^2(X,\mathbb{C}) = H_2^{2,0} \oplus H_2^{1,1} \oplus H_2^{0,2}$ is the Hodge decomposition, then $H_2^{2,0} = \oplus_i H^{2,0}(X_i)$. The weight-graded quotient in degree one is

$$\mathrm{Gr}_1^W H^2(X) \ = \ \mathrm{coker}\left\{ d_1 : \bigoplus_i H^1(X_i) \ \to \ \bigoplus_{i<j} H^1(C_{ij}) \right\} .$$

5.4 Smooth quasi-projective varieties

Recommended reference: [PS08, §4].

Let U be a nonsingular algebraic variety. It follows from [Hir64, Nag62] that U may be realized as $U = X \backslash D$, for some nonsingular, complete algebraic variety X and a simple normal crossing divisor $D \subset X$. Let $j : U \hookrightarrow X$ denote the inclusion.

Definition 5.4.1. Let $\Omega_X^p(kD) \supset \Omega_X^p$ be the sheaf of meromorphic p-forms on X that are holomorphic on U, and have a pole of order $\le k$ on D. The sheaf of *log p-forms* is

$$\Omega_X^p(\log D) \ = \ \{\omega \in \Omega_X^p(D) \text{ s.t. } \mathrm{d}\omega \in \Omega_X^{p+1}(D)\} \ \subset \ \Omega_X^p(D) .$$

The *logarithmic de Rham complex* is $(\Omega_X^\bullet(\log D), \mathrm{d}) \subset (j_*\Omega_U^\bullet, \mathrm{d})$.

As a complex manifold, every point $x \in X$ admits a local coordinate chart $z : V \xrightarrow{\simeq} \Delta^m$ centered at x so that $D \cap V = \{z_1 \cdots z_k = 0\}$.

Exercise 5.4.2. (a) Show that $\Omega_X^1(\log D)(V)$ is the $\mathcal{O}_X(V)$–module generated by $\mathrm{d}\log z_1, \ldots, \mathrm{d}\log z_k, \mathrm{d}z_{k+1}, \ldots, \mathrm{d}z_m$.

(b) Show that $\Omega_X^p(\log D)(V) = \bigwedge^p \Omega_X^1(\log D)(V)$.

Define a filtration

$$W_\ell\, \Omega_X^p(\log D) \ \stackrel{\mathrm{dfn}}{=\!=} \ \Omega_X^{p-\ell} \wedge \Omega_X^\ell(\log D) .$$

Let

$$F^k\, \Omega_X^p(\log D) \ \stackrel{\mathrm{dfn}}{=\!=} \ \begin{cases} 0 , & p < k , \\ \Omega_X^p(\log D) , & p \ge k \end{cases}$$

be the trivial filtration on the complex.

Theorem 5.4.3. *Deligne's mixed Hodge structure on $H^k(U,\mathbb{C})$ is given as follows.*

(i) *We have $H^k(U,\mathbb{C}) = \mathbb{H}^k(X,\Omega^\bullet_X(\log D))$.*

(ii) *The weight filtration is*

$$W_\ell\, H^k(U,\mathbb{C}) \;=\; \mathrm{im}\{\mathbb{H}^k(X, W_{\ell-k}\,\Omega^\bullet_X(\log D)) \;\to\; H^k(U,\mathbb{C})\}\,.$$

We have $W_\ell\, H^k(U) = 0$ for all $\ell < k$, and $W_k\, H^k(U) = \mathrm{im}\{H^k(X) \to H^k(U)\}$.

(iii) *The Hodge filtration is*

$$F^p\, H^k(U,\mathbb{C}) \;=\; \mathrm{im}\{\mathbb{H}^k(X, F^p\,\Omega^\bullet_X(\log D)) \;\to\; H^k(U,\mathbb{C})\}\,.$$

And $\mathrm{Gr}^p_F H^{p+q}(U,\mathbb{C}) = H^q(X,\Omega^p_X(\log D))$. (This is a consequence of the fact that a spectral sequence associated with the filtration $F^k\Omega^p_X(\log D)$ collapses at the E_1 term, and $E_1^{p,q} = H^q(X,\Omega^p_X(\log D))$.)

(iv) *The Hodge numbers $h^{p,q}$ of $H^k(U)$ are nonzero only when $p,q \le k \le p+q$. This is a consequence of the fact that $\mathrm{Gr}^W_{k+\ell}H^k(U,\mathbb{Q})$ is a subquotient of $H^{k-\ell}(D^{(\ell)},\mathbb{Q})(-\ell)$.* (The latter cohomology coincides with the E_1 term of the spectral sequence associated with the filtration $W^{-\ell} = W_\ell$ that collapses at the E_2 term.)

The proof of Theorem 5.4.3 is much more complicated than the construction of the mixed Hodge structure on $H^k(Y,\mathbb{Q})$ when Y is a simple normal crossing variety (summarized in §5.3.3). This is perhaps not surprising since the choice of completion $X \supset U$ is neither unique nor canonical. We will discuss (in §§5.4.1–5.4.3) the proof in the very special case that D is smooth hypersurface.

5.4.1 Residue map

Let $D \subset X$ be a smooth hypersurface. Define a map

$$\mathrm{Res}_D : \Omega^p_X(\log D) \;\to\; \Omega^{p-1}_D \tag{5.4.4}$$

as follows. Given $x \in D$, fix a local coordinate chart $z : V \xrightarrow{\simeq} \Delta^m$ centered at x so that $D \cap V = \{z_1 = 0\}$. Any $\omega \in \Omega^p_X(\log D)(V)$ may be written as $\omega = \eta \wedge (\mathrm{d}\log z_1) + \eta'$, with $\eta \in \Omega^{p-1}_X(V)$ and $\eta' \in \Omega^p_X(V)$ not involving $\mathrm{d}z_1$.

Exercise 5.4.5. Show that $\eta|_D \in \Omega^{p-1}_D(D \cap V)$ is independent of our choice of local coordinates z. Conclude that (5.4.4) is well-defined by $\mathrm{Res}_D(\omega) = \eta|_D$.

Exercise 5.4.6. Show that the residue map commutes with d.

Exercise 5.4.7. Show that

$$0 \;\to\; \Omega^p_X \;\to\; \Omega^p_X(\log D) \;\xrightarrow{\mathrm{Res}}\; \Omega^{p-1}_D \;\to\; 0$$

is an SES of sheaves.

Remark 5.4.8. The residue map admits the following coordinate-free interpretation: Fix a Riemannian metric on X. For sufficiently small $\varepsilon > 0$, the tubular neighborhood $T_{2\varepsilon}(D) = \{x \in X \text{ s.t. } \mathrm{dist}(x, D) < 2\varepsilon\}$ is diffeomorphic to a neighborhood of the zero section in the real normal bundle $ND = TD^\perp \subset TX|_D$. Then $S_\varepsilon(D) = \partial T_\varepsilon(D) = \{x \in X \text{ s.t. } \mathrm{dist}(x, D) = \varepsilon\}$ maybe viewed as an S^1–bundle over D. Let $i : S_\varepsilon(D) \to T_{2\varepsilon}(D)$ be the inclusion. Let $\rho_D : H^k(S_\varepsilon(D)) \to H^{k-1}(D)$ be the map given by fibre-wise integration. Then

$$\mathrm{Res}_D(\eta) \;=\; \rho_D\left(\tfrac{1}{2\pi\mathbf{i}}\, i^*\eta\right) .$$

This interpretation is dual to the following. Let $j : D \to T_{2\varepsilon}(D)$ denote the inclusion, and $\pi : S_\varepsilon(D) \to D$ the projection. The *Gysin LES* is

$$\cdots \to\; H_k(S_\varepsilon(D)) \;\xrightarrow{i_*}\; H_k(T_{2\varepsilon}(D)) \;\xrightarrow{j^!}\; H_{k-2}(D) \;\xrightarrow{\mathrm{Res}^*_D}\; H_{k-1}(S_\varepsilon(D)) \;\to \cdots$$

Informally the map $j^!$ is given by intersecting the cycle $c \in H_k(T_{2\varepsilon}(D))$ with D. Formally, the map is defined by

$$\langle \tau \wedge \pi^*\varphi \,,\, c\rangle \;=\; \langle \varphi \,,\, j^!(c)\rangle \,, \quad \forall\, \varphi \in H^{k-2}(D)$$

where τ_D is the Thom class of the normal bundle ND.

Exercise 5.4.9. The SES of HW 5.4.7 induces a LES in cohomology

$$\begin{array}{ccccccccc} H^0(X,\Omega^p_X) & \hookrightarrow & H^0(X,\Omega^p_X(\log D)) & \xrightarrow{\text{Res}} & H^0(D,\Omega^{p-1}_D) & \xrightarrow{\delta} & H^1(X,\Omega^p_X) & \longrightarrow & \cdots \\ & & & & \shortparallel & & \shortparallel & & \\ & & & & H^{p-1,0}(D) & & H^{p,1}(X) & & \end{array}$$

Let $i : D \hookrightarrow X$ denote the inclusion. Show that the connecting map $\delta : H^{p-1,0}(D) \to H^{p,1}(X)$ is the (restriction of the) Gysin map $i_! : H^{p-1}(D,\mathbb{C}) \to H^{p+1}(X,\mathbb{C})$. [*Hint.* Remark 5.4.8 and Stokes' Theorem.]

Exercise 5.4.10. Show that

$$0 \ \to \ H^0(V,\Omega^p_V) \ \hookrightarrow \ H^0(V,\Omega^p_V(\log D\cap V)) \ \xrightarrow{\text{Res}} \ H^0(D\cap V,\Omega^{p-1}_{D\cap V}) \ \to \ 0$$

is exact. [*Hint.* HW 5.4.7 and Cartan's Theorem B (§A.4.4).]

5.4.2 Hypercohomology

Hypercohomology is a generalization of sheaf cohomology that takes as its input not a single sheaf, but a complex of sheaves. For this discussion, we take X to be an arbitrary complex manifold. (While some of the discussion holds in greater generality, this suffices for our purposes.) Suppose that $(\mathcal{K}^\bullet, d)$ is a complex of sheaves (bounded below).

Definition 5.4.11. The *cohomology sheaf* (not to be confused with sheaf cohomology) is the sheafification of the presheaf

$$U \ \mapsto \ \frac{\ker\{d : \mathcal{K}^q(U) \to \mathcal{K}^{q+1}(U)\}}{\mathrm{im}\,\{d : \mathcal{K}^{q-1}(U) \to \mathcal{K}^q(U)\}}\,;$$

that is,

$$\mathcal{H}^q(\mathcal{K}^\bullet,d)(U) \ = \ \left\{ \sigma_\alpha \in \mathcal{K}^q(U_\alpha) \ \middle| \ \begin{array}{l} \{U_\alpha\} \text{ is an open cover of } U\,,\ d\sigma_\alpha = 0 \\ \text{and } \sigma_\alpha|_{U_{\alpha\beta}} - \sigma_\beta|_{U_{\alpha\beta}} \in d(\mathcal{K}^{q-1}(U_{\alpha\beta})) \end{array} \right\} \Big/ \sim\,,$$

where $\{\sigma_\alpha \in \mathcal{K}^q(U_\alpha)\} \sim \{\sigma'_\mu \in \mathcal{C}^q(U'_\mu)\}$ if for all $x \in U_\alpha \cap U'_\mu$ there exists an open $x \in V \subset U_\alpha \cap U'_\mu$ so that $\sigma_\alpha|_V - \sigma'_\mu|_V \in d(\mathcal{K}^{q-1}(V))\}$.

Definition 5.4.12. A morphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ of complexes is a *quasi-isomorphism* if the induced maps $\mathcal{H}^k(\mathcal{K}_1^\bullet) \to \mathcal{H}^k(\mathcal{K}_2^\bullet)$ are isomorphisms. This is a local property that can be checked at the level of stalks (HW A.4.4).

Remark 5.4.13. If the complex $(\mathcal{K}^\bullet, d)$ satisfies a Poincaré Lemma, then $\mathcal{H}^q(\mathcal{K}^\bullet, d) = 0$ for all $q > 0$, essentially by definition.

Example 5.4.14. Any resolution $\mathcal{F} \hookrightarrow \mathcal{S}^\bullet$ is a quasi-isomorphism of $\mathcal{S}^\bullet$ with the trivial complex $\mathcal{F} \to 0 \to 0 \to \cdots$.

Example 5.4.15 ([Voi07])**.** Suppose that $(\mathcal{K}^\bullet, d)$ is a complex, $(\mathcal{C}^{\bullet,\bullet}, \delta_1, \delta_2)$ is a double complex ($\delta_i^2 = 0$, $\delta_1\delta_2 + \delta_2\delta_1 = 0$), and there exists a morphism of complexes $i : (\mathcal{K}^\bullet, d) \hookrightarrow (\mathcal{C}^{\bullet,0}, \delta_1)$ so that $i_p : \mathcal{K}^p \hookrightarrow \mathcal{C}^{p,0}$ is an injection, and $\mathcal{K}^p \overset{i_p}{\hookrightarrow} (\mathcal{C}^{p,\bullet}, \delta_2)$ is a resolution. Let $(\mathcal{C}^\bullet, \delta)$ be the associated simple complex $\mathcal{C}^k = \oplus_{p+q=k}\mathcal{C}^{p,q}$ and $\delta = \delta_1 + \delta_2$. Then the induced map $i : (\mathcal{K}^\bullet, d) \to (\mathcal{C}^\bullet, \delta)$ is a quasi-isomorphism.

One such example is given by $(\mathcal{K}^p, d) = (\Omega_X^p, \partial)$ and $(\mathcal{C}^{p,q}, \delta_1, \delta_2) = (\mathcal{E}_X^{p,q}, \partial, \overline{\partial})$, for which we have $(\mathcal{C}^k, \delta) = (\mathcal{E}_X^k, \mathrm{d})$. Another example is given by the Čech resolution ...

Definition 5.4.16. Let $(\mathcal{C}^p(\{U_i\}, \mathcal{K}^q), \delta)$ denote the Čech resolution of $\mathcal{K}^q$ (Example A.4.14). Let $d : \mathcal{C}^p(\{U_i\}, \mathcal{K}^q) \to \mathcal{C}^p(\{U_i\}, \mathcal{K}^{q+1})$ be the induced map. Then $(\mathcal{C}^{p,q} = \mathcal{C}^p(\{U_i\}, \mathcal{K}^q); \delta, d)$ is a double complex (Example 5.4.15). Let $(\mathcal{C}^\bullet, D)$ be the associated simple complex. The *hypercohomology* of the complex is

$$\mathbb{H}^k(X, \mathcal{K}^\bullet) \overset{\text{dfn}}{=\!=} \varinjlim_{\{U_i\}} H^k(\Gamma(X, \mathcal{C}^\bullet), D)\,. \tag{5.4.17}$$

(See §A.4.6 for a more general discussion.)

Remark 5.4.18. There are two spectral sequences abutting to the hypercohomology ($\mathbb{H}^k \simeq \oplus_{p+q=k} E_\infty^{p,q}$) [GH94, p. 442]. The second pages are

$${}'E_2^{p,q} \;=\; \check{H}^p(X, \mathcal{H}^q) \quad \text{and} \quad {}''E_2^{p,q} \;=\; H_d^p(\check{H}^q(X, \mathcal{K}^\bullet)) \tag{5.4.19}$$

where $\mathcal{H}^q = \mathcal{H}^q(\mathcal{K}^\bullet, d)$ are the cohomology sheaves (§A.4.6). The first spectral sequence may be used to show: if $\phi : (\mathcal{K}_1^\bullet, d_1) \to (\mathcal{K}_2^\bullet, d_2)$ is a quasi-isomorphism (induces an isomorphism of cohomology sheaves), then $\mathbb{H}^k(X, \mathcal{K}_1^\bullet) = \mathbb{H}^k(X, \mathcal{K}_2^\bullet)$. The second spectral sequence implies that in the computation (5.4.17) it suffices to work with a cover $\{U_i\}$ that is acyclic with respect to all the $\mathcal{K}^q$.

Example 5.4.20 (Hypercohomology of a trivial complex)**.** Fix a sheaf $\mathcal{S}$, and let $(\mathcal{K}^\bullet, d)$ be the trivial complex $\mathcal{S} \to 0 \to 0 \to \cdots$. Then $\mathcal{H}^0 = \mathcal{S}$ and $\mathcal{H}^q = 0$ for all $q > 0$. Both $'E_2^{p,q}$ and $''E_2^{p,q}$ are zero for $q > 0$. Since these spectral sequences are supported in the positive quadrant, the differentials d_2' and d_2'' vanish. So $\mathbb{H}^k(X, \mathcal{K}^\bullet) = H^k(X, \mathcal{S})$ is the usual sheaf cohomology.

Example 5.4.21 (Dolbeault isomorphism)**.** The $\overline{\partial}$–Poincaré Lemma implies that the trivial complex $\Omega_X^k \to 0 \to 0 \to \cdots$ is quasi-isomorphic to the Dolbeault complex $(\mathcal{E}_X^{k,\bullet}, \overline{\partial})$. So $H^q(X, \Omega^k) = \mathbb{H}^q(X, \mathcal{E}_X^{k,\bullet})$ (Example 5.4.20).

The $\overline{\partial}$–Poincaré Lemma also implies that the cohomology sheaves of $(\mathcal{E}_X^{k,\bullet}, \overline{\partial})$ are $\mathcal{H}^0 = \Omega_X^k$ and $\mathcal{H}^q = 0$ for all $q > 0$. So $'E_2^{p,0} = H^p(X, \Omega_X^k)$, and $'E_2^{p,q} = 0$ for all $q > 0$. Likewise, the sheaves $\mathcal{E}_X^{k,\bullet}$ are fine, so that $\check{H}^q(X, \mathcal{E}^{k,\bullet}) = 0$ for all $q > 0$ (Example A.4.13), so that $''E_2^{p,0} = H^p_{\overline{\partial}}(H^0(X, \mathcal{E}_X^{k,\bullet})) = H^{k,p}_{\overline{\partial}}(X)$ and $''E_2^{p,q} = 0$ for all $q > 0$. Since these spectral sequences are supported in the positive quadrant, the differentials d_2' and d_2'' vanish. Whence we obtain the Dolbeault isomorphism $H^p(X, \Omega^k) = H^{k,p}_{\overline{\partial}}(X)$.

Example 5.4.22. The holomorphic de Rham complex $(\Omega_X^\bullet, \mathrm{d})$ is a resolution of the constant sheaf $\mathbb{C}_X$, but *not* an acyclic resolution. The complex satisfies a Poincaré Lemma, so that the inclusion of the trivial complex $\mathbb{C}_X \to 0 \to 0 \to 0 \cdots$ into $(\Omega_X^\bullet, \mathrm{d})$ is a quasi-isomorphism. It follows (Example 5.4.20) that

$$H^k(X, \mathbb{C}) \;=\; \mathbb{H}^k(X, \Omega_X^\bullet)\,.$$

(Keeping Example 5.4.15 in mind, we have $\mathbb{H}^k(X, \Omega_X^\bullet) = H^k_{\mathrm{d}}(H^0(X, \mathcal{E}_X^\bullet))$, which is consistent.) We have $''E_2^{p,q} = H^p_{\mathrm{d}}(\check{H}^q(X, \Omega_X^\bullet))$.

(a) If X is compact Kähler, then the differential d is trivial on $\check{H}^q(X, \Omega^p) = H^{p,q}_{\overline{\partial}}(X)$. So $''E_2^{p,q} = \check{H}^q(X, \Omega^p) = {}''E_1^{p,q}$, and $\mathbb{H}^k(X, \Omega_X^\bullet) \simeq \oplus_{k=p+q} \check{H}^q(X, \Omega^p)$.

(b) If X is Stein, then Cartan's Theorem B (§A.4.4) implies $\check{H}^q(X, \Omega_X^p) = 0$ for all $q > 0$. It follows that $H^k(X, \mathbb{C}) = H_{\mathrm{d}}^k(H^0(X, \Omega_X^\bullet))$.

Remark. This argument also implies $\mathbb{H}^p(X, \Omega_X^{\bullet-1}) = H_{\mathrm{d}}^{p-1}(H^0(X, \Omega_X^\bullet)) = H^{p-1}(X, \mathbb{C})$.

Exercise 5.4.23. Prove Theorem 5.4.24. [*Hint.* Remark 5.4.18.]

Theorem 5.4.24 (de Rham)**.** *Let X be a topological space and $(\mathcal{K}^\bullet, d)$ and exact complex of sheaves on X.*

(i) *Define $\mathcal{S} = \ker\{d : \mathcal{K}^0 \to \mathcal{K}^1\}$. We have a canonical identification $H^q(X, \mathcal{S}) = \mathbb{H}^q(X, \mathcal{K}^\bullet)$.*

(ii) *If $H^q(X, \mathcal{K}^p) = 0$ for all p and all $q > 0$, then we have $\mathbb{H}^q(X, \mathcal{K}^\bullet) \simeq H_{\mathrm{dR}}^p(X, \mathcal{K}^\bullet) \overset{\mathrm{dfn}}{=\!=} H^p(\Gamma(X, \mathcal{K}^\bullet), d)$.*

5.4.3 The logarithmic de Rham complex

Assume that X is a compact complex manifold and $D \subset X$ is a simple normal crossing divisor.

The local picture

The following lemma will play the role of a Poincaré lemma

Lemma 5.4.25. *The inclusion $\Omega^\bullet(\log D) \hookrightarrow j_*\mathcal{E}_U^\bullet$ is a quasi-isomorphism.*

Corollary 5.4.26. *We have $\mathbb{H}^k(X, \Omega_X^\bullet(\log D)) = \mathbb{H}^k(X, j_*\mathcal{E}_U^\bullet) = H^k(U, \mathbb{C})$.*

Proof. Recall that the property of being quasi-isomorphic can be checked at the level of stalks. If $x \in U$, then the stalks are $\Omega_x^\bullet = \Omega(\log D)_x$ and $\mathcal{E}_x^\bullet = (j_*\mathcal{E}_U^\bullet)_x$. The smooth and holomorphic Poincaré lemmas imply that these stalks are zero.

It remains to consider the case that $x \in D$. For simplicity we *assume that D is smooth.* (For the general argument, see [GH94, p. 451].) Then the local coordinate chart $z : V \xrightarrow{\simeq} \Delta^m$ satisfies $U \cap V \simeq \Delta^* \times \Delta^{m-1}$. The latter deformation retracts onto a circle S^1, so that

$$H^q_{\mathrm{dR}}(U \cap V, \mathbb{C}) \ = \ H^q_{\mathrm{dR}}(S^1, \mathbb{C}) \ = \ \textstyle\bigwedge^q H^1_{\mathrm{dR}}(S^1, \mathbb{C})$$

with $H^1_{\mathrm{dR}}(S^1, \mathbb{C}) = \mathbb{C}\, \mathrm{d}\log z_1$. This yields $\mathcal{H}^q(j_*\mathcal{E}^\bullet_U)_x = \bigwedge^q H^1_{\mathrm{dR}}(S^1, \mathbb{C})$.

It is clear that the stalk $\mathcal{H}^q(\Omega^\bullet_X(\log D))_x$ maps *onto* $\mathcal{H}^q(j_*\mathcal{E}^\bullet_U)_x$. We need to show that these maps are injective. Exercises 5.4.6 and 5.4.7 imply that we have a SES

$$0 \ \to \ \Omega^\bullet_V \ \to \ \Omega^\bullet_V(\log D \cap V) \ \xrightarrow{\mathrm{Res}} \ \Omega^{\bullet-1}_{D\cap V} \ \to \ 0$$

of complexes. This induces a LES in hypercohomology, that Example 5.4.22 and HW 5.4.27 allow us to write as

$$\begin{array}{l} 0 \longrightarrow H^0(V, \mathbb{C}) \longrightarrow \mathcal{H}^0(\Omega^\bullet_X(\log D))_x \longrightarrow 0 \\ 0 \to \mathcal{H}^1(\Omega^\bullet_X(\log D))_x \longrightarrow H^0(D \cap V, \mathbb{C}) \longrightarrow 0 \\ 0 \to \mathcal{H}^p(\Omega^\bullet_X(\log D))_x \longrightarrow 0\,, \qquad\qquad \forall\, p \geq 2\,. \end{array}$$

□

Exercise 5.4.27. Show that

$$\mathbb{H}^p(V, \Omega^\bullet_V(\log D \cap V)) \ = \ H^p_{\mathrm{d}}(H^0(V, \Omega^\bullet_V(\log D \cap V))) \ = \ \mathcal{H}^p(\Omega^\bullet_X(\log D))_x\,.$$

[*Hint.* The second equality is essentially by definition of the cohomology sheaves. For the first equality, consider the second page $''E_2^{p,q}$ in the second spectral sequence computing the hypercohomology (5.4.19), keeping Example 5.4.22(b) in mind.]

The global picture

Assume that $D \subset X$ is a smooth hypersurface. As noted above HW 5.4.6 and 5.4.7 imply that we have a SES

$$0 \ \to \ \Omega^\bullet_X \ \to \ \Omega^\bullet_X(\log D) \ \xrightarrow{\mathrm{Res}} \ \Omega^{\bullet-1}_{D\cap V} \ \to \ 0$$

of complexes. This induces a LES in hypercohomology, that Example 5.4.22 and Corollary 5.4.26 allow us to write as

$$0 \longrightarrow H^0(X,\mathbb{C}) \longrightarrow H^0(U,\mathbb{C}) \longrightarrow 0$$

$$\begin{array}{l} 0 \longrightarrow H^1(X,\mathbb{C}) \xrightarrow{j^*} H^1(U,\mathbb{C}) \xrightarrow{\text{Res}} H^0(D,\mathbb{C}) \xrightarrow{\text{Gys}} H^2(X,\mathbb{C}) \\ \xrightarrow{j^*} H^2(U,\mathbb{C}) \xrightarrow{\text{Res}} H^1(D,\mathbb{C}) \xrightarrow{\text{Gys}} H^3(X,\mathbb{C}) \xrightarrow{j^*} H^3(U,\mathbb{C}) \\ \xrightarrow{\text{Res}} H^2(D,\mathbb{C}) \xrightarrow{\text{Gys}} H^4(X,\mathbb{C}) \xrightarrow{j^*} H^2(U,\mathbb{C}) \xrightarrow{\quad\text{Res}\quad} \cdots \end{array} \tag{5.4.28}$$

Example 2.3.6 and HW 5.2.4 and 5.4.9 imply that the maps $H^{p-1}(D)(-1) \to H^{p+1}(X)$ are weight zero morphisms of Hodge structure. HW 5.2.5(c) implies

$$W_k\, H^k(U) \ \stackrel{\text{dfn}}{=\!=} \ \text{im}\,\{H^k(X) \to H^k(U)\} \ = \ \text{coker}\,\{H^{k-2}(D)(-1) \to H^k(X)\}$$

carries a pure Hodge structure of weight k. Likewise, setting $W_{k+1}H^k(U) = H^k(U)$, the weight graded quotient

$$\text{Gr}^W_{k+1} H^k(U) \ = \ \text{coker}\{H^k(X) \to H^k(U)\} \ = \ \ker\,\{H^{k-1}(D)(-1) \to H^{k+1}(X)\}$$

carries a pure Hodge structure of weight $k+1$.

This completes our discussion of Theorem 5.4.3. We now turn to some important corollaries.

5.4.4 Global Invariant Cycle Theorem

Let $f : \mathcal{X} \subset \mathbb{P}^m \times S \to S$ be as in Example 3.5.3. The stalks of the higher direct image are $R^n f_*(\mathbb{Q}_{\mathcal{X}})_s = H^n(X_s,\mathbb{Q})$, cf. Example A.4.10 and HW A.4.11. Fix $s_o \in S$ and let $\rho : \pi_1(S,s_o) \to \text{Aut}(H^n(X_{s_o},\mathbb{Q}),Q)$ be the monodromy representation (3.1.3). As a local system, we have

$$\begin{array}{c} R^n f_*(\mathbb{Q}_{\mathcal{X}}) \ = \ \tilde{S} \times_\rho \ H^n(X_{s_o},\mathbb{Q}) \\ \downarrow \\ S \end{array}$$

Let

$$H^n(X_{s_o},\mathbb{Q})^{\pi_1} \;=\; \{\eta\in H^n(X_{s_o},\mathbb{Q}) \text{ s.t. } \rho(\gamma)\cdot\eta=\eta \;\forall\,\gamma\in\pi_1(S,s_o)\}$$

denote the subspace upon which the monodromy acts trivially. We have a natural identification

$$H^n(X_{s_o},\mathbb{Q})^{\pi_1} \;\simeq\; H^0(S,R^nf_*(\mathbb{Q}_{\mathcal{X}})) \tag{5.4.29}$$

of the invariant subspace with the global sections. Let $i_s : X_s \hookrightarrow \mathcal{X}$ be the inclusion map, and note that restriction to fibres defines a map

$$H^n(\mathcal{X},\mathbb{Q}) \;\to\; H^0(S,R^nf_*(\mathbb{Q}_{\mathcal{X}}))\,, \qquad \xi\mapsto(s\mapsto i_s^*\xi)\,. \tag{5.4.30}$$

Remark 5.4.31. Corollary A.4.16 implies that (5.4.30) is surjective. Then the identification (5.4.29) yields the following generalization of HW 3.1.13(b)

$$H_n(X_{s_o},\mathbb{Q}) \;=\; \ker\{H_n(X_{s_o},\mathbb{Q})\to H_n(\mathcal{X},\mathbb{Q})\} \;\oplus\; H_n(X_{s_o},\mathbb{Q})^{\pi_1}\,.$$

Theorem 5.4.32 (Global Invariant Cycle Thm, aka Thm of the Fixed Part [Del71])**.**

(i) *The invariant subspace $H^n(X_{s_o},\mathbb{Q})^{\pi_1}$ inherits a weight n Hodge structure from $H^n(X_{s_o},\mathbb{Q})$.*

(ii) *The induced Hodge structure on $\Gamma(S,R^nf_*(\mathbb{Q}_{\mathcal{X}}))$ does not depend on the choice of $s_o\in S$.*

(iii) *If $j:\mathcal{X}\hookrightarrow\overline{\mathcal{X}}$ is a smooth compactification (with $\overline{\mathcal{X}}\backslash\mathcal{X}$ simple normal crossing), then the composition*

$$H^n(\overline{\mathcal{X}},\mathbb{Q}) \;\stackrel{j^*}{\to}\; H^n(\mathcal{X},\mathbb{Q}) \;\to\; H^0(S,R^nf_*(\mathbb{Q}_{\mathcal{X}})) \;=\; H^n(X_{s_o},\mathbb{Q})^{\pi_1}$$

is surjective.

Proof. The pullback $i_{s_o}^*$ decomposes as

$$i_{s_o}^* : H^n(\mathcal{X},\mathbb{Q}) \;\twoheadrightarrow\; H^0(S,R^nf_*(\mathbb{Q}_{\mathcal{X}})) \;\simeq\; H^n(X_{s_o},\mathbb{Q})^{\pi_1} \;\hookrightarrow\; H^n(X_{s_o},\mathbb{Q})\,.$$

Since i_s^* is a morphism of Hodge structures (Theorem 5.1.1), the first claim of Theorem 5.4.32 follows from Remark 5.4.31. As a quotient of $H^n(\mathcal{X}, \mathbb{Q})$, the Hodge structure on $H^0(S, R^n f_*(\mathbb{Q}_\mathcal{X}))$ is clearly independent of s_o. It remains to establish the third claim. This follows from HW 5.4.33 below. $\square$

Exercise 5.4.33. Set $j_s = j \circ i_s : X_s \to \overline{\mathcal{X}}$. Both $H^n(\overline{\mathcal{X}}, \mathbb{Q})$ and $H^n(X_s, \mathbb{Q})$ carry weight n Hodge structures. By Theorem 5.4.3(ii) we have a mixed Hodge structure on $H^n(\mathcal{X}, \mathbb{Q})$ with

$$W_n H^n(\mathcal{X}, \mathbb{Q}) \ = \ \mathrm{im}\{j^* : H^n(\overline{\mathcal{X}}, \mathbb{Q}) \to H^n(\mathcal{X}, \mathbb{Q})\} \, .$$

Use Theorem 5.1.1(i) and strictness of morphisms of MHS (HW 5.2.11) to show that

$$\mathrm{im}\{i_s^* : H^n(\mathcal{X}, \mathbb{Q}) \to H^n(X_s, \mathbb{Q})\} \ = \ \mathrm{im}\,\{j_s^* : H^n(\overline{\mathcal{X}}, \mathbb{Q}) \to H^n(X_s, \mathbb{Q})\} \, .$$

Theorem 5.4.34 (Complete reducibility [Del71])**.** *The monodromy representation on $H^n(X_{s_o})$ is completely reducible.*

Remark 5.4.35. The elements of $\ker\{H_n(X_{s_o}, \mathbb{Z}) \to H_n(\mathcal{X}, \mathbb{Z})\}$ are the *vanishing cycles.*

Remark 5.4.36. Let

$$\begin{array}{l} \mathcal{I}^n \ = \ \tilde{S} \times_\rho \ H^n(X_{s_o}, \mathbb{Q})^{\pi_1} \ \subset \ R^n f_*(\mathbb{Q}_\mathcal{X}) \\ \downarrow \\ S \end{array}$$

be the local subsystem defined by $H^n(X_{s_o}, \mathbb{Q})^{\pi_1} \subset H^n(X_{s_o}, \mathbb{Q})$. This local subsystem is a constant sheaf. If η is a section of $\mathcal{I}^n$, and is of Hodge type (p, q) at some point $s \in S$, then Theorem 5.4.32 implies η is of Hodge type (p, q) at every point of S. With more work, one may use the Global Invariant Cycle Theorem obtain results Hodge classes [CS14].

5.4.5 Hodge structure on a smooth projective hypersurface

Let $X = \{f = 0\} \subset \mathbb{P}^{n+1} = \mathbb{P}$ be a smooth projective hypersurface of degree d. Our goal is to compute the Hodge decomposition of $H^k_{\mathrm{prim}}(X, \mathbb{Q})$.

The Lefschetz hyperplane theorem (§A.3.7) implies $H^k(X, \mathbb{Z}) = H^k(\mathbb{P}, \mathbb{Z})$ for all $k < n$. We have seen (Example 2.2.3) that $H^\bullet(\mathbb{P}^{n+1}, \mathbb{Z}) = \oplus_{k=0}^{n+1} \mathbb{Z}\, \omega^k$, where $\omega \in H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \cap H^{1,1}(\mathbb{P}^{n+1})$ is (a multiple of) the Fubini-Study Kähler form. Then Poincaré duality implies $H^k(X, \mathbb{Z}) \simeq H^{2n-k}(X, \mathbb{Z})$. So we need only compute $H^n_{\mathrm{prim}}(X, \mathbb{Z})$.

Set $U = \mathbb{P} \backslash X$. In this setting the LES (5.4.28) reads

$$
\begin{aligned}
0 \longrightarrow H^{2k-1}(U, \mathbb{C}) &\xrightarrow{\mathrm{Res}} H^{2k-2}(X, \mathbb{C}) \xrightarrow{\mathrm{Gys}} H^{2k}(\mathbb{P}, \mathbb{C}) \\
&\xrightarrow{j^*} H^{2k}(U, \mathbb{C}) \xrightarrow{\mathrm{Res}} H^{2k-1}(X, \mathbb{C}) \longrightarrow 0\,.
\end{aligned}
$$

Lemma 5.4.37. *The pullback $j^* : H^{2k}(\mathbb{P}, \mathbb{Z}) \to H^{2k}(U, \mathbb{Z})$ is the zero map if $k > 0$.*

Proof. By duality it suffices to show that $j_* : H_{2k}(U, \mathbb{Z}) \to H_{2k}(\mathbb{P}, \mathbb{Z})$ is the zero map. Given $\alpha \in H_{2k}(U, \mathbb{Z})$, define $m \in \mathbb{Z}$ by $j_*(\alpha) = m[\mathbb{P}^k] \in H_{2k}(\mathbb{P}, \mathbb{Z})$. We have a disjoint union $\mathbb{P} = U \sqcup X$, with $[X] = d[\mathbb{P}^n] \subset H_{2n}(\mathbb{P}, \mathbb{Z})$. So $0 = j_*(\alpha) \cdot [X] = dm[\mathbb{P}^{n+k-(n+1)}]$ forces $m = 0$ so long as $k \geq 1$. □

The lemma updates the LES to

$$
\begin{aligned}
&0 \longrightarrow H^{2k-1}(U, \mathbb{C}) \xrightarrow{\mathrm{Res}} H^{2k-2}(X, \mathbb{C}) \xrightarrow{\mathrm{Gys}} H^{2k}(\mathbb{P}, \mathbb{C}) \longrightarrow 0\,, \\
&0 \longrightarrow H^{2k}(U, \mathbb{C}) \xrightarrow{\mathrm{Res}} H^{2k-1}(X, \mathbb{C}) \longrightarrow 0\,.
\end{aligned}
$$

Exercise 5.4.38. Use the Gysin map (Definition 5.2.3) to show that the ker $\{H^n(X, \mathbb{Q}) \to H^{n+2}(\mathbb{P})\} = H^n_{\mathrm{prim}}(X, \mathbb{Q})$.

Exercises 5.2.4 and 5.4.38 imply

$$
H^n_{\mathrm{prim}}(X, \mathbb{Q}) \;\simeq\; H^{n+1}(U, \mathbb{Q})(1)\,. \tag{5.4.39}
$$

So to describe the Hodge decomposition of $H^n_{\mathrm{prim}}(X,\mathbb{Q})$, it suffices to describe the mixed Hodge structure on $H^{n+1}(U,\mathbb{Q})$. Theorem 5.4.3(ii) and Lemma 5.4.37 imply that $W_{n+1}H^{n+1}(U,\mathbb{Q}) = 0$. Then Theorem 5.4.3(iv) implies $W_{n+2}H^{n+1}(U,\mathbb{Q}) = H^{n+1}(U,\mathbb{Q})$; this is, $H^{n+1}(U,\mathbb{Q})$ is pure Hodge structure of weight $n+2$. Moreover, Theorem 5.4.3(iii) asserts

$$\mathrm{Gr}^p_F H^{p+q}(U) \;=\; H^q(\mathbb{P},\Omega^p_{\mathbb{P}}(\log X))\,. \tag{5.4.40}$$

Exercise 5.4.41. Show that the sequence

$$0 \;\to\; \Omega^p_{\mathbb{P}}(\log X) \;\hookrightarrow\; \Omega^p_{\mathbb{P}}(X) \;\stackrel{\mathrm{d}}{\to}\; \frac{\Omega^{p+1}_{\mathbb{P}}(2X)}{\Omega^{p+1}_{\mathbb{P}}(X)} \;\stackrel{\mathrm{d}}{\to}\; \frac{\Omega^{p+2}_{\mathbb{P}}(3X)}{\Omega^{p+2}_{\mathbb{P}}(2X)} \;\stackrel{\mathrm{d}}{\to}\; \cdots$$

is exact.

Set

$$\mathcal{K}^k \;=\; \frac{\Omega^{p+k}_{\mathbb{P}}((k+1)X)}{\Omega^{p+k}_{\mathbb{P}}(kX)}\,.$$

Exercise 5.4.41 implies that $(\mathcal{K}^\bullet,\mathrm{d})$ is a resolution of $\Omega^p_{\mathbb{P}}(\log X)$. Keeping §5.4.2 in mind, this implies

$$H^q(\mathbb{P},\Omega^p_{\mathbb{P}}(\log X)) \;=\; \mathbb{H}^q(\mathbb{P},\mathcal{K}^\bullet)\,.$$

Lemma 5.4.42. *We have* $\mathbb{H}^q(\mathbb{P},\mathcal{K}^\bullet) = \dfrac{\ker\{\mathrm{d}: H^0(\mathbb{P},\mathcal{K}^q)\to H^0(\mathbb{P},\mathcal{K}^{q+1})\}}{\mathrm{im}\{\mathrm{d}: H^0(\mathbb{P},\mathcal{K}^{q-1})\to H^0(\mathbb{P},\mathcal{K}^q)\}}$.

Proof. Bott vanishing (§A.3.10) asserts $H^q(\mathbb{P},\Omega^p_{\mathbb{P}}(k)) = 0$ for all $q,k>0$ and $p\ge 0$. This implies that the LES associated to the SES

$$0 \;\to\; \Omega^p_{\mathbb{P}}(kX) \;\to\; \Omega^p_{\mathbb{P}}((k+1)X) \;\to\; \frac{\Omega^p_{\mathbb{P}}((k+1)X)}{\Omega^p_{\mathbb{P}}(kX)} \;\to\; 0$$

is

$$0 \;\to\; H^0(\mathbb{P},\Omega^p_{\mathbb{P}}(kX)) \;\to\; H^0(\mathbb{P},\Omega^p_{\mathbb{P}}((k+1)X)) \;\to\; H^0\left(\mathbb{P},\frac{\Omega^p_{\mathbb{P}}((k+1)X)}{\Omega^p_{\mathbb{P}}(kX)}\right) \;\to\; 0$$

for $k>0$. In particular,

$$H^0\left(\mathbb{P},\frac{\Omega^p_{\mathbb{P}}((k+1)X)}{\Omega^p_{\mathbb{P}}(kX)}\right) \;=\; \frac{H^0(\mathbb{P},\Omega^p_{\mathbb{P}}((k+1)X))}{H^0(\mathbb{P},\Omega^p_{\mathbb{P}}(kX))}$$

and

$$H^q\left(\mathbb{P}, \frac{\Omega^p_{\mathbb{P}}((k+1)X)}{\Omega^p_{\mathbb{P}}(kX)}\right) \;=\; 0\,, \quad \forall\, q>0\,.$$

The lemma now follows from (5.4.19). □

In the case $p+q=n+1$, the lemma yields

$$H^q(\mathbb{P},\Omega^p_{\mathbb{P}}(\log X)) \;=\; \frac{H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((q+1)X))}{H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}(qX)) + \mathrm{d}H^0(\mathbb{P},\Omega^n_{\mathbb{P}}(qX))}\,. \tag{5.4.43}$$

Exercise 5.4.44. Let $(z_0,\ldots,z_{n+1})$ be coordinates on $\mathbb{C}^{n+2}$. Define $E=\sum z_i\partial_{z_i}$, and set

$$\varrho \;=\; i_E(\mathrm{d}z_0\wedge\cdots\wedge \mathrm{d}_{n+1}) \;=\; \sum(-1)^i z_i\,\mathrm{d}z_0\wedge\cdots\widehat{\mathrm{d}z_i}\cdots\wedge\mathrm{d}_{n+1} \;\in\; \Omega^{n+1}_{\mathbb{C}^{n+2}}\,.$$

Let $A,B\in\mathbb{C}[z_0,\ldots,z_{n+1}]$ be two homogeneous polynomials. Show that $\varphi=\dfrac{A}{B}\,\varrho$ descends to a well-defined $(n+1)$-form on $\mathbb{P}$ if and only if $\deg A+(n+2)=\deg B$.

Remark 5.4.45. Exercise 5.4.44 implies

$$H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}(kX)) \;=\; \left\{\frac{A}{f^k}\varrho \text{ s.t. } A\in\mathrm{Sym}^{kd-n-2}\mathbb{C}^{n+2}\right\},$$

explicitly realizing the Bott–Borel–Weil assertion $H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}(kX))\simeq\mathrm{Sym}^{kd-n-2}\mathbb{C}^{n+2}$.

Exercise 5.4.46. Keeping the notation of Exercise 5.4.44, show that

$$\psi \;=\; \frac{1}{B}\sum_{i<j}(-1)^{i+j}A_{ij}\mathrm{d}z_0\wedge\cdots\wedge\cdots\widehat{\mathrm{d}z_i}\cdots\wedge\cdots\widehat{\mathrm{d}z_j}\cdots\wedge\mathrm{d}_{n+1} \;\in\; \Omega^n_{\mathbb{C}^{n+2}}$$

descends to a well-defined n–form on $\mathbb{P}$ if and only if $\deg A_{ij}+n=\deg B$ and $A_{ij}=z_iA_j-z_jA_i$ (the latter is equivalent to $i_E(\psi)=0$).

Set $B=f^q$, and compute

$$\mathrm{d}\psi \;=\; \left(\frac{q}{f^{q+1}}\sum A_j\frac{\partial f}{\partial z_j} \;-\; \frac{1}{f^q}\sum\frac{\partial A_j}{\partial z_j}\right)\varrho\,.$$

From this we see that $\varphi = \frac{A}{f^{q+1}}\,\varrho \in H^0(\mathbb{P}, \Omega^{n+1}_{\mathbb{P}}((q+1)X)$ lies in the denominator of the right-hand side of (5.4.43) if and only if A lies in the Jacobian ideal $J_f \subset \mathbb{C}[z_0, \ldots, z_{n+1}]$ generated by the partial derivatives $\partial f/\partial z_i$. Let $R_f = \mathbb{C}[z_0, \ldots, z_{n+1}]/J_f$ be the Jacobian ring. Setting $t'(p) = d(q+1) - (n+2) = d(n+2-p) - (n+2)$, we have

$$\begin{aligned} R_f^{t'(p)} \ &\overset{(5.4.43)}{=\!=\!=} \ H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(\log X)) \ \overset{(5.4.40)}{=\!=\!=} \ \mathrm{Gr}^p_F H^{n+1}(U, \mathbb{C}) \\ &\overset{(5.4.39)}{=\!=\!=} \ \mathrm{Gr}^{p-1}_F H^n_{\mathrm{prim}}(X, \mathbb{C})\,. \end{aligned}$$

Set $t(p) = t'(p+1) = d(n+1-p) - (n+2)$ for

$$\dim_{\mathbb{C}} R_f^{t(p)} \ = \ h^{p,q}_{\mathrm{prim}}(X)\,. \tag{5.4.47}$$

Remark 5.4.48. The Hodge numbers are independent of our choice of smooth hypersurface $X = \{f = 0\}$ of degree d (Example 3.5.3). So we might as well take the Fermat hypersurface

$$f \ = \ z_0^d + \cdots + z_{n+1}^d\,,$$

which has the computational advantage that the Jacobian ring $J_f = \langle z_0^{d-1}, \ldots, z_{n+1}^d\rangle$ is quite simply presented. Using this one may show that

$$h^{p,q}_{\mathrm{prim}}(X) \ = \ \#\{\lambda \in \mathbb{Z}^{n+1} \text{ s.t. } qd < \textstyle\sum \lambda_i < (q+1)d\,,\ 1 \le \lambda_i \le d-1\}\,.$$

Remark 5.4.49. Steenbrink extended the arguments here (§5.4.5) to quasi-smooth hypersurfaces of weighted projective space $\mathbb{P}(a_0, \ldots, a_{n+1})$. Here $1 \le a_0 \le \cdots \le a_{n+1} \in \mathbb{Z}$ and $\gcd(a_0, a_{n+1}) = 1$. In general, the weighted projective space will be singular. Every weighted projective space is isomorphic to a *well-formed* weighted projective space.[1] The latter are characterized by $\gcd(a_0, \ldots, \hat{a}_j, \ldots, a_{n+1}) = 1$ for all $0 \le j \le n+1$, and have singular locus

$$\mathrm{Sing}\,\mathbb{P}(a_0, \ldots, a_{n+1}) \ = \ \bigcup_{p \text{ prime}} \{x \in \mathbb{P}(a_0, \ldots, a_{n+1}) \text{ s.t. } x_j \ne 0 \implies p | a_j\}\,.$$[2]

[1] For example, $\mathbb{P}(a, b) \simeq \mathbb{P}^1$.

[2] In general, $\mathbb{P}(a_0, \ldots, a_{n+1})$ is a normal irreducible projective algebraic variety; the singularities are all cyclic quotient singularities; and a nonsingular $\mathbb{P}(a_0, \ldots, a_{n+1})$ is isomorphic to $\mathbb{P}^{n+1}$, [Dol82].

For example,

$$\operatorname{Sing}\mathbb{P}(1,1,2,5) \;=\; \{(0:0:1:0)\,,\ (0:0:0:1)\}\;.$$

A hypersurface $X \subset \mathbb{P}(a_0,\ldots,a_{n+1})$ cut out by a homogeneous polynomial $f \in \mathbb{C}[x_0,\ldots,x_{n+1}]_d$ of weighted degree d is *quasi-smooth* if the only singularity of $\{f = 0\} \subset \mathbb{C}^{n+2}$ is the vertex $0 \in \mathbb{C}^{n+2}$. Set $|a| = \sum a_j$ and $w(p) = d(n+1-p) - |a|$. Then [Dol82, Theorem 4.3.2]

$$h^{p,q}_{\mathrm{prim}}(X) \;=\; \dim_{\mathbb{C}} R_f^{w(p)}\,. \tag{5.4.50}$$

5.4.6 Hodge structure of a blow-up

Let $Y \subset X$ be smooth projective varieties of dimensions $n-m \le n$. Let

$$\pi : X' = \mathrm{Bl}_Y(X) \;\to\; X$$

denote the blow-up of X along Y (Definition 1.1.21). Our goal here is to sketch how one computes the Hodge numbers of X' using Mayer–Vietoris exact sequences.

Let $U = X\backslash Y$, and let $Y \subset V \subset X$ be a tubular neighborhood of Y. Then V deformation retracts onto Y, and $U \cap V$ deformation retracts to $S^{2m-1}\times Y$. Set $U' = \pi^{-1}(U) \simeq U$ and $V' = \pi^{-1}(V)$. Note that $U'\cap V' \simeq U\cap V$, and V' deformation retracts to $\mathbb{P}(N_{X/Y})$. The Mayer–Vietoris sequences are

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_{k+1}(X) & \longrightarrow & H_k(U\cap V) & \longrightarrow & H_k(U)\oplus H_k(V) & \longrightarrow & H_k(X) & \longrightarrow \cdots \\
& \big\downarrow{\scriptstyle \pi^*} & & \| & & \| \quad \big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \pi^*} & \\
\cdots \longrightarrow & H_{k+1}(X') & \longrightarrow & H_k(U'\cap V') & \longrightarrow & H_k(U')\oplus H_k(V') & \longrightarrow & H_k(X') & \longrightarrow \cdots
\end{array}$$

Since V deformation retracts on to Y, we have

$$H_k(V) \;=\; H_k(Y)\,.$$

Likewise, the Künneth formula yields

$$H_k(U'\cap V') \;=\; H_k(U\cap V) \;=\; H_k(S^{2m-1}\times Y) \;=\; H_k(Y)\oplus H_{k+1-2m}(Y)\,.$$

And the Leray–Hirsh Theorem yields

$$H_k(V') \;=\; H_k(\mathbb{P}(N_{X/Y})) \;=\; \bigoplus_{a+2b=k} H_a(Y) \otimes H_{2b}(\mathbb{P}^{m-1})\,.$$

This allows us to refine the Mayer–Vietoris sequences to

$$\begin{array}{ccccccccc}
\cdots \to & H_{k+1}(X) & \longrightarrow & H_{k+1-2m}(Y) & \longrightarrow & H_k(U) & \longrightarrow & H_k(X) & \to \cdots \\
 & \downarrow \pi^* & & \downarrow & & \downarrow & & \downarrow \pi^* & \\
\cdots \to & H_{k+1}(X') & \to & \left\{ \begin{array}{c} H_k(Y) \oplus \\ H_{k+1-2m}(Y) \end{array} \right\} & \to & \left\{ \begin{array}{c} H_k(U) \oplus \\ H_k(\mathbb{P}(N_{X/Y})) \end{array} \right\} & \to & H_k(X') & \to \cdots
\end{array}$$

The maps are all morphisms of mixed Hodge structures. And this allows us to deduce

$$h^{p,q}(X') \;=\; h^{p,q}(X)\,, \quad \forall\ |p-q| > \dim Y\,.$$

Chapter 6

Degenerations of algebraic varieties

6.1 Schmid's nilpotent orbit theorem

Recommended reference: [Sch73].

Let

$$\begin{array}{ccc} \mathcal{X}^* & \hookrightarrow & \mathbb{P} \times \Delta^* \\ {\scriptstyle f}\downarrow & \swarrow & \\ \Delta^* & & \end{array} \tag{6.1.1a}$$

be a commutative diagram with the property that $f : \mathcal{X}^* \to \Delta^*$ is a smooth proper surjective morphism of complex manifolds. Then each $X_t = f^{-1}(t)$, $t \in \Delta^*$ is nonsingular projective variety. Fix a point $t_o \in \Delta^*$. Set

$$H_{\mathbb{Q}} \ = \ H^n_{\mathrm{prim}}(X_{t_o}, \mathbb{Q})\,.$$

Let

$$\rho : \pi_1(\Delta^*, t_o) \ = \ \mathbb{Z} \ \twoheadrightarrow \ \Gamma \ \subset \ \mathrm{Aut}(H_{\mathbb{Q}}, Q)$$

be the monodromy representation (§3.1). And let

$$\begin{array}{ccccc} z & \in & \mathcal{H} & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow & & \downarrow & & \downarrow \\ e^{2\pi \mathbf{i} z} = t & \in & \Delta^* & \xrightarrow{\Phi} & \Gamma\backslash D \end{array} \tag{6.1.1b}$$

be the period map induced by the smooth family $f : \mathcal{X}^* \to \Delta^*$ (Example 3.5.3). Here $\mathcal{H} = \{z \in \mathbb{C} \text{ s.t. } \mathrm{Im}(z) > 0\}$ is the upper-half plane, and D is the period domain parameterizing Q–polarized Hodge structures of weight n on $H_{\mathbb{Z}} = H^n_{\mathrm{prim}}(X_{t_o}, \mathbb{Z})$. Note that $t \to 0$ if and only if $\mathrm{Im}\, z \to +\infty$.

Let $\gamma(s) = t_o\, e^{2\pi \mathbf{i} s}$, $0 \le s \le 1$, be a counter-clockwise generator of $\pi_1(\Delta^*, t_o) = \mathbb{Z}$; and let $T = \rho(\gamma) \in \Gamma$ be the associated Picard–Lefschetz transformation (Definition 3.1.5). The generator γ acts on the universal cover $\mathcal{H}$ by $\gamma \cdot z = z + 1$, and

$$\tilde{\Phi}(z+1) \;=\; \tilde{\Phi}(\gamma \cdot z) \;=\; T \cdot \tilde{\Phi}(z)\,. \tag{6.1.2}$$

Lemma 6.1.3 (Borel [Sch73])**.** *The eigenvalues of T are roots of unity.*

Idea of the proof. Curvature properties of horizontal maps into D imply that $\tilde{\Phi}$ is distance non-increasing relative to the Poincaré metric on $\mathcal{H}$ and a suitably normalized $G_{\mathbb{R}}$–invariant hermitian metric on D. The points $\mathbf{i}k$ and $1 + \mathbf{i}k$, $k \in \mathbb{N}$ have distance $1/k$ in $\mathcal{H}$. Then (6.1.2) implies the conjugacy class of T in $G_{\mathbb{R}}$ has an accumulation point in the compact subgroup $\mathrm{Stab}_{G_{\mathbb{R}}}(\tilde{\Phi}(\mathbf{i}))$. This forces the eigenvalues of T to have norm one. On the other hand $T \in \mathrm{Aut}(H_{\mathbb{Z}}, Q)$ implies that the eigenvalues are algebraic integers (roots of a monic polynomial in integer coefficients). The lemma now follows from a result of Kronecker. $\square$

The lemma implies T is quasi-unipotent. Let $0 < m \in \mathbb{Z}$ be the smallest positive integer so that T^m is unipotent. Let

$$N \;=\; \frac{1}{m} \log T^m$$

be the nilpotent *logarithm of monodromy.*

Theorem 6.1.4 (Local monodromy, Landman)**.** *The action of N on $H_{\mathbb{Z}} = H^n_{\mathrm{prim}}(X_{t_o}, \mathbb{Z})$ satisfies $N^{n+1} = 0$.*

Exercise 6.1.5. Show that $\tilde{\Psi}(z) \overset{\mathrm{dfn}}{=\!=} \exp(-mzN) \cdot \tilde{\Phi}(mz)$ descends to a well-defined map $\Psi : \Delta^* \to \check{D}$ satisfying $\tilde{\Psi}(z) = \Psi(e^{2\pi \mathbf{i} z})$.

Definition 6.1.6. Fix $F \in \check{D}$ and a nilpotent $N \in \mathfrak{g}_\mathbb{Q}$ with the property $N(F^p) \subset F^{p-1}$. Define a map $\theta : \mathbb{C} \to \check{D}$ by $\theta(z) \stackrel{\text{dfn}}{=\!=} \exp(zN) \cdot F$. We say θ is a *nilpotent orbit* if $\theta(z) \in D$ for $\operatorname{Im} z \gg 0$. In this case we say that (F, N) *defines a nilpotent orbit.*

Theorem 6.1.7 (Nilpotent Orbit Theorem [Sch73])**.** *The map Ψ extends holomorphically over the origin $0 \in \Delta$. The pair $(\Psi(0), N)$ is a defines a nilpotent orbit $\theta(z)$ that asymptotically approximates $\tilde{\Phi}$ in the sense that there exists $\beta \geq 0$ so that*

$$\operatorname{dist}(\theta(z), \tilde{\Phi}(z)) \ \leq \ (\operatorname{Im} z)^\beta e^{-2\pi m^{-1} \operatorname{Im} z}$$

for $\operatorname{Im} z \gg 0$.

It turns out that a nilpotent orbit is equivalent to a polarized mixed Hodge structure (Theorem 6.1.10).

Exercise 6.1.8. Let $N : H_\mathbb{Q} \to H_\mathbb{Q}$ be a nilpotent linear operator.

(a) Define $0 \leq k \in \mathbb{Z}$ by $N^k \neq 0$ and $N^{k+1} = 0$. Show that there is a unique filtration

$$W(N)_{-k} \ \subset \ W(N)_{1-k} \ \subset \cdots \subset \ W(N)_{k-1} \ \subset \ W(N)_k \ = \ H_\mathbb{Q}$$

so that $N(W(N)_\ell) \ \subset \ W(N)_{\ell-2}$, for all ℓ, and the induced map $N^a : \operatorname{Gr}_a^{W(N)} \to \operatorname{Gr}_{-a}^{W(N)}$ is an isomorphism, for all $a \geq 0$. [*Hint.* To get you started, note that $W_{-k} = N^k(H_\mathbb{Q})$ and $W_{k-1} = \ker N^k$. From here one may pass to a quotient space and argue inductively. (It may be instructive to work out the cases $k = 1, 2$.)]

(b) Suppose that $N \in \mathfrak{g}_\mathbb{Q} = \operatorname{End}(H_\mathbb{Q}, Q)$; that is, $0 = Q(Nu, v) + Q(u, Nv)$ for all $u, v \in H_\mathbb{Q}$. Show that $W(N)_\bullet$ is Q–isotropic: $Q(W(N)_{-a}, W(N)_{a-1}) = 0$ and the induced bilinear form $Q_\ell : \operatorname{Gr}_{-a}^{W(N)} \times \operatorname{Gr}_a^{W(N)} \to \mathbb{Q}$ is nondegenerate.

(c) Show that $\ker\{N : H_\mathbb{Q} \to H_\mathbb{Q}\} \subset W(N)_0$.

Definition 6.1.9. Let (W, F) be a mixed Hodge structure with $F \in \check{D}$. We say a nilpotent element $N \in \mathfrak{g}_\mathbb{Q}$ *polarizes* (W, F) if:

(i) We have $N^{n+1} = 0$.

(ii) The nilpotent operator and weight filtration are related by $W(N)_\ell = W_{n+\ell}$. We express this as $W = W(N)[-n]$. Note that $\mathrm{Gr}^W_{n+\ell} = \mathrm{Gr}^{W(N)}_\ell$.

(iii) The nilpotent operator and Hodge filtration are related by $N(F^p) \subset F^{p-1}$; equivalently $N \in \mathfrak{g}_\mathbb{Q} \cap F^{-1}(\mathfrak{g}_\mathbb{C})$.

(iv) The induced weight $n + a$ Hodge structure on

$$P(N)_{n+a} \overset{\text{dfn}}{=\!=} \ker\{N^{a+1} : \mathrm{Gr}^W_{n+a} \to \mathrm{Gr}^W_{n-a-2}\}$$

is polarized by $Q(\cdot, N^a \cdot)$.

We call (W, F, N) a *polarized mixed Hodge structure.* And, since N determines the weight filtration W, we say the pair (F, N) *defines a polarized mixed Hodge structure.*

Theorem 6.1.10 ([Sch73, CKS86])**.** *The pair (F, N) defines a nilpotent orbit if and only if the pair defines a polarized mixed Hodge structure.*

Remark 6.1.11. We may interpret Theorems 6.1.7 and 6.1.10 as saying that the family $\Phi(t)$ of (Γ–equivalence classes of) Hodge structures on $H_\mathbb{Q} = H^n_{\mathrm{prim}}(X_{t_o}, \mathbb{Q})$ degenerates to a polarized mixed Hodge structure $(\Psi(0), N)$ on $H_\mathbb{Q}$ as $t \to 0$. Be aware that the Hodge filtration $F_{\mathrm{lim}} = \Psi(0)$ depends our choice of coordinates; only N and the nilpotent orbit $\theta(z)$ are independent of this choice.

Exercise 6.1.12. Let (W, F, N) be a polarized mixed Hodge structure (Definition 6.1.9). Show that N is a weight 0 morphism of mixed Hodge structure (Definition 5.2.6).

Definition 6.1.13. Suppose that the family $f : \mathcal{X}^* \to \Delta^*$ of (6.1.1a) extends over the origin as

$$\begin{array}{ccccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X} & \hookrightarrow & \mathbb{P} \times \Delta \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & \swarrow & \\ \Delta^* & \hookrightarrow & \Delta & & \end{array} \tag{6.1.14}$$

with $f : \mathcal{X} \to \Delta$ is a proper, flat[1] holomorphic map of complex manifolds. Then we say $f : \mathcal{X} \to \Delta$ is a *degeneration* (of nonsingular projective varieties).

Example 6.1.15. For many naturally arising families the total space $\mathcal{X}$ will fail to be smooth. Suppose that $F, G, H \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ are homogeneous polynomials with $\deg F = (\deg G)(\deg H)$. Consider the family (6.1.14) defined by

$$\mathcal{X} \ = \ \{tF + GH = 0\} \ \subset \ \mathbb{P}^{n+1} \times \Delta\,.$$

For generic choice of F, G, H, the restriction of f to $\mathcal{X}^* = f^{-1}(\Delta^*) \to \Delta^*$ is smooth morphism of smooth manifolds; and the central fibre $X_0 = f^{-1}(0)$ is a simple normal crossing hypersurface in $\mathbb{P}^{n+1}$. However, $\mathrm{Sing}\,(\mathcal{X}) = \{t, F, G, H = 0\}$. In order to obtain a degeneration in the sense of Definition 6.1.13, we need to resolve these singularities.

Question. What is the relationship between Deligne's mixed Hodge structure (§5) on $H^n(X_0, \mathbb{Q})$, and Schmid's limiting mixed Hodge structure (Remark 6.1.11) on $H^n(X_{t_o}, \mathbb{Q})$? When the family (6.1.14) is a semistable degeneration the answer is given by the Clemens–Schmid exact sequence. (For results in more general settings, see [KLS21, KL24].)

6.2 Clemens–Schmid exact sequence

Recommended reference: [Mor84].

Definition 6.2.1. The degeneration $f : \mathcal{X} \to \Delta$ of (6.1.14) is *semistable* if the central fibre $X_0 = f^{-1}(0)$ is simple normal crossing (Definition 5.3.1). This implies

[1] Suppose that $f : X \to Y$ is a morphism of smooth algebraic varieties with equidimensional fibres. Then *miracle flatness* states that f is *flat*. This means that the induced map on stalks $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,f(x)}$–module: taking the tensor product $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}}$ preserves exact sequences. The fibres of a flat morphism have constant Hilbert polynomial, cf. [Har77, Proposition III.9.9] or [Vak24]. And so, for example, a (nontrivial) blow-up is not flat.

that every point $x_o \in X_0$ admits local coordinates (x_j) on $\mathcal{X}$ with respect to which the projection f is given by $x_1 \cdots x_k = t$.

Theorem 6.2.2 (Semistable reduction [KKMSD73]). *Given a degeneration $f : \mathcal{X} \to \Delta$, there exists a base change $b : \Delta \to \Delta$, mapping $t \mapsto t^m$, a semistable degeneration $g : \mathcal{Y} \to \Delta$ and a diagram*

$$\begin{array}{ccccc} \mathcal{Y} & \dashrightarrow & \mathcal{X}_b & \longrightarrow & \mathcal{X} \\ & \searrow^{g} & \downarrow & & \downarrow^{f} \\ & & \Delta & \xrightarrow{b} & \Delta \end{array}$$

so that $\mathcal{Y} \dashrightarrow \mathcal{X}_b$ is a bimeromorphic (birational) map obtained by blowing up and blowing down subvarieties of the central fibre.

Theorem 6.2.3 ([Cle77]). *If $f : \mathcal{X} \to \Delta$ is a semistable degeneration, then there is a retraction $\mathcal{X} \to X_0$. In particular, $H^n(\mathcal{X}, \mathbb{Q}) \simeq H^n(X_0, \mathbb{Q})$ has a mixed Hodge structure.*

Theorem 6.2.4 ([Lan73]). *Let $f : \mathcal{X} \to \Delta$ be a degeneration.*

(i) *The Picard–Lefschetz transformation is quasi-unipotent, with index of unipotency at most n. That is, there exists $0 < m \in \mathbb{Z}$ so that T^m is unipotent* (we assume m is minimal with this property), *and $(T^m - \mathrm{Id})^{n+1} = 0$.*

(ii) *If $f : \mathcal{X} \to \Delta$ is semistable, then T is unipotent ($m = 1$).*

Let $i : X_{t_o} \hookrightarrow \mathcal{X}$ denote the inclusion, and $i^* : H^n(\mathcal{X}) \to H^n(X_{t_o})$ the pullback. Note that i^* is a weight 0 morphism of mixed Hodge structures, and $N : H^n(X_{t_o}, \mathbb{Q}) \to H^n(X_{t_o}, \mathbb{Q})$ is a weight -2 morphism of mixed Hodge structures (Definition 5.2.6).

Theorem 6.2.5 (Clemens–Schmid exact sequence [Cle77]). *Assume that* (6.1.14) *is a semistable degeneration, and the fibres X_t have dimension d. Then there is an exact sequence*

$$\cdots \longrightarrow H_{2d+2-n}(\mathcal{X}, \mathbb{Q}) \xrightarrow{\alpha} H^n(\mathcal{X}, \mathbb{Q}) \xrightarrow{i^*} H^n(X_{t_o}, \mathbb{Q}) \xrightarrow{N} H^n(X_{t_o}, \mathbb{Q})$$
$$\xrightarrow{\beta} H_{2d-n}(\mathcal{X}, \mathbb{Q}) \xrightarrow{\alpha} H^{n+2}(\mathcal{X}, \mathbb{Q}) \longrightarrow \cdots$$

of morphisms of mixed Hodge structure. The morphisms i^* and N are of weight 0 and -2, respectively. The maps α and β are compositions of inclusions and Poincaré duality maps, and are of weight $d+1$ and $-d$, respectively.

Exercise 6.2.6. Use the Clemens–Schmid exact sequence to show that $H^0(X_0, \mathbb{Q}) \simeq H^0(\mathcal{X}, \mathbb{Q}) \xrightarrow{i^*} H^0(X_{t_o}, \mathbb{Q})$ is an isomorphism.

The statement of Theorem 6.2.5 contains the Local Invariant Cycle Theorem:

Corollary 6.2.7 (Local Invariant Cycle Theorem)**.** *The sequence*

$$H^n(\mathcal{X}, \mathbb{Q}) \ \xrightarrow{i^*} \ H^n(X_{t_o}, \mathbb{Q}) \ \xrightarrow{N} \ H^n(X_{t_o}, \mathbb{Q})$$

is exact. That is, all cocycles in $H^n(X_{t_o}, \mathbb{Q})$ that are invariant under the Picard–Lefschetz transformation come from cocycles on $\mathcal{X}$.

Exercise 6.2.8. Recall (§5.3.3) that the weight filtration on $H^n(\mathcal{X}, \mathbb{Q}) = H^n(X_0, \mathbb{Q})$ satisfies

$$0 \subset W_0 H^n(X_0) \ \subset \ W_1 H^n(X_0) \ \subset \cdots \subset \ W_n H^n(X_0) \ = \ H^n(X_0)\,.$$

(a) Show that the Deligne's weight filtration on $H_{n-2d-2}(\mathcal{X}, \mathbb{Q}) \simeq H_{n-2d-2}(X_0, \mathbb{Q})$ is

$$\begin{aligned} 0 \ &\subset \ W_{n-2d-2} H_{n-2d-2}(\mathcal{X}) \ \subset \ W_{n-2d-1} H_{n-2d-2}(\mathcal{X}) \ \subset \cdots \subset \ W_{-1} H_{n-2d-2}(\mathcal{X}) \\ &\subset \ W_0 H_{n-2d-2}(\mathcal{X}) \ = \ H_{n-2d-2}(\mathcal{X})\,. \end{aligned}$$

(b) Show that

$$\mathrm{im}\,\{\alpha : H_{2d+2-n}(\mathcal{X}, \mathbb{Q}) \to H^n(\mathcal{X}, \mathbb{Q})\} \ = \ \alpha(W_{n-2d-2} H_{2d+2-n}(\mathcal{X}, \mathbb{Q}))\,.$$

[*Hint.* Strictness (Exercise 5.2.11) may be useful here.]

(c) Conclude that the restriction of $i^* : H^n(\mathcal{X}, \mathbb{Q}) \to H^n(X_{t_o}, \mathbb{Q})$ to $W_{n-1} H^n(\mathcal{X}, \mathbb{Q})$ is injective.

Corollary 6.2.9. *Given $k > 0$, $N^k : H^n(X_{t_o}, \mathbb{Q}) \to H^n(X_{t_o}, \mathbb{Q})$ is the zero map if and only if $W_{n-k}(H^n(X_0, \mathbb{Q})) = 0$. In particular, N^{n+1} is always zero, and $N^n = 0$ if and only if $H^n(|\Gamma(X_0)|) = 0$.* (Here $\Gamma(X_0)$ is the dual complex, cf. Definition 5.3.8.)

Exercise 6.2.10. Prove Corollary 6.2.9. [*Hint.* Corollary 5.3.9, Exercise 6.1.8, Definition 6.1.9, and Exercise 6.2.8.]

6.2.1 Degree one cohomology groups

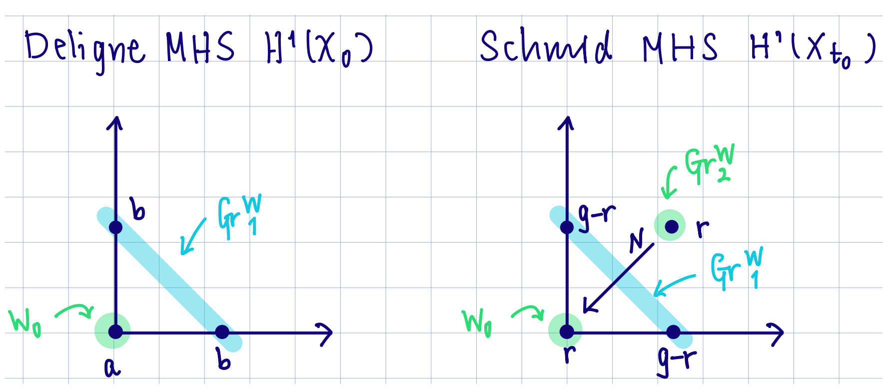
Deligne's mixed Hodge structure

Recall (§5.3.3) that Deligne's weight filtration $W_0 \subset W_1$ on $H^1(X_0, \mathbb{Q})$ is

$$\begin{aligned} W_0 H^1(X_0, \mathbb{Q}) &\simeq H^1(|\Gamma(X_0)|)\,, \\ \mathrm{Gr}_1^W H^1(X_0, \mathbb{Q}) &\simeq \ker\{d_1 : H^1(X_0^{(1)}, \mathbb{Q}) \to H^1(X_0^{(2)}, \mathbb{Q})\}\,. \end{aligned}$$

Here $X_0^{(1)} = \sqcup X_0^j$ is the disjoint union of the irreducible components X_0^j of $X_0 = \cup X_0^j$, and $X_0^{(2)} = \sqcup_{j<k} X_0^j \cap X_0^k$; and $\Gamma(X_0)$ is the dual complex of the simple normal crossing variety X_0 (Definition 5.3.8 and Corollary 5.3.9). The Hodge diamond (Definition 5.2.14) of this mixed Hodge structure is given in Figure 6.1.

Figure 6.1: Hodge diamonds for H^1



Schmid's mixed Hodge structure

In degree one, the nilpotent logarithm of monodromy $N : H^1(X_{t_o},\mathbb{Q}) \to H^1(X_{t_o},\mathbb{Q})$ satisfies $N^2 = 0$ (Theorem 6.2.4), and the Schmid's weight filtration $W = W(N)[-1]$ is

$$\begin{aligned} W_0 H^1(X_{t_o},\mathbb{Q}) &= \operatorname{im} N \\ W_1 H^1(X_{t_o},\mathbb{Q}) &= \ker N \\ W_2 H^1(X_{t_o},\mathbb{Q}) &= H^1(X_{t_o},\mathbb{Q})\,. \end{aligned}$$

Implications of the Clemens–Schmid exact sequence

The Clemens–Schmid sequence yields

$$0 \xrightarrow{\alpha} H^1(\mathcal{X},\mathbb{Q}) \xrightarrow{i^*} H^1(X_{t_o},\mathbb{Q}) \xrightarrow{N} H^1(X_{t_o},\mathbb{Q})\,.$$

So

$$H^1(X_0,\mathbb{Q}) \simeq \ker\{N : H^1(X_{t_o},\mathbb{Q}) \to H^1(X_{t_o},\mathbb{Q})\} = W_1 H^1(X_{t_o},\mathbb{Q})\,,$$

and

$$W_0 H^1(X_0,\mathbb{Q}) \simeq W_0 H^1(X_{t_o},\mathbb{Q})\,.$$

In the notation of Figure 6.1, this implies $a = r = h^1(|\Gamma(X_0)|)$ and $b = g - r$.

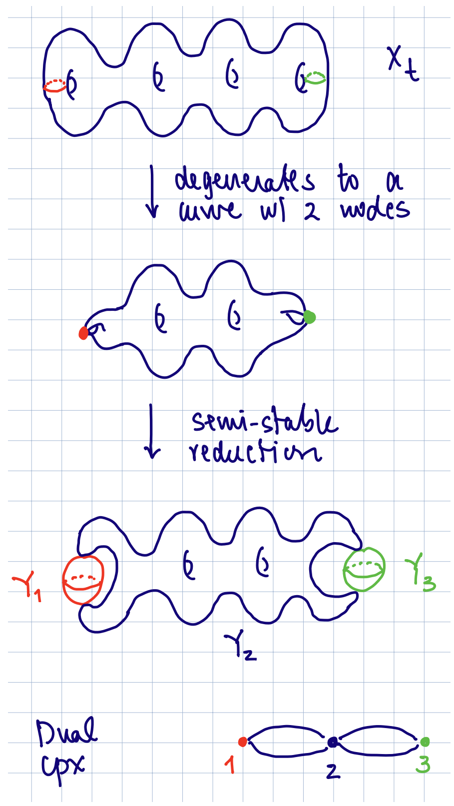
Degenerations of curves

If we further assume that $d = 1$, so that $\mathcal{X}$ is a family of curves, then $X_0^{(2)}$ is the set of double points $P_{jk} = X_0^j \cap X_0^k$, and we have (§5.3.2)

$$\mathrm{Gr}_1^W H^1(X_{t_o},\mathbb{Q}) \simeq \mathrm{Gr}_1^W H^1(X_0,\mathbb{Q}) \simeq H^1(X_0^{(1)},\mathbb{Q})\,.$$

Let $g = h^{1,0}(X_{t_o})$ denote the genus of the smooth fibres. Fix generators $\{\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g\}$ of $H_1(X_t,\mathbb{Z})$ satisfying $\alpha_i\cdot\alpha_j = 0 = \beta_i\cdot\beta_j$ and $\alpha_i\cdot\beta_j = 1$. Suppose that as $t \to 0$ some of the cycles α_j collapse to nodes (as in Figure 6.2). Passing to the semistable

Figure 6.2: Degeneration of curves



reduction replaces those nodes with $\mathbb{P}^1$'s. Then $r + \sum g_j = g$, where $g_j = h^{1,0}(X_0^j)$ is the genus of X_0^j and r is the number of nodes (the number of vanishing cycles α_j). Note that $N = 0$ if and only if the dual graph has no cycles.

6.2.2 Degenerations of surfaces

Deligne's mixed Hodge structure

If $d = 2$ (the fibres X_t are surfaces), then $X_0^{(2)}$ is the disjoint union of the double curves $C_{jk} = X_0^i \cap X_0^j$, and $X_0^{(3)}$ is the union of the triple points $P_{ijk} = X_0^i \cap X_0^j \cap X_0^k$.

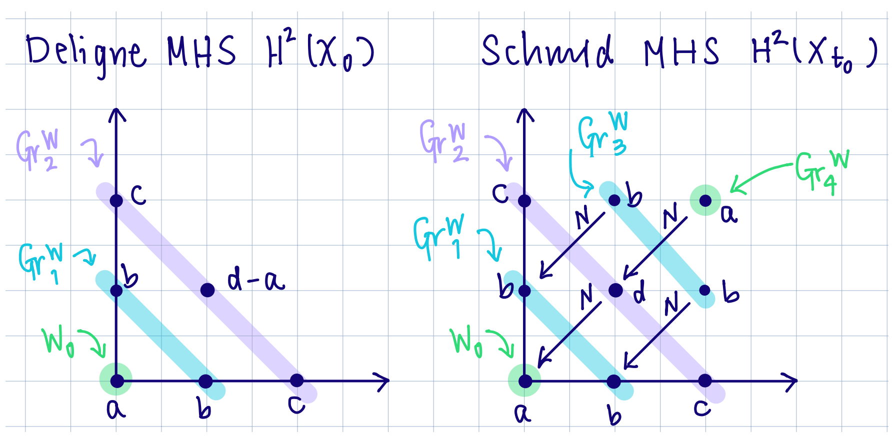
Deligne's weight filtration $W_0 \subset W_1 \subset W_2 = H^2(X_0, \mathbb{Q})$ has graded quotients (§5.3.4)

$$\begin{aligned} W_0 H^2(X_0, \mathbb{Q}) &= H^2(|\Gamma(X_0)|) \\ \mathrm{Gr}_1^W H^2(X_0, \mathbb{Z}) &= \frac{\oplus_{j<k} H^1(C_{jk})}{\mathrm{im}\,\{d_1 : \oplus_j H^1(X_0^j) \to \oplus_{j<k} H^1(C_{jk})\}} \\ \mathrm{Gr}_2^W H^2(X_0, \mathbb{Z}) &= \ker\{d_1 : \oplus_j H^2(X_0^j) \to \oplus_{j<k} H^2(C_{jk})\}\,. \end{aligned} \tag{6.2.11}$$

In the notation of Figure 6.3

$$\begin{aligned} a &= h^2(|\Gamma(X_0)|)\,, \\ c &= \sum h^{2,0}(X_0^j)\,, \\ d - a &\le \sum h^{1,1}(X_0^j)\,. \end{aligned}$$

Figure 6.3: Hodge diamonds for H^2



Schmid's mixed Hodge structure

In degree two, the nilpotent logarithm of monodromy $N : H^2(X_{t_o}, \mathbb{Q}) \to H^2(X_{t_o}, \mathbb{Q})$ satisfies $N^3 = 0$ (Theorem 6.2.4), and the Schmid's weight filtration $W = W(N)[-1]$

is

$$
\begin{aligned}
W_0H^2(X_{t_o},\mathbb{Q}) &= \operatorname{im} N^2 \\
W_1H^2(X_{t_o},\mathbb{Q}) &= (\operatorname{im} N) \cap (\ker N) \\
W_2H^2(X_{t_o},\mathbb{Q}) &= (\operatorname{im} N) + (\ker N) \\
W_3H^2(X_{t_o},\mathbb{Q}) &= \ker N^2 \\
W_4H^2(X_{t_o},\mathbb{Q}) &= H^2(X_{t_o},\mathbb{Q})\,.
\end{aligned}
$$

Implications of the Clemens–Schmid exact sequence

The Clemens–Schmid exact sequence yields

$$
0 \xrightarrow{\alpha} H^2(\mathcal{X}) \xrightarrow{i^*} H^2(X_{t_o}) \xrightarrow{N} H^2(X_{t_o}) \xrightarrow{\beta} H_2(\mathcal{X}) \xrightarrow{\alpha} 0\,,
$$

so that

$$
H^2(X_0,\mathbb{Q}) \simeq \ker\{N : H^2(X_{t_o},\mathbb{Q}) \to H^2(X_{t_o},\mathbb{Q})\}\,.
$$

In the notation of Figure 6.3

$$
\begin{aligned}
a+b+c &= h^{2,0}(X_{t_o})\,, \\
2b+d &= h^{1,1}(X_{t_o})\,,
\end{aligned}
$$

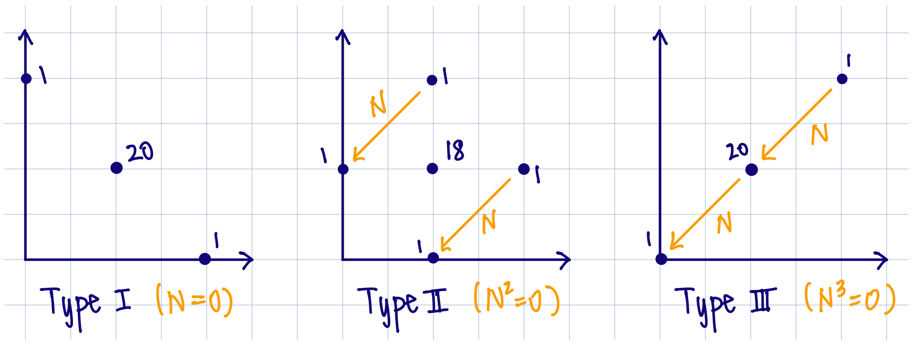
and

$$
\sum_{j<k} h^{1,0}(C_{jk}) - \sum_j h^{1,0}(X_0^j) \le b \le \sum_{j<k} h^{1,0}(C_{jk})\,.
$$

6.3 Degeneration of K3 surfaces

Smooth K3 surfaces S are characterized by $K_S = \mathcal{O}_S$ and $q(S) = 0$ (§A.2.8). By Exercise A.2.3, the Hodge numbers of $H^2(S)$ are $\mathbf{h} = (1, 20, 1)$. This allows for only three possible types of Schmid MHS (W, F, N) on $H^3(X_{t_o},\mathbb{Q})$; these types are indexed by their Hodge diamonds in Figure 6.4. Each of these three types can be realized geometrically by a semistable degeneration.

Figure 6.4: Hodge diamond of Schmid's MHS on $H^2(\mathrm{K3})$ by Kulikov type



6.3.1 Eg. geometric realization of Type I degeneration

Any smooth hypersurface $X \subset \mathbb{P}^3$ of degree four is K3 surface. A popular example is give the Fermat quartic, which is cut out by

$$F \;=\; x_0^4 + x_1^4 + x_2^4 + x_3^4 \,.$$

For a generic choice of degree four homogeneous polynomials $P_0, P_1 \in \mathbb{C}[x_0, x_1, x_2, x_3]$, the hypersurfaces $\{P_j = 0\}$ will be smooth, and $\mathrm{d}P_1$ and $\mathrm{d}P_2$ will be point-wise linearly independent along $\{P_0, P_1 = 0\}$. Then the family

$$\mathcal{X} \;=\; \{P_0 + tP_1 = 0\} \;\subset\; \mathbb{P}^3 \times \Delta$$

is a semistable degeneration that geometrically realizes Schmid's Type I MHS (via the Clemens–Schmid exact sequence).

6.3.2 Eg. geometric realization of Type II degeneration

For a generic choice of homogeneous polynomials $Q_1, Q_2, P \in \mathbb{C}[x_0, x_1, x_2, x_3]$ of degrees $\deg Q_j = 2$ and $\deg P = 4$, the hypersurfaces $\{Q_j = 0\}$ and $\{P = 0\}$ are smooth, and the differentials $\mathrm{d}P, \mathrm{d}Q_1, \mathrm{d}Q_2$ are point-wise linearly independent along

$\{Q_1, Q_2, P = 0\}$. The family

$$\mathcal{X}' \;=\; \{tP + Q_1 Q_2 = 0\} \;\subset\; \mathbb{P}^3 \times \Delta$$

has the properties that X'_t is smooth for all $t \neq 0$, and $X'_0 = \{Q_1 Q_2 = 0\} = X'_{01} \cup X'_{02}$ is simple normal crossing. The double curve $C'_{12} = X'_{01} \cap X'_{02}$ is an elliptic curve (Example A.3.14). However this is not a semistable degeneration because $\mathcal{X}'$ is not smooth; there are 16 isolated singularities along the central fibre $\mathrm{Sing}\,\mathcal{X}' = \{t, P, Q_1, Q_2 = 0\} \subset X'_0$.

Let $\pi : \mathcal{Y} \to \mathbb{P}^3 \times \Delta$ be the blow-up of $\mathbb{P}^3 \times \Delta$ at each of the 16 points in $\mathrm{Sing}\,\mathcal{X}'$.

Lemma 6.3.1. *The strict transform*

$$\mathcal{X} \;\stackrel{\text{dfn}}{=\!=}\; \overline{\pi^{-1}(\mathcal{X}' \backslash \mathrm{Sing}\,\mathcal{X}')} \;\subset\; \mathcal{Y}$$

$\mathcal{X}'$ is smooth, and a semistable degeneration. The fibre of $\rho \stackrel{\text{dfn}}{=\!=} \pi|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}'$ over a singular point $s \in \mathrm{Sing}\,\mathcal{X}'$ is a smooth quadric surface X_s. The central fibre of $\mathcal{X} \to \Delta$ is a simple normal crossing $X_0 = \rho^{-1}(X'_0) = X_{01} \cup X_{02} \cup (\cup_s X_s)$, with $X_{0j} = \overline{\rho^{-1}(X'_{0j} \backslash \mathrm{Sing}\,\mathcal{X}')}$ smooth quadric surfaces. The double curves are the elliptic $C_{12} = X_{01} \cap X_{02}$, and $X_{0j} \cap X_s \simeq \mathbb{P}^1$. We have

$$\begin{aligned} W_0 H^2(X_0, \mathbb{Q}) &= H^2(|\Gamma(X_0)|) \;=\; 0 \\ \mathrm{Gr}^W_1 H^2(X_0, \mathbb{Z}) &= H^1(C_{12}) \\ \mathrm{Gr}^W_2 H^2(X_0, \mathbb{Z}) &= \ker\left\{ d_1 : \oplus_j H^2(X_0^j) \;\to\; H^2(C_{12}) \right\} . \end{aligned}$$

Remark 6.3.2. The Hodge numbers of the smooth quadric hypersurfaces $X_{0j} \subset \mathbb{P}^3$ are $\mathbf{h}^2(X_0^j) = (0, 2, 0)$, by either the Künneth formula (every smooth quadric surface in $\mathbb{P}^3$ is isomorphic to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$), or Griffiths' Jacobian ring computation (Remark 5.4.48).

Proof. Fix a point $s \in \mathrm{Sing}\,\mathcal{X}'$, and choose $U_a = \{x_a \neq 0\} \subset \mathbb{P}^3$ so that $s \in U_a \times \Delta$. Define $p, q_j : U_a \to \mathbb{C}$ by

$$\begin{aligned} p(x_0 : x_1 : x_2 : x_3) &\stackrel{\text{dfn}}{=\!=} P\left(\tfrac{x_0}{x_a}, \tfrac{x_1}{x_a}, \tfrac{x_2}{x_a}, \tfrac{x_3}{x_a}\right) \\ q_j(x_0 : x_1 : x_2 : x_3) &\stackrel{\text{dfn}}{=\!=} Q_j\left(\tfrac{x_0}{x_a}, \tfrac{x_1}{x_a}, \tfrac{x_2}{x_a}, \tfrac{x_3}{x_a}\right) . \end{aligned}$$

The condition that the differentials $\mathrm{d}P, \mathrm{d}Q_1, \mathrm{d}Q_2$ are point-wise linearly independent along $\{Q_1, Q_2, P = 0\}$ imply that $(p, q_1, q_2, t) : U \to \mathbb{C}^4$ are local coordinates on some neighborhood $s \in U \subset \mathbb{P}^3 \times \Delta$. In this coordinate neighborhood the blow-up $\pi^{-1}(U)$ is the closure of the graph of $U \backslash \{0\} \xrightarrow{(p:q_1:q_2:t)} \mathbb{P}^3$ in $U \times \mathbb{P}^3$. The blow-up $\pi^{-1}(U)$ is covered by four coordinate charts:

- $(t, z_1, z_2, z_3) \mapsto ((tz_1, tz_2, tz_3, t)\,;\,(z_1 : z_2 : z_3 : 1)) \in \pi^{-1}(U)$. The exceptional divisor is cut out by $t = 0$, and $\mathcal{X}$ is cut out by $z_1 + z_2\, z_3 = 0$ (which is clearly smooth).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = t$, so that X_0 is locally cut out by $\{t,\ z_1 + z_2\, z_3 = 0\}$. The equation $z_1 + z_2\, z_3 = 0$ defines a (a Zariski open subset of) a Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$.

- $(q_2, z_1, z_2, z_3) \mapsto ((q_2 z_1, q_2 z_2, q_2, q_2 z_3)\,;\,(z_1 : z_2 : 1 : z_3)) \in \pi^{-1}(U)$. The exceptional divisor is cut out by $q_2 = 0$, and $\mathcal{X}$ is cut out by $z_2 + z_3\, z_1 = 0$ (which is clearly smooth).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = q_2 z_3$, so that X_0 is locally cut out by $\{q_2 z_3,\ z_2 + z_3\, z_1 = 0\}$. If $q_2 = 0$, then we get (a Zariski open subset of) the Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$. If $z_3 = 0$, then $z_2 = 0$ gives us (a Zariski open subset of) $X_{01} \simeq X'_{01} = \{Q_1 = 0\}$. These two surfaces intersect along (a Zariski open subset of) $X_{01} \cap X_s = \mathbb{P}^1 \subset \mathbb{P}^3$.

- $(q_1, z_1, z_2, z_3) \mapsto ((q_1 z_1, q_1, q_1 z_2, q_1 z_3)\,;\,(z_1 : 1 : z_2 : z_3)) \in \pi^{-1}(U)$. The exceptional divisor is cut out by $q_1 = 0$, and $\mathcal{X}$ is cut out by $z_2 + z_3\, z_1 = 0$ (which is clearly smooth).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = q_1 z_3$, so that X_0 is locally cut out by $\{q_1 z_3,\ z_2 + z_3\, z_1 = 0\}$. If $q_1 = 0$, then we get (a Zariski open subset of) the Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$. If $z_3 = 0$, then $z_2 = 0$ gives us (a Zariski open subset of) $X_{02} \simeq X'_{02} = \{Q_2 = 0\}$. These two surfaces intersect along (a Zariski open subset of) $X_{02} \cap X_s = \mathbb{P}^1 \subset \mathbb{P}^3$.

- $(p, z_1, z_2, z_3) \mapsto ((p, pz_1, pz_2, pz_3)\,;\,(1 : z_1 : z_2 : z_3)) \in \pi^{-1}(U)$. The exceptional divisor is cut out by $p = 0$, and $\mathcal{X}$ is cut out by $z_3 + z_1\, z_2 = 0$ (which is clearly smooth).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = pz_3$, so that X_0 is locally cut out by $\{pz_3,\ z_3 + z_1\, z_2 = 0\}$. If $p = 0$, then we get (a Zariski open subset of) the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. If $z_3 = 0$, then $z_1\, z_2 = 0$ gives us (a Zariski open subset of) $X_{01} \cup X_{02} \simeq X'_{01} \cup X'_{02} = \{Q_1\, Q_2 = 0\}$. These two surfaces intersect (in a Zariski open subset of) $X_s \cap (X_{01} \cup X_{02}) \simeq \mathbb{P}^1 \cup \mathbb{P}^1$

□

6.3.3 Eg. geometric realization of Type III degeneration

The family

$$\mathcal{X}' \;=\; \{x_0x_1x_2x_3 \;+\; t(x_0^4 + x_1^4 + x_2^4 + x_3^4) \;=\; 0\} \;\subset\; \mathbb{P}^3 \times \Delta$$

has the properties that X'_t is smooth for all $t \neq 0$, and $X'_0 = \cup_{j=0}^{3} X'_{0j}$ is the "tetrahedron" formed by the coordinate planes $X'_{0j} = \{x_j = 0\}$ (and simple normal crossing). However this is not a semistable degeneration because $\mathcal{X}'$ is not smooth; the singular locus

$$\begin{aligned}\operatorname{Sing}\mathcal{X}' \;=\;& \big\{(1 : \alpha : 0 : 0)\,,\ (1 : 0 : \alpha : 0)\,,\ (1 : 0 : 0 : \alpha)\\ & (0 : 1 : \alpha : 0)\,,\ (0 : 1 : 0 : \alpha)\,,\ (0 : 0 : 1 : \alpha)\ \text{ s.t. } \alpha^4 = -1\big\} \;\subset\; X'_0\end{aligned}$$

consists of 24 points on the central fibre (Figure 6.5).

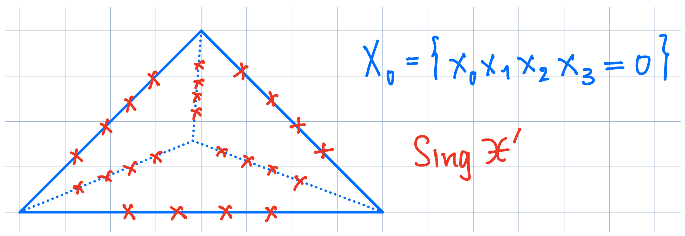
Let $\pi : \mathcal{Y} \to \mathbb{P}^3 \times \Delta$ be the blow-up of $\mathbb{P}^3 \times \Delta$ at each of the 24 points in $\operatorname{Sing}\mathcal{X}'$.

Lemma 6.3.3. *The strict transform*

$$\mathcal{X} \;\stackrel{\text{dfn}}{=\!=}\; \overline{\pi^{-1}(\mathcal{X}' \backslash \operatorname{Sing}\mathcal{X}')} \;\subset\; \mathcal{Y}$$

of $\mathcal{X}'$ is smooth, and a semistable degeneration. The fibre of $\rho \stackrel{\text{dfn}}{=\!=} \pi|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}'$ over a singular point s is a smooth quadric surface X_s. The central fibre of $\mathcal{X} \to$

Figure 6.5: Sing $\mathcal{X}'$



Δ is a simple normal crossing $X_0 = \rho^{-1}(X_0') = (\cup_{j=0}^3 X_{0j}) \cup (\cup_s X_s)$. The $X_0^j = \overline{\rho^{-1}(X_{0j}' \backslash \mathrm{Sing}\,\mathcal{X}')}$ are rational surfaces. The double curves are the $C_{jk} = X_{0j} \cap X_{0k} \simeq \mathbb{P}^1$ and $C_{js} = X_{0j} \cap X_s \simeq \mathbb{P}^1$. We have

$$\begin{aligned} W_0 H^2(X_0, \mathbb{Q}) &= H^2(|\Gamma(X_0)|) = \mathbb{Q}\,, \\ \mathrm{Gr}_1^W H^2(X_0, \mathbb{Z}) &= 0 \\ \mathrm{Gr}_2^W H^2(X_0, \mathbb{Z}) &= \ker \left\{ d_1 : H^2(X_0^{(1)}) \simeq \mathbb{Q}^{52} \to H^2(X_0^{(2)}) \simeq \mathbb{Q}^{54} \right\} \simeq \mathbb{Q}^{19}\,. \end{aligned}$$

Proof. Fix a singular point $s = (1 : \alpha : 0 : 0) \in X_0'$. Let $U_0 = \{x_0 \neq 0\} \subset \mathbb{P}^3$ be an affine coordinate chart, and define $(v_1, v_2, v_3) : U_0 \to \mathbb{C}^3$ by $v_1 = \frac{1}{x_0^4}(x_0^4 + x_1^4 + x_2^4 + x_3^4)$, $v_2 = x_2/x_0$ and $v_3 = x_3/x_0$. Then the differentials $\mathrm{d}v_1, \mathrm{d}v_2, \mathrm{d}v_3$ are linearly independent at s. It follows that (v_1, v_2, v_3, t) are local coordinates (centered at s) on some neighborhood $s \in U \subset \mathbb{P}^3 \times \Delta$. Let $\xi(v_1, v_2, v_3)$ be the local coordinate expression for the function x_1/x_0. Then $\mathcal{X}'$ is locally cut out by $\{tv_1 + \xi v_2 v_3 = 0\}$. Note that $\xi(0,0,0) = \alpha$ at s, so we may assume that ξ is nowhere zero on U.

Over the coordinate neighborhood the blow-up $\pi^{-1}(U)$ is the closure of the graph of $U \backslash \{0\} \xrightarrow{(v_1:v_2:v_3:t)} \mathbb{P}^3$ in $U \times \mathbb{P}^3$. The blow-up $\pi^{-1}(U)$ is covered by four coordinate charts:

- $(t, z_1, z_2, z_3) \mapsto ((tz_1, tz_2, tz_3, t)\,;\,(z_1 : z_2 : z_3 : 1)) \in \pi^{-1}(U)$. The exceptional divisor is cut out by $t = 0$, and $\mathcal{X}$ is cut out by $z_1 + \xi(tz_1, tz_2, tz_3)\, z_2\, z_3 = 0$ (which is clearly smooth).

The map $f : \mathcal{X} \to \Delta$ is locally given by $f = t$, so that X_0 is locally cut out by $\{t,\, z_1 + \alpha\, z_2\, z_3 \,=\, 0\}$. The equation $z_1 + \alpha\, z_2\, z_3 \,=\, 0$ defines a (a Zariski open subset of) a Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$.

- $(v_3, z_1, z_2, z_3) \;\mapsto\; ((v_3z_1, v_3z_2, v_3, v_3z_3)\,;\,(z_1 : z_2 : 1 : z_3)) \;\in\; \pi^{-1}(U)$. The exceptional divisor is cut out by $v_3 = 0$, and $\mathcal{X}$ is cut out by $z_3\, z_1 + \xi(v_3z_1, v_3z_2, v_3)\, z_2 = 0$ (which is smooth since ξ is nowhere zero).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = v_3z_3$, so that X_0 is locally cut out by $\{v_3z_3,\, z_3\, z_1 + \xi(v_3z_1, v_3z_2, v_3)\, z_2 \,=\, 0\}$. If $v_3 = 0$, then the equation $z_3z_1 + \alpha z_2 \,=\, 0$ defines (a Zariski open subset of) the Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$. If $z_3 = 0$, then $z_2 = 0$ gives us (a Zariski open subset of) $X_{02} \simeq X'_{02} = \{x_2 = 0\} \subset \mathbb{P}^2$. These two surfaces intersect in (a Zariski open subset of) $C_{2s} = \mathbb{P}^1 \subset \mathbb{P}^3$.

- $(v_2, z_1, z_2, z_3) \;\mapsto\; ((v_2z_1, v_2, v_2z_2, v_2z_3)\,;\,(z_1 : 1 : z_2 : z_3)) \;\in\; \pi^{-1}(U)$. The exceptional divisor is cut out by $v_2 = 0$, and $\mathcal{X}$ is cut out by $z_3\, z_1 + \xi(v_2z_1, v_2, v_2z_2)\, z_2 = 0$ (which is smooth since ξ is nowhere zero).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = v_2z_3$, so that X_0 is locally cut out by $\{v_2z_3,\, z_3\, z_1 + \xi(v_2z_1, v_2, v_2z_2)\, z_2 \,=\, 0\}$. If $v_2 = 0$, then the equation $z_3z_1 + \alpha z_2 \,=\, 0$ defines (a Zariski open subset of) the Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$. If $z_3 = 0$, then $z_2 = 0$ gives us (a Zariski open subset of) $X_{03} \simeq X'_{03} = \{x_3 = 0\}$. These two surfaces intersect in (a Zariski open subset of) $C_{3s} = \mathbb{P}^1 \subset \mathbb{P}^3$

- $(v_1, z_1, z_2, z_3) \;\mapsto\; ((v_1, v_1z_1, v_1z_2, v_1z_3)\,;\,(1 : z_1 : z_2 : z_3)) \;\in\; \pi^{-1}(U)$. The exceptional divisor is cut out by $v_1 = 0$, and $\mathcal{X}$ is cut out by $z_3 + \xi(v_1, v_1z_1, v_1z_2)\, z_1\, z_2 = 0$ (which is clearly smooth).

  The map $f : \mathcal{X} \to \Delta$ is locally given by $f = v_1z_3$, so that X_0 is locally cut out by $\{v_1z_3,\, z_3 + \xi(v_1, v_1z_1, v_1z_2)\, z_1\, z_2 \,=\, 0\}$. If $v_1 = 0$, then the equation $z_3 + \alpha\, z_1\, z_2 \,=\, 0$ cuts out (a Zariski open subset of) the Segre embedding $X_s = \{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\}$. If $z_3 = 0$, then $z_1z_2 = 0$ gives us (a Zariski open subset of)

$X_{02} \cup X_{03} \simeq X'_{02} \cup X'_{03} = \{x_2 x_3 = 0\}$. These two surfaces intersect in (a Zariski open subset of) $C_{2s} \cup C_{3s} = \mathbb{P}^1 \cup \mathbb{P}^1$.

A similar analysis applies to the other singular points. □

6.3.4 Kulikov degenerations

Each of the three types of Schmid MHS may be geometrically realized by a semistable degeneration of a particularly nice form:

Theorem 6.3.4 (Kulikov [Kul77a], Persson–Pinkham [PP81])**.** *A semistable degeneration of K3 surfaces is birational to one for which the central fibre X_0 is one of the three types.*

Type I: X_0 is a smooth K3 surface.

Type II: $X_0 = X_0^0 \cup X_0^1 \cup \cdots \cup X_0^{k+1}$. Each X_0^a meets only $X_0^{a\pm 1}$. Each double curve $C_a = X_0^a \cap X_0^{a+1}$ is an elliptic curve. The "tails" X_0^0 and X_0^{k+1} are rational surfaces. The X_0^a, with $1 \leq a \leq k$, are ruled; and both $X_0^{a\pm 1}$ are sections of the ruling.

Type III: all components X_0^j of X_0 are rational surfaces; $X_0^i \cap (\cup_{j\neq i} X_0^j)$ is a cycle of rational curves, and $|\Gamma(X_0)| = S^2$.

Remark 6.3.5. The Type II example in §6.3.2 is not in Kulikov form: the central fibre X_0 does not have the property that X_0^a meets only $X_0^{a\pm 1}$. And some of the double curves are rational (not elliptic).

Similarly the Type III example in §6.3.3 is not in Kulikov form: the central fibre does not have the property that $X_0^i \cap (\cup_{j\neq i} X_0^j)$ is a cycle; nor is $|\Gamma(X_0)| = S^2$.

Chapter 7

Horikawa surfaces of general type

Recommended references: [BHPVdV04, Hor78, Hor79]. This chapter is adapted from the notes [GGLR17].

7.1 I-surfaces

Definition 7.1.1. An *I-surface* is a smooth, regular minimal surface X of general type such that $K_X^2 = 1$ and $p_g(X) = 2$. Cf. [BHPVdV04, Chapter VII].

Exercise 7.1.2. (a) Use Noether's formula (§A.2.1) to show that $h^{1,1}(X) = 29$.

(b) Since $2 = p_g(X) = h^{2,0}(X) = \dim H^0(X, K_X)$, we see that $|K_X| = \mathbb{P}\, H^0(X, K_X) \simeq \mathbb{P}^1$ is a pencil. Use the genus formula (§A.2.5) to show that $p_a(X) = 2$ for every $C \in |K_X|$.

7.1.1 Projective realization

Fix projective coordinates $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3$. Let

$$Q_0 \;=\; \{x_0 x_2 = x_1^2\} \;\subset\; \mathbb{P}^3$$

be quadric with singular point

$$p \;=\; (0:0:0:1) \;\in\; Q_0\,.$$

Proposition 7.1.3 ([GGLR17]). *A general I-surface is realized via the bi-canonical map* $\varphi_{2K_X} : X \to \mathbb{P}\,H^0(X, 2K_X)^\vee = \mathbb{P}^3$ *as a 2:1 covering of* Q_0 *branched over* p *and* $V \cap Q_0$ *where* $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$ *is a general quintic surface not passing through* p.

Proof. Given $C \neq C' \in |K_X|$ we have $C \cdot C' = K_X^2 = 1$. So any two distinct canonical divisors intersect at a unique point with multiplicity one. This point is the base locus of the linear system. Bertini's theorem (§A.3.4) asserts that a general $C \in |K_X|$ is smooth away from the base locus. Since X is general, we may assume that C is smooth. The genus formula (§A.2.5) implies $g(C) = 2$.

Fix a basis $\{t_0, t_1\} \in H^0(X, K_X)$ with $C = \{t_0 = 0\}$. Since $P_2 = 4$ (Exercise A.2.5), we see that there exists $u \in H^0(X, 2K_X)$ that completes $\{t_0^2, t_0t_1, t_1^2\}$ to a basis. The adjunction formula (§A.3.6) implies $K_C = 2K_X|_C$. The SES

$$0 \;\to\; K_X \;\xrightarrow{t_0}\; 2K_X \;\to\; K_C \;\to\; 0$$

induces a LES

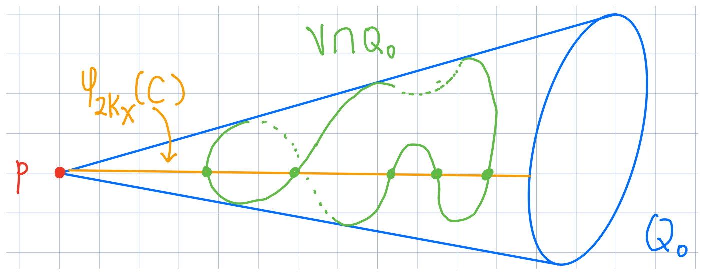
$$0 \;\to\; H^0(X, K_X) \;\to\; H^0(X, 2K_X) \;\to\; H^0(C, K_C) \;\to\; H^1(X, K_X) \;=\; H^{2,1}(X) \;=\; 0\,,$$

where the vanishing of the last term is due to $h^{2,1}(X) = q(X) = 0$. It follows that the restrictions of t_1^2, u to C give a basis of $H^0(C, K_C)$. This in turn implies that $|2K_X|$ is base-point free. Using the basis $\{t_0^2, t_0t_1, t_1^2, u\} \subset H^0(X, 2K_X)$ as homogeneous coordinates on $\mathbb{P}\,H^0(X, 2K_X)^\vee$, the bi-canonical map is

$$\varphi_{2K_X} : X \;\to\; Q_0 \;\subset \mathbb{P}^3\,.$$

Since $t_0(p) = t_1(p) = 0$, it follows that $u(p) \neq 0$. So $\varphi_{K_C} : C \to \mathbb{P}^1$ is given by t_1^2/u near p. It follows that $\varphi_{K_C} = \varphi_{2K_X}|_C$ is a 2:1 covering of one of the rulings $\mathbb{P}^1 \subset Q_0$ that is branched at p. The Riemann–Hurwitz formula (§A.1.4) implies that φ_{K_C} is branched over an additional 5 points in the ruling $\mathbb{P}^1$. It follows that φ_{2K_X} is branched over $p + V$, where $V \in |Q_0(5)|$ does not pass through p (Figure 7.1). $\square$

Figure 7.1: Bi-canonical realization of general I-surface



Proposition 7.1.4 ([GGLR17])**.** *A general I-surface is realized via the 5-canonical map* $\varphi_{5K_X} : X \to \mathbb{P}^N$ *as a hypersurface*

$$z^2 \;=\; F_5(t_0, t_1, u)\, v \;+\; F_{10}(t_0, t_1, u)$$

in $\mathbb{P}(1, 1, 2, 5)$ *with homogeneous coordinates* $(t_0 : t_1 : u : v)$ *and* F_k *a weighted homogeneous polynomial of degree* k*.*

Proof. We have seen that $H^0(X, 2K_X)$ has dimension 4, and that a basis is given by the weighted degree 2 monomials in t_0, t_1, u, where t_0, t_1 have weight 1 and u has weight 2.

By Exercise A.2.5, $P_3 = \dim H^0(X, 3K_X) = 6$. We see that a basis is given by the weighted degree 3 monomials in t_0, t_1, u.

Likewise $P_4 = \dim H^0(X, 4K_X) = 9$. A basis is given by the weighted degree 4 monomials in t_0, t_1, u.

Next $P_5 = \dim H^0(X, 5K_X) = 13$. The weighted degree 5 monomials in t_0, t_1, u span a codimension 1 subspace. So there exists a weighted degree five $v \in H^0(X, 5K_X)$ completing the monomials to a basis.

Exercise 7.1.5. Let $R_X = \oplus_{m \geq 0} H^0(X, mK_X)$ denote the pluri-canonical ring. We have $\mathbb{C}[t_0, t_1, u] \oplus v\, \mathbb{C}[t_0, t_1, u] \subset R_X$. Show that equality holds. [*Hint.* Both rings are graded. Show that the dimension agree (and are finite) in each graded degree.]

It follows that for R_X there is a generating relation $v^2 = F_5(t_0,t_1,u)\,v + F_{10}(t_0,t_1,u)$.

Exercise 7.1.6. Show that φ_{5K_X} contracts an irreducible curve $E \subset X$ to a point if and only if E is a (-2)-curve (§A.2.6). [*Hint.* §A.2.5.]

A priori the five-canonical map φ_{5K_X} could contract some (-2) curves. Nonetheless, $Y = \varphi_{5K_X}(X) \subset \mathbb{P}(1,1,2,5)$ is an I-surface: Remark 5.4.49 yields Hodge numbers $\mathbf{h}^2_{\mathrm{prim}}(Y) = (2,28,2)$. And the Lefschetz hyperplane theorem implies that Y is regular. □

Corollary 7.1.7. *Since $X = \operatorname{Proj} R_X$ it follows that $\varphi_{5K_X}(X) \subset \mathbb{P}(1,1,2,3)$ is a smooth surface isomorphic to X.*

Remark 7.1.8. The two realizations of X given in Propositions 7.1.3 and 7.1.4 are related as follows: Begin with the second

$$X \;\simeq\; \{v^2 = vF_5(t_0,t_1,u) + F_{10}(t_0,t_1,u)\} \;\subset\; \mathbb{P}(1,1,2,5)\,.$$

Define a rational map $\mathbb{P}(1,1,2,5) \dashrightarrow \mathbb{P}(1,1,2)$ by $(t_0 : t_1 : u : v) \mapsto (t_0 : t_1 : u)$. Let $\rho : X \to \mathbb{P}(1,1,2)$ denote the restriction of this map to X. The quadratic formula implies that ρ is branched over $\{F_5(t_0,t_1,u)^2 - 4F_{10}(t_0,t_1,u) = 0\} \subset \mathbb{P}(1,1,2)$, a curve of genus 16 (Remark 5.4.49). The second veronese map $v_2 : \mathbb{P}(1,1,2) \to \mathbb{P}^3$, sending $(t_0 : t_1 : u) \mapsto (t_0^2 : t_0t_1 : t_1^2 : u)$, identifies $\mathbb{P}(1,1,2)$ with the singular quadric $Q_0 = \{x_0x_2 = x_1^2\} \subset \mathbb{P}^3$. And there exists a unique homogeneous polynomial $G \in \mathbb{C}[x_0,x_1,x_2,x_3]$ of degree five so that $F_5^2 - 4F_{10} = \rho^*(G)$; that is, $F_5(t_0,t_1,u)^2 - 4F_{10}(t_0,t_1,u) = G(t_0^2, t_0t_1, t_1^2, u)$.

7.1.2 Moduli

The automorphism group $G = \operatorname{Aut}\mathbb{P}(1,1,2,5)$ naturally acts on the locus $\mathcal{U} \subset \mathbb{P}\,\mathbb{C}[t_0,t_1,u,v]_{10}$ of quasi-smooth hypersurfaces of weighted degree 10. The stabilizer of $X \in \mathcal{U}$ in G is finite:

Theorem 7.1.9 ([Bun21]). *The automorphism group of weight projective space* $\mathbb{P}(a_0, \ldots, a_{n+1})$ *acts on a quasi-smooth hypersurface of weighted degree* $\geq \max\{a_0, \ldots, a_{n+1}\} + 2$ *with finite stabilizer.*

It follows from [KM97] that the algebraic stack $[\mathcal{U}/G]$ admits a coarse moduli space $\mathcal{M}$ as an algebraic space.

Proposition 7.1.10 ([GGLR17]). *The coarse moduli space* $\mathcal{M}$ *of I-surfaces is of dimension* 28*.*

Proof. The space of weighted degree 10 polynomials in $\mathbb{C}[t_0, t_1, u, v]$ is spanned by $\{v^2,\ vu^2P_1(t_0, t_1),\ vuP_3(t_0, t_1),\ vP_5(t_0, t_1),\ u^5,\ u^4P_2(t_0, t_1),\ u^3P_4(t_0, t_1),\ u^2P_6(t_0, t_1),\ uP_8(t_0, t_1),\ P_{10}(t_0, t_1)\}$, where the $P_k \in \mathbb{C}[t_0, t_1]$ are homogeneous polynomials of degree k. In particular, $\dim \mathbb{C}[t_0, t_1, u, v]_{10} = 49$.

Any automorphism $\phi \in G$ of $\mathbb{P}(1, 1, 2, 5) = \operatorname{Proj} \mathbb{C}[t_0, t_1, u, v]$ is determined by the induced $\phi^* : \mathbb{C}[t_0, t_1, u, v] \to \mathbb{C}[t_0, t_1, u, v]$. The automorphisms of $\mathbb{P}(1, 1, 2, 5)$ are given by

$$\begin{aligned} t_0 &\mapsto P_1(t_0, t_1) \\ t_1 &\mapsto Q_1(t_0, t_1) \\ u &\mapsto au + U_2(t_0, t_1) \\ v &\mapsto bv + u^2V_1(t_0, t_1) + uV_3(t_0, t_1) + V_5(t_0, t_1) \end{aligned}$$

with $P_d, Q_d, U_d, V_d \in \mathbb{C}[t_0, t_1]_d$, $\mathrm{d}P_1 \wedge \mathrm{d}Q_1 \neq 0$ and $0 \neq a, b \in \mathbb{C}$. In particular, $\dim \operatorname{Aut} \mathbb{P}(1, 1, 2, 5) = 21$. □

Remark 7.1.11. According to [Bun21], $\mathcal{M}$ will be quasi-projective if $\mathbb{P}(1, 1, 2, 5)$ satisfies "condition ($\mathfrak{C}$)". Unfortunately, condition ($\mathfrak{C}$) appears to fail for $\mathbb{P}(1, 1, 2, 5)$. In the notation of that paper, $Z_{\min} = Z_{\min}^{\mathrm{ss}, L_G}$ is point $(0 : \cdots : 0 : 1)$ corresponding to the hypersurface $z^2 = 0$ (Lemma 5.5 loc.cit). The stabilizer of $Z_{\min}$ in the unipotent radical U_G of G is not trivial: We have $G = L_G \ltimes U_G$, with $U_G = \{P_1 = t_0, Q_1 = t_1, a, b = 1\}$ the unipotent radical of G, and $L_G = \{U_d, V_d = 0\}$ a reductive Levi factor. The stabilizer of $Z_{\min}$ in U_G is the nontrivial $\{V_d = 0\} \subset U_G$. (See Remark 5.19 for discussion of possible approach when condition ($\mathfrak{C}$) fails.)

7.1.3 Local Torelli

The primitive Hodge numbers are $\mathbf{h}^2_{\mathrm{prim}}(X) = (2, 28, 2)$. The associated period domain D has dimension 57, and the infinitesimal period relation is a contact system on D.

Proposition 7.1.12 ([GGLR17]). *The period map $\Phi : \mathcal{M} \to \Gamma\backslash D$ satisfies the local Torelli property. In particular, the period map is a maximal integral manifold of the IPR.*

Proof. Very similar to the arguments of §4.4.

Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_{n+1})$ be a weighted projective space as in Remark 5.4.49. One must show that $R^d_f \times R^{w(2)}_f \to R^{d+w(2)}_f$ is non-degenerate. For a quasi-smooth hypersurface $\{f = 0\}$ of weighted degree $d = 10$, we have $w(2) = 1$ and it is a calculation to verify that $R^{10}_f \times R^1_f \to R^{11}_f$ is nondegenerate. □

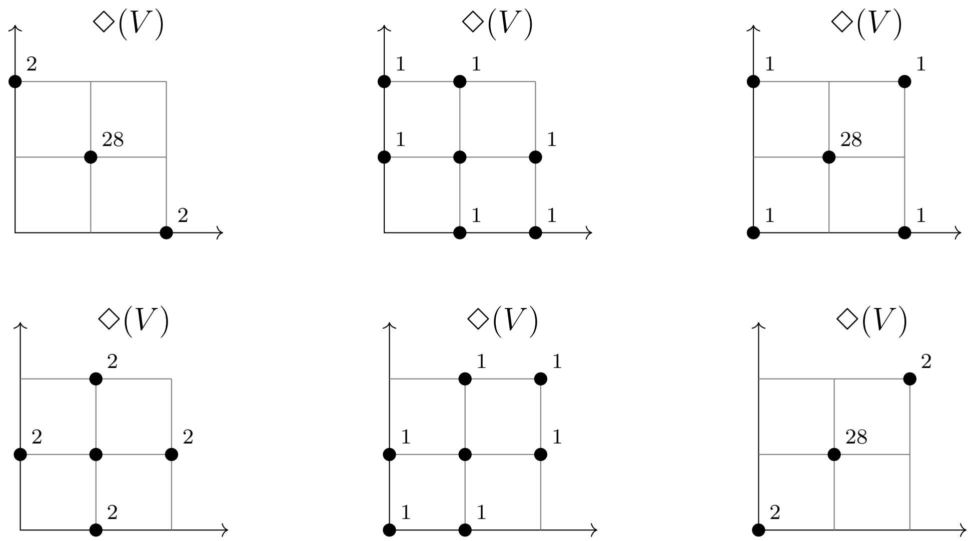
Remark 7.1.13. There is also a generic global Torelli result for these surfaces [PZ19].

7.1.4 Degenerations of I-surfaces

There are six types of Schmid PMHS associated to the period domain (Figure 7.2. Each of those types may be realized (via the Clemens–Schmid exact sequence) by a degeneration of I-surfaces [CFPR22]. Moreover, the various polarized relations [KPR19] between the PMHS may also be realized geometrically (loc. cit.). These geometric realizations may all be given by double covers of Q_0 branched over $Q \cap V$, where $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$ is a union of hyperplanes [CFPR22].

Here we will consider degenerations not coming from hyperplane arrangements.

Figure 7.2: Hodge diamonds of Schmid's PMHS



Example

Given $s \in \Delta$, let

$$F_s(t_0, t_1, u, v) \;=\; v^2 + s(u^5 + t_1^{10} + t_0^{10}) + (u + t_1^2 + t_0^2)(u^4 + t_1^8 + t_0^8) \;\in\; \mathbb{C}[t_0, t_1, u, v]_{10}\,,$$

and consider the family

$$\mathcal{X}' \;=\; \{F_s = 0\} \;\subset\; \mathbb{P}(1,1,2,5) \times \Delta\,.$$

Notice that the central fibre

$$X_0' \;=\; \{F_0 = 0\}$$

does not pass through either of the singular points of $\mathbb{P}(1,1,2,5)$. Shrining Δ if necessary, we may assume that none of the fibres X_s' pass through the two singular points of $\mathbb{P}(1,1,2,5)$.

[*To be continued…*]

7.2 H-surfaces

Appendix A

Some results from complex algebraic geometry

A.1 Curves

There is an overwhelming volume of literature on algebraic curves. Expository accounts, from a variety of perspectives, include: [Cle03, Gri89, Mir95].

Terminology

All "curves" are *algebraic curves* over $\mathbb{C}$. In particular, they have $\dim_{\mathbb{C}} = 1$ and $\dim_{\mathbb{R}} = 2$. A smooth curve is a Riemann surface; and a morphism of smooth curves is a holomorphic map of Riemann surfaces.

A.1.1 Genus

The *geometric genus* $p_g(C)$ of a curve C is the *topological genus* of C viewed as a surface of of real dimension two: the number of handles or donut holes. The *arithmetic genus* is $p_a(C) = 1 - \chi(\mathcal{O}_C)$. When the curve is smooth, the geometric and arithmetic genus agree and are denoted $g(C)$. In this case

$$g(C) \;=\; h^{1,0}(C) \;=\; \dim H^0(C, K_C)\,.$$

Here $K_C = T_C^\vee$ is the *canonical line bundle*.

If C is not smooth, the geometric genus $p_g(C)$ is the genus $g(C')$ of the normalization $C' \to C$. If the curve is singular, with only ordinary singularities, then $p_g(C) < p_a(C)$. More precisely, an ordinary singularity of multiplicity r decreases the genus by $\frac{1}{2}r(r-1)$.

A.1.2 Degree–genus formula

If $C \subset \mathbb{P}^2$ is a curve of degree d, then the arithmetic genus of C is $p_a = \frac{1}{2}(d-1)(d-2)$. See Example 2.2.18.

A.1.3 Bezout's Theorem

Suppose that $C_1, C_2 \subset \mathbb{P}^2$ are curves with no common component. Then $C_1 \cdot C_2 = (\deg C_1) \cdot (\deg C_2)$.

Remark A.1.1. Bezout's theorem holds for curves defined over any algebraically closed field, and generalizes to hypersurfaces $X_1, \ldots, X_n \subset \mathbb{P}^n$.

Exercise A.1.2. Let $f : C \to C'$ be a morphism of smooth curves. Fix $p \in C$, and show that there exist local coordinates on C and C', centered at p and $f(p)$, respectively, so that $f(z) = z^n$.

Definition A.1.3. The integer n in HW A.1.2 is the *ramification index* r_p of f at p. The map f is *ramified* at p if $r_p \geq 2$.

A.1.4 Riemann–Hurwitz formula

Let $f : C \to C'$ be a morphism of smooth curves. Then

$$\chi(C) \;=\; (\deg f)\,\chi(C') \;-\; \sum_{p\in C}(r_p - 1)\,.$$

A.1.5 Hyperelliptic curves

Hyperelliptic curves are characterized by the existence of a degree two morphism $C \to \mathbb{P}^1$. (The field of functions is a quadratic extension of the field of rational functions.) They may be realized as hypersurfaces $C = \{z = f(x,y)\} \subset \mathbb{P}(1,1,d)$ in weighted projective space. Here $f(x,y)$ is a homogenous polynomial of degree d with the property that $\{f(x,y) = 0\} \subset \mathbb{P}^2$ consists of d distinct solutions.

Exercise A.1.4. Show that $d = 2g(C) + 2$. [*Hint.* Riemann–Hurwitz formula (§A.1.4.)

Exercise A.1.5. Show that a curve C of genus $g \geq 1$ is hyperelliptic if and only if C admits a base point free (§A.3.4) line bundle L with $h^0(C,L) = 2$.

Exercise A.1.6. Let L be the line bundle of HW A.1.5, and recall the notations of §A.3.4. Note that $\phi_{L^{\otimes(g-1)}} = \phi_{\mathcal{O}_{\mathbb{P}^1}(g-1)} \circ \phi_L$

$$C \xrightarrow{\;\phi_L\;} \mathbb{P}^1 \xrightarrow{\;\phi_{\mathcal{O}_{\mathbb{P}^1}(g-1)}\;} \mathbb{P}^{g-1}\,.$$

(a) Show that $L^{\otimes(g-1)}$ has degree $2g-2$.

(b) Show that the pullback $H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) \to H^0(C, L^{\otimes(g-1)})$ is injective. Conclude that $h^0(C, L^{\otimes(g-1)}) \geq g$.

(c) Show that $h^0(C, L^{\otimes(g-1)}) = g$. Deduce that $L^{\otimes(g-1)} = K_C$. [*Hint.* Riemann–Roch (§A.1.6).]

(d) Conclude that $\kappa : \phi_{K_C} : C \to \mathbb{P}^{g-1}$ is 2:1 onto the image $\kappa(C) \simeq \mathbb{P}^1$.

A.1.6 Riemann–Roch for line bundles

Let C be a smooth curve equipped with a line bundle L.

Exercise A.1.7. Suppose that $C \neq \mathbb{P}^1$. Suppose that $\deg L = 1$. Show that $h^0(C, L) < 2$. [*Hint.* If $h^0(C, L) \geq 2$, then C admits a degree one meromorphic function.]

The *Riemann–Roch formula* is

$$h^0(C, L) \; - \; h^0(C, L^{-1} \otimes K_C) \; = \; \deg L \; + \; 1 \; - \; g(C)\,.$$

The *Riemann–Roch inequality* is $h^0(C, L) \geq \deg L + 1 - g$.

Example A.1.8. (a) Taking $L = \mathcal{O}$ yields $h^0(C, K_C) = g(C)$.

(b) Taking $L = K_C$ yields $\deg K_C = 2g(C) - 2$.

Exercise A.1.9. Show that any line bundle of degree zero is trivial.

Exercise A.1.10. Compute the *plurigenera* $P_n = \dim_{\mathbb{C}} H^0(C, K_C^{\otimes n})$.

Exercise A.1.11. Suppose that $\deg L > 2g(C) - 2$. Prove that $h^0(C, L) = \deg L + 1 - g(C)$.

A.1.7 Riemann–Roch for divisors

Given a divisor D on a smooth curves C, let $\ell(D) = \dim H^0(C, [D])$ be the dimension of the vector space of meromorphic functions f on C so that $(f) + D \geq 0$ (cf. HW 1.3.16(c)). Then

$$\ell(D) \; - \; \ell(K_C - D) \; = \; \deg D \; + \; 1 \; - \; g\,.$$

The *Riemann–Roch inequality* is $\ell(D) \geq \deg D + 1 - g$.

Exercise A.1.12 (Kodaira embedding for curves)**.** Let D be an effective divisor (§1.3) on C. With the terminology and notations of §A.3.4:

(a) Show that $\ell(D) - \ell(D-p) \in \{0,1\}$. [*Hint.* Let $L = [D]$ and consider the SES $0 \to L(-p) \to L \to L_p \to 0$.]

(b) Show that the complete linear system $|D|$ is base point free if $\ell(D-p) = \ell(D)-1$ for all $p \in C$.

(c) Show that the map $\phi_{|D|} : C \to \mathbb{P}\, H^0(C,[D])^\vee$ is a closed embedding if $\ell(D-p-q) = \ell(D) - 2$ for all $p, q \in C$ (including $p = q$). Equivalently, the line bundle $[D]$ is very ample.

A.2 Surfaces

Recommended reference: [BHPVdV04, Bea96].

Terminology

All "surfaces" are *algebraic surfaces* over $\mathbb{C}$. In particular, they have $\dim_{\mathbb{C}} = 2$ and $\dim_{\mathbb{R}} = 4$.

A.2.1 Noether's formula

The *holomorphic Euler characteristic* of a smooth projective surface S is

$$\chi_S(\mathcal{O}_S) \;=\; 1 \,-\, h^{0,1}(S) \,+\, h^{0,2}(S)\,.$$

The *topological Euler characteristic* is

$$\begin{aligned} e(S) \;&=\; c_2(S) \;=\; 2 \,-\, 2b_1(S) \,+\, b_2(S) \\ &=\; 2 \,-\, 4\,h^{1,0}(S) \,+\, 2\,h^{2,0}(S) \,+\, h^{1,1}(S)\,. \end{aligned}$$

The *canonical bundle* of S is $K_S = \bigwedge^2 T_S^\vee$. The first Chern class satisfies $c_1(S)^2 = K_S \cdot K_S$. These quantities are related by *Noether's formula*

$$\chi_S(\mathcal{O}_S) \;=\; \frac{c_1(S)^2 + c_2(S)}{12} \;=\; \frac{K_S \cdot K_S + e(S)}{12}\,.$$

A.2.2 Irregularity

The *irregularity* $q(S) = h^{1,0}(S)$ of S is the dimension of the Albanese variety (§2.4.1). The surface is *regular* if $q(S) = 0$.

A.2.3 Genus

The *geometric genus* is $p_g(S) = h^{2,0}$. The *arithmetic genus* is $p_a(S) = p_g(S) - q(S) = h^{2,0} - h^{1,0}$.

A.2.4 Riemann–Roch for surfaces

The *holomorphic Euler characteristic* of a divisor D on S is

$$\begin{aligned} \chi_S(D) &= \dim H^0(S, \mathcal{O}(D)) \; - \; \dim H^1(S, \mathcal{O}(D)) \; + \; \dim H^2(S, \mathcal{O}(D)) \\ &= h^{0,0}(S, D) \; - \; h^{0,1}(S, D) \; + \; h^{0,2}(S, D) \,. \end{aligned}$$

If D is a divisor on a smooth projective surface S, then

$$\chi_S(D) \; = \; \chi_S(\mathcal{O}_S) \; + \; \tfrac{1}{2} D \cdot (D - K_S) \,.$$

A.2.5 Genus formula

Assume S is smooth, and $C \subset S$ is reduced and irreducible. Then $2\, p_a(C) - 2 = C \cdot C + K_S \cdot C$. In the smooth case, this can be deduced from the adjunction formula (§A.3.6) and the Riemann–Roch formula (§A.1.6). Alternatively, see §2.2.2.

A.2.6 Exceptional curves

Given a nonsingular surface X, we say a compact, reduced, connected curve $C \subset X$ is *exceptional* if there is a birational map $\pi : X \to Y$ that contracts C to a points $y \in Y$ and there exist neighborhoods $C \subset U \subset X$ and $y \subset V \subset Y$ so that π restricts to an isomorphism $U \backslash C \to V \backslash \{y\}$.

Grauert's criterion

A reduced, compact connected curve C with irreducible components C_i on a smooth surface is exceptional if and only if the intersection matrix $(C_i \cdot C_j)$ is negative definite.

Exercise A.2.1. A (-1)-*curve* is a nonsingular rational curve with self-intersection -1. Show that an irreducible curve $C \subset X$ is a (-1) curve if and only if $C^2 < 0$ and $K_X \cdot C < 0$.

A (-2)-*curve* is a nonsingular rational curve with self-intersection $C^2 = -2$.

A.2.7 Minimal models and Castelnuovo's Theorem

Every irreducible projective curve C is birational to a unique smooth projective curve C', the minimal model. In this sense, the theory of minimal models is trivial for curves.

A smooth surface S is *minimal* if every birational morphism $S \to S'$ of smooth surfaces is necessarily an isomorphism. For example, a blowup $\mathrm{Bl}_p S$ is never minimal.

Castelnuovo's theorem describes the process of constructing a minimal model of S. A *(-1)-curve* on S is a smooth rational curve C with $C \cdot C = -1$. (Eg. the exceptional curve of a blowup $S = \mathrm{Bl}_p(S')$.) *Castelnuovo's theorem* asserts that every nontrivial birational morphism $S \to S'$ must contract a (-1)-curve; and conversely every such curve can be smooth contracted: there exists a smooth surface S' and a birational morphism $S \to S'$ that contracts C to a point and is an isomorphism away from C.

A.2.8 Enriques classification

The *plurigenera* $P_n = \dim H^0(S, K_S^n)$ are birational invariants. The *Kodaira dimension* $\kappa(S)$ is $-\infty$ if $P_n = 0$ for all n; otherwise, $\kappa(S)$ is the smallest number so that

P_n/n^κ is bounded for all n. Enriques showed that

$$\begin{aligned}
\kappa = -\infty &\iff P_{12} = 0\,,\\
\kappa = 0 &\iff P_{12} = 1\,,\\
\kappa = 1 &\iff P_{12} > 1 \quad \text{and} \quad K\cdot K = 0\,,\\
\kappa = 2 &\iff P_{12} > 1 \quad \text{and} \quad K\cdot K > 0\,.
\end{aligned}$$

Up to birational equivalence, every smooth algebraic surface over a field of characteristic zero is of one of the following types: ruled surface (which includes rational surfaces), abelian variety, K3 surface, elliptic surface, surface of general type.

Rational surfaces

A *rational surface* S is any surface birationally equivalent to $\mathbb{P}^2$. Examples include $\mathbb{P}^1 \times \mathbb{P}^1$ and Hirzebruch surfaces $\Sigma_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-r))$.

Exercise A.2.2. (a) Show that $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(b) Show that Σ_1 is isomorphic to the blow-up of $\mathbb{P}^2$ at point.

The plurigenera $P_n(S)$ all vanish, and the fundamental group is trivial. Castelu-ovo's Rationality Criterion classifies rational surfaces as those with second plurigenera $P_2(S) = 0$ and irregularity $q(S) = 0$.

Every smooth rational surface may be realized by successively blowing-up a minimal rational surface. The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces Σ_r with $r = 0, 2, 3, 4, \ldots$. The nonzero Hodge numbers of a smooth rational surface are $h^{0,0}(S) = h^{2,2} = 1$, and $h^{1,1}(S) = 1 + m$. We have $h^{1,1}(\mathbb{P}^2) = 1$, $h^{1,1}(\Sigma_r) = 2$, and $h^{1,1}(S) > 2$ for all other smooth rational surfaces. The Picard group is the odd unimodular lattice $\mathrm{I}_{1,n}$; expect in the case of the Hirzebruch surfaces Σ_{2m}, where it is the even unimodular lattice $\mathrm{II}_{1,1}$.

Ruled surface

A *ruled surface* of genus g is any smooth surface that is birationally equivalent to $\mathbb{P}^1 \times C$, with C a smooth curve of genus $g \geq 0$. A *geometrically ruled surface* of genus g is morphism $\pi : S \to C$ with fibres $\pi^{-1}(x)$. Every geometrically ruled surface is of the form $S = \mathbb{P}_C(E)$ with $E \to C$ a rank two vector bundle. (The geometrically ruled surfaces of genus $g = 0$ are the Hirzebruch surfaces.) Moreover, $S \simeq S'$ if and only if $E' = E \otimes L$ for some line bundle L. Every geometrically ruled surface admits a section (Noether–Enriques Theorem). These surfaces have $\mathrm{Pic}(S) = \pi^*(C) \oplus \mathbb{Z}^\sigma$, where σ is the class of some section; irregularity $q(S) = g(C)$; Hodge numbers $h^{2,0} = 0$ and $h^{1,1} = 2$; and plurigenera $P_{12} = 0$.

Any smooth minimal ruled surface is geometrically ruled. If S' is a minimal ruled surface, then:

(i) There exists a curve E such that $E \cdot K_{S'} < 0$.

(ii) For any divisor D on S', there exists n_o so that the linear system $|D + nK_{S'}| = \emptyset$ for all $n \geq n_o$.

Abelian variety

The surface S may be realized as a complex torus $\mathbb{C}^2/\Lambda$. These surfaces are characterized $p_{12} = 1$, $p_g = 1$ and $p_a = -1$.

K3 surface

These surfaces are characterized by $K_S = \mathcal{O}_S$ and $q(S) = 0$.

Exercise A.2.3. (a) Use Riemann–Roch (§A.2.4) to show that $p_g(S) = 1$ and $\chi(\mathcal{O}_S) = 2$.

(b) Use Noether's formula (§A.2.1) to show that the second Betti number is $b_2(S) = 22$, and Euler characteristic $e(S) = 24$.

(c) Use the genus formula (§A.2.5) to show that the arithmetic genus of an irreducible curve $C \subset S$ is $p_a(C) = 1 + \frac{1}{2}C^2$.

Elliptic surface

The surface admits a morphism $\pi : S \to C$, onto a smooth curve C, with the property that the generic fibre of π is a smooth elliptic curve. The surfaces are characterized by $K_S^2 = 0$ and $p_{12} \geq 2$; or $p_{12} = 1$ and $p_g(S) = 0$.

The Euler characteristic satisfies $e(S) = \sum_{x \in C} e(\pi^{-1}(x))$. Elliptic surfaces admit unique minimal models, characterized by the property that the fibre of π does not contain any exceptional curve of arithmetic genus 1.

Surface of general type

These surfaces are characterized by $K_S^2 > 0$ and $p_{12} \geq 2$.

Theorem A.2.4 ([BHPVdV04]). *Fix $0, 1 \neq m \in \mathbb{Z}$. A surface of general type is minimal if and only if $H^1(S, mK_S) = 0$.*

Exercise A.2.5. Let S be a surface of general type.

(a) Use Kodaira–Serre duality (§A.3.12) to show that $H^2(S, mK_X) = 0$ for all $m \geq 2$.

(b) Assume S is a minimal surface of general type. Use Riemann–Roch (§A.2.4) and Theorem A.2.4 to show that the plurigenera are $P_m = \chi(S) + \frac{1}{2}m(m-1)K_S^2$.

Suppose S is a minimal surface of general type. We have $q(S) \leq p_g(S)$. Moreover, $p_g(S) = 2 + \frac{1}{2}K_S^2$ if K_S^2 is even, and $p_g(S) = \frac{1}{2}(K_S^2 + 3)$ if K_S^2 is odd. The *Bombieri–Kodaira theorem* asserts that the pluricanonical map $\varphi_{mK_S} : S \to \mathbb{P}^{P_m - 1}$ is a birational morphism onto its image for all $m \geq 5$. We have $K_S \cdot D \geq 0$ for every effective divisor D. If $C \subset S$ is an irreducible curve, then $K_S \cdot C = 0$ if and only if C is a (-2) curve. The number of (-2) curves on S is bounded above by $\rho(S) - 1$, where $\rho(S) = \dim H^{1,1}(S) \cap H^2(S, \mathbb{Z})$ is the Picard number. The intersection form is negative definite on the subspace of $H^2(S, \mathbb{Z})$ spanned by the (-2) curves. The pluricanonical map φ_{mK_S} is injective and of maximal rank away from the (-2) curves as long as $m \geq 5$. It follows from [Gra62] that the image is a normal variety.

A.3 Complex geometry

Recommended reference: [Ara12, GH94, Huy05, Voi07].

The purpose of this section is to collect some of the standard results in complex geometry that we will utilize. In general, M will denote a complex manifold, and X an algebraic variety. We will write $X \subset \mathbb{P}$ to indicate that X is projective algebraic.

A.3.1 Commutative algebra

Recommended reference: [GH94].

Unique factorization domain

An integral domain R is a *unique factorization domain* if every may be expressed as a product of a unit with a finite number of irreducible elements.

Gauss's Lemma

If R is a UFD, then so is $R[z]$.

Example A.3.1. The polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ is a UFD.

Euclidean algorithm

If R is a unique factorization domain and $u, v \in R[t]$ are relatively prime, then there exist relatively prime $\alpha, \beta \in R[t]$ and $\gamma \neq 0 \in R$ so that $\alpha u + \beta v = \gamma$.

Hilbert basis theorem

If R is a commutative noetherian ring, then so is $R[z]$.

A commutative ring R is *noetherian* if every sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ stabilizes: there exists $m \geq 1$ so that $I_m = I_n$ for all $n \geq m$.

Example A.3.2. Every field is noetherian ring. As a corollary of the Hilbert basis theorem, we see that $\mathbb{C}[z_1, \ldots, z_n]$ is noetherian.

Hilbert's Nullstellensatz

If $V = V(I)$, then $\{f \in \mathbb{C} \mid f|_V \equiv 0\} = \sqrt{I}$. Abbr. $I(V(I)) = \sqrt{I}$.

A.3.2 Weierstrass theorems

Recommended reference: [GH94, §0.1].

Definition A.3.3. Let $\mathcal{O}_n$ denote the sheaf of *analytic functions* on $\mathbb{C}^n$, with respect to the (usual) analytic topology. This topology has neighborhood basis given by *polydiscs*

$$\Delta^n_{a,\varepsilon} \;=\; \{z \in \mathbb{C} \text{ s.t. } |z_j - a_j| < \varepsilon\}$$

of radius $\varepsilon > 0$ centered at $a \in \mathbb{C}^n$. Let $\mathcal{O}_{n,a}$ be the ring of *germs of analytic functions at $a \in \mathbb{C}^n$*. That is, $\mathcal{O}_{n,a}$ consists of equivalence classes $[f, U]$ of analytic functions $f : U \to \mathbb{C}$ defined on a neighborhood $a \in U \subset \mathbb{C}^n$, with $f_1 \sim f_2$ if $f_1 = f_2$ on $U_1 \cap U_2$.

Exercise A.3.4. The ideal $\mathfrak{m}_{n,a} = \{f \in \mathcal{O}_{n,a} \mid f(a) = 0\}$ of germs vanishing at a is the unique maximal ideal of $\mathcal{O}_{n,a}$.

Definition A.3.5. Let $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$. A *Weierstrass polynomial* is a holomorphic function of the form

$$p(z, w) \;=\; w^d \,+\, a_1(z)\, w^{d-1} \,+ \cdots +\, a_d(z)$$

with $a_j(z)$ holomorphic functions defined in a neighborhood $0 \in U \subset \mathbb{C}^{n-1}$, and $a_j(0) = 0$. We regard $p(z, w)$ as an element of $\mathcal{O}_{n-1,0}[w]$.

Weierstrass preparation theorem

Suppose that $f(z,w)$ is holomorphic in a neighborhood of $(0,0) \in \mathbb{C}^{n-1} \times \mathbb{C}$, that $f(0,0) = 0$, and that $f(0,w)$ is not identically zero. Then in some neighborhood of the origin, f can be uniquely factored as $f(z,w) = g(z,w)p(z,w)$ with $p(z,w)$ a Weierstrass polynomial, and $g(z,w)$ holomorphic and no-where vanishing in a neighborhood of $(0,0) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

Exercise A.3.6. Use Gauss's Lemma (§A.3.1) and the Weierstrass preparation theorem to show that $\mathcal{O}_{n,0}$ is a unique factorization domain.

Exercise A.3.7. (a) Show that $\mathcal{O}_{1,0}$ is noetherian.

(b) Use the Hilbert basis and Weierstrass preparation theorems to deduce that $\mathcal{O}_{n,0}$ is noetherian.

Exercise A.3.8. Use the Euclidean algorithm (§A.3.1) to show that if f, g are relatively prime in $\mathcal{O}_{n,0}$, then they are relatively prime in $\mathcal{O}_{n,z}$ for $|z| < \varepsilon$.

Weierstrass divison theorem

Given any Weierstrass polynomial $p(z,w)$ of degree d and a holomorphic function $f \in \mathcal{O}_{n,0}$, we can write $f(z,w) = g(z,w)p(z,w) + r(z,w)$ with $r(z,w) \in \mathcal{O}_{n-1,0}[w]$ a polynomial of degree $\leq d-1$.

Exercise A.3.9. Use the Euclidean algorithm and the Weierstrass theorems to prove the *Weak Nullstellensatz*: if $f \in \mathfrak{m}_{n,0}$ is irreducible, and $h \in \mathcal{O}_{n,0}$ vanishes on the set $\{f = 0\}$, then f divides h.

A.3.3 Positivity of transverse intersections

The intersection number of two analytic varieties meeting transversely is always positive (≥ 1), cf. [GH94, §0.4]. In fact something stronger is true: if M is a compact, complex manifold, and $A, B \subset M$ are analytic subvarieties such that $\dim A + \dim B = \dim M$, and $A \cap B \neq \emptyset$ is a finite set of points, then $A \cdot B \geq \#(A \cap B) \geq 1$.

(More generally, we recall that given two cycles, $a \in H_k(M, \mathbb{Z})$ and $b \in H_{2n-k}(M, \mathbb{Z})$, $\dim_{\mathbb{R}} M = 2n$, intersection number may be computed by integrating the fundamental classes $\pi_a \in H^{2n-k}_{\mathrm{d}}(M)$ and $\pi_b \in H^k_{\mathrm{d}}(M)$: $a \cdot b = \int_b \pi_a = \int_M \pi_a \wedge \pi_b$. Here the first equality is essentially the definition of π_a via Poincaré duality; the second equality is the assertion that the intersection pairing is dual to the cup product of the fundamental classes.)

A.3.4 Bertini's theorem

A *complete linear system* is the collection $|D| = \mathbb{P}\, H^0(X, [D])$ of effective divisors that are linearly equivalent to a fixed divisor D. A *linear system* is a projective linear subspace $\mathfrak{d} = \mathbb{P}\,\lambda$, with $\lambda \subset H^0(X, [D])$ a linear subspace. The *base locus* of $\mathfrak{d}$ is the set of points $x \in X$ with the property that every section $s \in \lambda$ vanishes at x; equivalently, it is the set of points $x \in X$ that are supported on every effective divisor $(s) \in \mathfrak{d}$. The linear system is *base point free* if the base locus is empty.

Bertini's theorem asserts: if $X \subset \mathbb{P}$ is smooth and quasi-projective, then a very general member of $\mathfrak{d}$ is smooth away from the base locus.

The linear system defines a rational map $\phi_{\mathfrak{d}} : X \to \mathfrak{d}^\vee = \mathbb{P}\,\lambda^\vee$, mapping $x \in X$ to the hyperplane $\mathbb{P}\{s \in \lambda \mid s(x) = 0\}$. The map is regular away from the base locus.

Exercise A.3.10. (a) Show that the image $\phi_{\mathfrak{d}}(X)$ is *linearly nondegenerate*; that is, $\phi_{\mathfrak{d}}(X)$ is not contained in a proper linear subspace of $\mathfrak{d}^\vee$.

(b) Fix a basis $\{s_0, \ldots, s_d\}$ of λ. Given $s \in \lambda$, and let $U = \{x \in X \text{ s.t. } s(x) \neq 0\}$. Show that $\phi_{\mathfrak{d}}|_U$ may be identified with the map $U \to \mathbb{P}^d$ sending $x \mapsto \left[\frac{s_0(x)}{s(x)} : \cdots : \frac{s_d(x)}{s(x)}\right]$.

Bertini's theorem is equivalent to the statement that $\phi_{\mathfrak{d}}^{-1}(H)$ is smooth away from the base locus (where $\phi_{\mathfrak{d}}$ is not defined) for all hyperplanes in some dense open subset of $\mathfrak{d}$.

Remark A.3.11. If a line bundle L over X admits a meromorphic section s (Remark 1.3.17), then we may take $D = (s)$. In this case the complete linear system is denoted $|L| = \mathbb{P}\, H^0(X, L)$, and the rational map $\phi_{|D|}$ is denoted $\phi_L : X \to \mathbb{P}\, H^0(X, L)$.

A.3.5 Canonical line bundle

Let M be a complex manifold of dimension n. The canonical line bundle is $K_M = \bigwedge^n T^\vee M$. We have

$$K_{\mathbb{P}^n} \;=\; \mathcal{O}_{\mathbb{P}^n}(-n-1)\,.$$

Theorem A.3.12 (Finite generation of the canonical ring)**.** *Let $R_X = \oplus_{m \geq 0} H^0(X, mK_X)$ be the canonical ring.*

(i) *If X is a surface of general type, then the R_X is a finitely generated noetherian ring* [BHPVdV04].

(ii) *If X is a smooth projective variety over field of characteristic zero, then R_X is finitely generated* [BCHM10, Siu08].

A.3.6 Adjunction formula

Let $i : D \hookrightarrow M$ be a smooth divisor in a complex manifold M. The normal bundle $N_{D/M} = i^*(T_M)/T_D$ extends to a line bundle $\mathcal{O}(D)$ on M. The ideal sheaf of D is the dual $\mathcal{O}(-D)$. In particular, the conormal bundle is $N^\vee_{D/M} = i^*\mathcal{O}(-D)$. We have

$$K_D \;=\; i^*(K_M \otimes \mathcal{O}(D))\,.$$

As canonical classes we have

$$K_D \;=\; \left.(K_M + D)\right|_D\,.$$

Example A.3.13. If $i : X \hookrightarrow \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d, then $K_X = i^* K_{\mathbb{P}^{n+1}} \otimes \mathcal{O}_X(d) \simeq \mathcal{O}_X(d-n-1)$.

Example A.3.14. Suppose that C is the complete intersection of hypersurfaces in $\mathbb{P}^{n+1}$ of degrees $d_1, \ldots, d_n$. The adjunction formula implies $K_C = \mathcal{O}_C(-n-2+\sum d_j)$. The degree of this line bundle is $\deg(K_C) = (-n-2+\sum d_j) \prod d_j$. Keeping in mind that $\deg K_C = 2g(C) - 2$, we find $g(C) = 1 - \frac{1}{2}(n+2-\sum d_j) \prod d_j$.

A.3.7 Lefschetz hyperplane theorem

Let Y be a hyperplane section of a smooth $X^{n+1} \subset \mathbb{P}$ so that $X \backslash Y$ is smooth. Then

$$\begin{aligned} H^k(X,\mathbb{Z}) &= H^k(Y,\mathbb{Z}) , \quad \forall\, k < n , \\ H^n(X,\mathbb{Z}) &\hookrightarrow H^n(Y,\mathbb{Z}) . \end{aligned}$$

A.3.8 Chern curvature

Let $L \to M$ be a holomorphic line bundle on a compact Kähler manifold, equipped with a hermitian metric h. The *Chern connection* form is the (1,0)-form $\partial \log h$; the *Chern curvature* is the (1,1)-form

$$\omega_L = \overline{\partial}\partial \log h .$$

The *Chern class* is

$$c_1(L) = \left[\tfrac{\mathbf{i}}{2\pi}\,\omega_L\right] \in H^2(M,\mathbb{R}) \cap H^{1,1}(M) .$$

The line bundle is *positive* (or *ample*) if $c_1(L) > 0$; that is, $\omega_L(v,\overline{v}) > 0$ for all $0 \neq v \in T_M$.

The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi\mathbf{i}} \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^\times \to 0$$

induces a long exact sequence in cohomology with boundary map

$$\cdots \to H^1(M,\mathcal{O}_M^\times) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \xrightarrow{i} H^2(M,\mathcal{O}_M) \to \cdots .$$

The cohomology group $H^1(M,\mathcal{O}_M^\times)$ parameterizes isomorphism classes of line bundles on M; and the boundary map is the sends L to the Chern class $c_1(L)$.

A.3.9 Lefschetz theorem on (1,1)-classes

Let M be a compact Kähler manifold. The boundary map $c_1 : H^1(M, \mathcal{O}_M^\times) \to H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$ is surjective.

Why. Hodge theory implies $H^2(M, \mathcal{O}_M) = H^{0,2}(M)$. The map $i : H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}_M)$ is the restriction of the projection $H^2(M, \mathbb{C}) \twoheadrightarrow H^{2,0}(M)$. □

A.3.10 Positivity implies vanishing

Kodaira vanishing

Let $L \to M$ be a positive holomorphic line bundle on a compact Kähler manifold. Then $H^q(M, K_M \otimes L) = 0$ for all $q > 0$.

Bott vanishing

Let $L \to X$ be an ample line bundle on a projective toric variety. Then $H^q(X, \Omega_X^p \otimes L) = 0$ for all $q > 0$ and $p \geq 0$ [BC94].

A.3.11 Kodaira embedding

Let $L \to M$ be a positive holomorphic line bundle on a compact Kähler manifold. There exists a holomorphic embedding $\phi : M \to \mathbb{P}$ so that $L^{\otimes m} = \phi^* \mathcal{O}_{\mathbb{P}}(1)$ for some $m > 0$.

Slogan. Positivity implies algebraicity.

A.3.12 Kodaira–Serre duality

Given a holomorphic vector bundle $E \to M$ over a compact, complex manifold of dimension d, we have

$$H^{p,q}(M, E) \simeq H^{d-p,d-q}(M, E^\vee)^\vee .$$

An useful corollary is the following: Given any algebraic vector bundle $E \to X$ over a smooth, proper (a.k.a. complete) algebraic variety of dimension d, we have

$$H^q(X, E) \;\simeq\; H^{d-q}(X, K_X \otimes E^\vee)^\vee \,.$$

More precisely, the natural trace map on $H^d(X, K_X)$ is a perfect pairing

$$H^q(X, E) \;\times\; H^{d-q}(X, K_X \otimes E^\vee) \;\to\; H^d(X, K_X) \;\to\; \mathbb{C}\,.$$

A.3.13 Picard variety

The *Picard group* of any ringed space $(Y, \mathcal{O}_Y)$ is the group $\mathrm{Pic}(Y) = H^1(Y, \mathcal{O}_Y^*)$ of isomorphism classes of line bundles (or invertible sheaves) on Y. The *Picard variety* of a smooth, proper algebraic variety $(X, \mathcal{O}_X)$ is the connected identity component $\mathrm{Pic}^0(X)$, and has the structure of an abelian variety. The *Neron–Severi group* $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is a finitely generated abelian group. The rank of $\mathrm{NS}(X)$ is the *Picard number* $\rho(X)$.

A.3.14 Bogomolov–Miyaoka–Yau inequality

If X is an n-dimensional minimal model of general type, then $(-1)^n(2n+2)\, c_1(X)^{n-2}\, c_2(X) \geq (-1)^n n\, c_1(X)^n$.

A.3.15 Riemann–Roch–Hirzebruch

The *holomorphic Euler characteristic* of a holomorphic vector bundle $E \to M$ on a compact Kähler manifold is

$$\chi(M, E) \;=\; \sum_{q \geq 0} (-1)^q \dim_{\mathbb{C}} H^q(M, E)\,.$$

The *Riemann–Roch–Hirzebruch formula* is

$$\chi(M, E) \;=\; \langle \mathbf{td}(M)\mathbf{ch}(E), [M] \rangle \;=\; \int_M \mathbf{td}(M)\mathbf{ch}(E)\,,$$

where $\mathbf{td}(M) = \mathbf{td}(T_M)$ is the Todd class of M and

$$\mathbf{ch}(E) \;=\; \left[\mathrm{tr}(\exp \tfrac{\mathbf{i}}{2\pi}\Omega)\right]$$

is the Chern character of E. The Chern character is an additive and multiplicative invariant: $\mathbf{ch}(E_1 \oplus E_2) = \mathbf{ch}(E_1) + \mathbf{ch}(E_2)$ and $\mathbf{ch}(E_1 \otimes E_2) = \mathbf{ch}(E_1)\,\mathbf{ch}(E_2)$. It is related to the Chern classes $c_k(E) \in H^{2k}(M)$ by

$$\sum_k c_k(E)t^k \;=\; \left[\det\left(\tfrac{\mathbf{i}}{2\pi}t\Omega + \mathrm{Id}\right)\right] \;=\; \prod_j (1 + \alpha_j(E)t)\,,$$

and

$$\begin{aligned}\mathbf{ch}(E) \;&=\; \mathrm{rank}(E) \,+\, c_1(E) \,+\, \tfrac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right)\\ &\quad + \tfrac{1}{6}\left(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\right) \,+\, \cdots\,.\end{aligned}$$

The Todd class is

$$\begin{aligned}\mathbf{td}(E) \;&=\; \prod_j \frac{\alpha_j(E)}{1 - e^{-\alpha_j(E)}}\\ &=\; 1 \,+\, \tfrac{1}{2}c_1(E) \,+\, \tfrac{1}{12}\left(c_1(E)^2 + c_2(E)\right) \,+\, \tfrac{1}{24}c_1(E)\,c_2(E)\\ &\quad + \tfrac{1}{720}\left(-c_1(E)^4 + 4c_1(E)^2c_2(E) + c_1(E)c_3(E) + 3c_2(E)^2 - c_4(E)\right) \,+\, \cdots\,.\end{aligned}$$

The Todd class is an exponential invariant in the sense that $\mathbf{td}(E_1 \oplus E_2) = \mathbf{td}(E_1)\mathbf{td}(E_2)$.

Given a SES $0 \to A \to B \to C \to 0$ of vector bundles we have

$$\mathbf{td}(B) \;=\; \mathbf{td}(A)\mathbf{td}(C) \quad \text{and} \quad \mathbf{ch}(B) \;=\; \mathbf{ch}(A) \,+\, \mathbf{ch}(C)\,. \tag{A.3.15}$$

A.4 Very little sheaf theory

Recommended references: [GH94, Voi07].

A.4.1 A few definitions

The defining property of a *presheaf* is that the restriction maps satisfy $\mathrm{res}_{U,U} = \mathrm{id}$, and $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ for all $W \subset V \subset U$. The defining properties of a *sheaf* are that given any open cover $\{U_\alpha\}$ of U we have: (i) if $\sigma, \tau \in \mathcal{S}(U)$ and $\sigma|_{U_\alpha} = \tau|_{U_\alpha}$ for all α, then $\sigma = \tau$; and (ii) if $\sigma_\alpha \in \mathcal{S}(U_\alpha)$ satisfy $\sigma_\alpha|_{U_{\alpha\beta}} = \sigma_\beta|_{U_{\alpha\beta}}$, then $\sigma_\alpha = \sigma|_{U_\alpha}$ for some $\sigma \in \mathcal{S}(U)$. Examples of presheaves that are not sheaves include: the presheaf of constant functions (whose sheafification is the sheaf of (locally) constant functions); and the sheaf of exact forms (whose sheafification is the sheaf of closed forms).

The *sheafification* of a presheaf $\mathcal{P}$ is the sheaf $\mathcal{P}^\#(U) = \{(\rho_x \in \mathcal{P}_x)_{x \in U}$ s.t. $\forall\, x \in U\,,\ \exists$ open $x \in V \subset U\,,\ \sigma \in \mathcal{P}(V)$ with $\sigma_y = \rho_y\ \forall\, y \in V\}$.

Exercise A.4.1. Suppose that $\mathcal{P}$ is a subpresheaf of a sheaf $\mathcal{S}$. Show that $\mathcal{P}^\#(U) = \{\sigma \in \mathcal{S}(U) \mid \exists$ open cover $\{U_\alpha\}$ of U s.t. $\sigma|_{U_\alpha} \in \mathcal{P}(U_\alpha)\ \forall\, \alpha\}$.

Exercise A.4.2. Let $\phi : \mathcal{S} \to \mathcal{T}$ be a morphism of sheaves of abelian groups.

(a) Show that $U \mapsto \ker\{\phi_U : \mathcal{S}(U) \to \mathcal{T}(U)\}$ is a sheaf, the *kernel sheaf* $\ker\phi \subset \mathcal{S}$. We have $(\ker\phi)_x = \ker(\phi_x)$.

(b) Show that $U \mapsto \mathrm{im}\{\phi_U : \mathcal{S}(U) \to \mathcal{T}(U)\}$ is a presheaf, but need not be a sheaf. [*Hint.* Consider the exponential map $\mathcal{O}_{\Delta^*} \to \mathcal{O}^\times_{\Delta^*}$.] The sheafification is a subsheaf $\mathrm{im}\,\phi \subset \mathcal{T}$ whose sections may be described as in HW A.4.1. We have $(\mathrm{im}\,\phi)_x = \mathrm{im}\,(\phi_x)$.

(c) Show that $U \mapsto \mathrm{coker}\{\phi_U : \mathcal{S}(U) \to \mathcal{T}(U)\}$ is a presheaf, but need not be a sheaf. The sheafification is

$$(\mathrm{coker}\,\phi)(U) \ = \ \left\{ \sigma_\alpha \in \mathcal{T}(U_\alpha) \ \middle|\ \begin{array}{c} \{U_\alpha\} \text{ is an open cover of } U \text{ and} \\ \sigma_\alpha|_{U_{\alpha\beta}} - \sigma_\beta|_{U_{\alpha\beta}} \in \phi_{U_{\alpha\beta}}(\mathcal{S}(U_{\alpha\beta})) \end{array} \right\} \Big/ \sim\,,$$

where $\{\sigma_\alpha \in \mathcal{T}(U_\alpha)\} \sim \{\sigma'_\mu \in \mathcal{T}(U'_\mu)\}$ if for all $x \in U_\alpha \cap U'_\mu$ there exists an open $x \in V \subset U_\alpha \cap U'_\mu$ so that $\sigma_\alpha|_V - \sigma'_\mu|_V \in \phi_V(\mathcal{S}(V))\}$. We have $(\mathrm{coker}\,\phi)_x = \mathrm{coker}\,(\phi_x)$.

Exercise A.4.3. Show that the following are equivalent:

(a) ϕ is injective;

(b) $\phi_U : \mathcal{S}(U) \to \mathcal{T}(U)$ is injective for all open U.

Exercise A.4.4. Show that the following are equivalent:

(a) ϕ is injective (resp. surjective, an isomorphism)

(b) $\phi_x : \mathcal{S}_x \to \mathcal{T}_x$ is injective (resp. surjective, an isomorphism) for all x.

A.4.2 Functors

Exact functors

A functor is *exact* if it preserves short exact sequences. A covariant functor F is *left-exact* if $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C$ exact implies $0 \to F(A) \xrightarrow{F(\phi)} F(B) \xrightarrow{F(\psi)} F(C)$ is exact. A contravariant functor F is *right-exact* if $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C$ exact implies $F(C) \xrightarrow{F(\psi)} F(B) \xrightarrow{F(\phi)} F(A) \to 0$ is exact

Example A.4.5. If V is a vector space over a field k, then $V^\vee = \mathrm{Hom}_k(V, k)$ is an exact (contravariant) functor on the category of k vector spaces.

Example A.4.6. If M is an abelian group, the (covariant) functor $A \mapsto \mathrm{Hom}(M, A)$ of the category of abelian groups to itself is left-exact.

Example A.4.7. If $\mathrm{Sh}(M, \mathbb{Z}_M)$ is the category of sheaves of abelian groups over a topological space M, then the (covariant) *global sections functor* $\Gamma(\mathcal{S}) = \mathcal{S}(M)$ is left-exact.

Let $(X, \mathcal{O}_X)$ be either an algebraic variety or a complex analytic space. Let $\mathrm{Sh}(X)$ denote the category of sheaves on X; let $\mathrm{Sh}(X, \mathbb{Z}_X)$ denote the category of sheaves of abelian groups on X, and let $\mathrm{Sh}(X, \mathcal{O}_X)$ denote the category of $\mathcal{O}_X$–modules on X.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism.

Direct image functor

The *direct image functor* $f_* : \mathrm{Sh}(X, \mathcal{O}_X) \to \mathrm{Sh}(Y, \mathcal{O}_Y)$ maps $\mathcal{S} \in \mathrm{Sh}(X, \mathcal{O}_X)$ to the *direct image* (or *pushforward*) *sheaf* defined by $(f_*\mathcal{S})(V) = \mathcal{S}(f^{-1}(V))$ for all open $V \subset Y$.

(i) If Y is a point, then f_* is the global sections functor.

(ii) The direct image functor is left-exact.

(iii) If $f : X \to Y$ is the inclusion of a closed subspace, then f_* is exact (preserves SES), and an equivalence of categories between $\mathrm{Sh}(X)$ and the category of sheaves on Y supported on X.

Direct image functor with compact support

The *direct image with compact support functor* $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ maps $\mathcal{S} \in \mathrm{Sh}(X)$ to the sheaf defined by $(f_!\mathcal{S})(V) = \{s \in \mathcal{S}(f^{-1}(V)) \text{ s.t. } f : \mathrm{supp}(s) \to V \text{ is proper}\}$ for all open $V \subset Y$.

(iv) If f is proper, then $f_! = f_*$.

(v) If f is an open embedding, then $f_!$ is the extension by zero functor.

Inverse image functor

The *inverse image functor* $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ maps $\mathcal{T} \in \mathrm{Sh}(Y)$ to the *inverse image* (or *pullback*) *sheaf* $f^{-1}\mathcal{T}$, the sheaf associated to the presheaf $U \mapsto \lim\limits_{V \supset f(U)} \mathcal{T}(V)$.

(vi) The stalks are $(f^{-1}\mathcal{T})_x = \mathcal{T}_{f(y)}$.

(vii) The functor f^{-1} is exact (preserves SES).

(viii) The direct image functor is right adjoint to the inverse image functor: $\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{T}, \mathcal{S}) = \mathrm{Hom}_{\mathrm{Sh}(X)}(T, f_*\mathcal{S})$.

If $\mathcal{T} \in \mathrm{Sh}(Y, \mathcal{O}_Y)$, then in general $f^{-1}\mathcal{T} \notin \mathrm{Sh}(X, \mathcal{O}_X)$. In this case it is better to work with the sheaf $f^*\mathcal{T} \overset{\text{dfn}}{=\!=} f^{-1}\mathcal{T} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

Example A.4.8. If f is the inclusion of a point $y \in Y$, then $f^{-1}(\mathcal{T})$ is the stalk $\mathcal{T}_y$. If $\mathcal{T} \in \mathrm{Sh}(Y, \mathcal{O}_Y)$, then $f^{-1}(\mathcal{O}_Y) = \mathcal{O}_{Y,y}$, and $f^*(\mathcal{O}_Y) = \mathbb{C}$.

(ix) In general f^* is right exact. If f^* is exact, we say f is *flat.*

A.4.3 Sheaf cohomology

Enough injectives

Fact. Let $\mathrm{Sh}(X, \mathcal{O}_X)$ denote the category of sheaves of $\mathcal{O}_X$–modules. This category *has enough injectives.*

Why. Because the category of R–modules has enough injectives (because the category of R–modules has injective hulls). One then uses Godement's construction to show that $\mathrm{Sh}(X, \mathcal{O}_X)$ has enough injectives: $\mathcal{S} \hookrightarrow \prod \mathcal{S}_x \hookrightarrow \prod I_x$.

Remark. Since abelian groups are $\mathbb{Z}$–modules, we see that the category $\mathrm{Sh}(X, \mathbb{Z}_X)$ of sheaves of abelian groups also has enough injectives.

Consequence: existence of injective resolutions. Every sheaf $\mathcal{S} \in \mathrm{Sh}(X, \mathcal{R}_X)$ admits an injective resolution: there exist injective objects $\mathcal{I}^k \in \mathrm{Sh}(X, \mathcal{R}_X)$, and maps $j : \mathcal{S} \hookrightarrow \mathcal{I}^0$ and $d^k : \mathcal{I}^k \to \mathcal{I}^{k+1}$ so that $\mathrm{im}\, j = \ker d^0$ and $\mathrm{im}\, d^k = \ker d^{k+1}$.

Right derived functors

Given a left-exact functor F, the right derived functors R^kF are

$$R^kF(\mathcal{S}) \;\overset{\text{dfn}}{=\!=}\; H^k(F(\mathcal{I}^\bullet)) \;=\; \frac{\ker\{Fd^k : F\mathcal{I}^k \to F\mathcal{I}^{k+1}\}}{\mathrm{im}\,\{Fd^{k-1} : F\mathcal{I}^{k-1} \to F\mathcal{I}^k\}}\,.$$

Remark. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a SES of sheaves, then $0 \to F(\mathcal{A}) \to F(\mathcal{B}) \to F(\mathcal{C})$ is exact. The "job" of the right derived functors are to complete this to a LES $0 \to F(\mathcal{A}) \to F(\mathcal{B}) \to F(\mathcal{C}) \to R^1F(\mathcal{A}) \to R^1F(\mathcal{B}) \to R^1F(\mathcal{C}) \to \cdots$. Injective sheaves have the property $R^kF(\mathcal{I}) = 0$ for all $k > 0$.

Example A.4.9 (Sheaf cohomology)**.** Sheaf cohomology is the right derived functor $R^k\Gamma$ associated to the global sections functor Γ:

$$H^k(X, \mathcal{S}) \;=\; R^k\Gamma(\mathcal{S})\,.$$

Example A.4.10 (Higher direct images)**.** Given a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, the *higher direct images* are the right derived functors $R^k f_*$. Given $\mathcal{S} \in \mathrm{Sh}(X, \mathcal{O}_X)$, $R^k f_*(\mathcal{S})$ is the sheaf associated to the presheaf $V \mapsto H^k(f^{-1}(V), \mathcal{S})$.

Exercise A.4.11. Let $f : \mathcal{X} \to S$ be a smooth surjective morphism of complex manifolds with compact fibres (as in §3.1). Show that the stalks are $R^k f_*(\mathbb{Q}_{\mathcal{X}})_s = H^k(X_s, \mathbb{Q})$, where $X_s = f^{-1}(s)$.

Acyclic resolutions

A sheaf $\mathcal{A}$ is *F-acyclic* if $R^q F(\mathcal{A}) = 0$ for all $q > 0$. In particular, a sheaf is Γ-acyclic if $H^q(X, \mathcal{A}) = 0$ for all $q > 0$. A resolution $\mathcal{S} \to \mathcal{A}^\bullet$ is *F-acyclic* if each $\mathcal{A}^k$ is F-acyclic. Injective objects are acyclic (for any functor). Any acyclic resolution is chain homotopic to an injective resolution $\mathcal{S} \to \mathcal{I}^\bullet$. This means that acyclic resolutions can be used to compute right-derived functors.

Example A.4.12. A sheaf $\mathcal{S}$ is *flasque* if the restriction map $\mathcal{S}(V) \to \mathcal{S}(U)$ is surjective for all open $U \subset V$. (Godement's construction is flasque.) Flasque sheaves are Γ-acyclic.

Example A.4.13. Let M be a paracompact Hausdorff space. A *fine* sheaf on M is a sheaf of $\mathcal{R}_M$–modules, where $\mathcal{R}_M$ is a sheaf of rings with the property that every open cover of M admits a subordinate partition of unity. For example, if M is a smooth manifold, then any sheaf of $\mathcal{C}^\infty_M$–modules is fine. Fine sheaves are Γ-acyclic. This yields the de Rham and Dolbeault theorems

$$H^n(M, \mathbb{C}) \;=\; \frac{\ker\{\mathrm{d} : \mathcal{E}^n(M) \to \mathcal{E}^{n+1}(M)\}}{\mathrm{im}\,\{\mathrm{d} : \mathcal{E}^{n-1}(M) \to \mathcal{E}^n(M)\}}$$

and

$$H^q(X,\Omega^p_X) \;=\; \frac{\ker\{\overline{\partial}:\mathcal{E}^{p,q}(X)\to\mathcal{E}^{p,q+1}(X)\}}{\operatorname{im}\{\overline{\partial}:\mathcal{E}^{p,q-1}(X)\to\mathcal{E}^{p,q}(X)\}}\,.$$

Given a smooth manifold M, let $\mathrm{Sh}(M,\mathcal{C}^\infty_M)$ denote the category of sheaves of $\mathcal{C}^\infty_M$–modules.

Example A.4.14 (Čech resolution)**.** Fix an open cover $\{U_i\}_{i\in\mathbb{N}}$. Given $I=\{i_0<i_1<\cdots<i_k\}\subset\mathbb{N}$, let $U_I=\cap_{i\in I}U_i$ denote the finite intersections, and $j_I:U_I\hookrightarrow X$ the inclusion. Define sheaves

$$\mathcal{C}^k(\{U_i\},\mathcal{S}) \;=\; \bigoplus_{|I|=k+1} j_{I*}(\mathcal{S}|_{U_I})\,,$$

and sheaf morphisms $\delta:\mathcal{C}^k(\{U_i\},\mathcal{S})\to\mathcal{C}^{k+1}(\{U_i\},\mathcal{S})$ by

$$(\delta\sigma)_{i_0\cdots i_{k+1}} \;=\; \sum(-1)^a\sigma_{i_0\cdots\hat{i}_a\cdots i_{k+1}}|_{U_{i_0\cdots i_{k+1}}}\,,\quad i_0<\cdots<i_{k+1}\,.$$

Then $\delta^2=0$, and the map $j:\mathcal{S}\to\mathcal{C}^0(\{U_i\},\mathcal{S})$ sending $\sigma\mapsto\sigma|_{U_i}$ realizes $(\mathcal{C}^\bullet(\{U_i\},\mathcal{S}),\delta)$ as a resolution of $\mathcal{S}$. The global sections of $\mathcal{C}^k(\{U_i\},\mathcal{S})$ are

$$\Gamma(X,\mathcal{C}^k(\{U_i\},\mathcal{S})) \;=\; \bigoplus_{|I|=k+1}\mathcal{S}(U_I)\,.$$

The resolution $\mathcal{S}\hookrightarrow\mathcal{C}^\bullet(\{U_i\},\mathcal{S})$ is Γ–acyclic if $\{U_i\}$ is a *Leray cover* $H^k(U_I,\mathcal{S})=0$ for all $k>0$.

A.4.4 Vanishing theorems

Cartan's Theorems A & B

Suppose that $\mathcal{S}$ is a coherent sheaf (Definition 1.1.40) on a Stein manifold X (a submanifold of $\mathbb{C}^m$). The sheaf is spanned by its global sections $H^0(X,\mathcal{S})$, and $H^q(X,\mathcal{S})=0$ for all $q>0$ [Car53].

Serre vanishing

Analogous results were established by Serre for a quasi-coherent sheaf over an affine scheme $(X, \mathcal{O}_X)$: if $\mathcal{S}$ is quasi-coherent, then $H^q(X, \mathcal{S}) = 0$ for all $q > 0$ [Har77].

A.4.5 Leray spectral sequence

Via de Rham cohomology

Suppose that $\pi : M \to N$ is a smooth fibre bundle with compact fibres (in particular, π is a submersion). Let $V = \ker \pi_* \subset TM$ denote the *vertical subbundle*. (Note that V_x the tangent space to the fibre $\pi^{-1}(\pi(x))$ through x.) Let $V^\perp \subset T^*M$ denote the *annihilator* $V_x^\perp = \{\eta \in T_x^*M \text{ s.t. } \eta(v) = 0 \ \forall\, v \in V_x\}$. If we fix local coordinates (u, v) on M so that $\pi(u, v) = u$, then $\{\frac{\partial}{\partial v_j}\}$ is a local framing of V, and $\{\mathrm{d}u_a\}$ is a local framing of $V^\perp$.

Define a filtration of $\bigwedge^n T^*M$ by

$$F^p(\textstyle\bigwedge^n T^*M) \ \stackrel{\text{dfn}}{=\!=} \ (\textstyle\bigwedge^p V^\perp) \wedge (\textstyle\bigwedge^{n-p} T^*M)\,.$$

Note that $\{\mathrm{d}u_I \wedge \mathrm{d}v_J \ \text{ s.t. } \ |I| \geq p\,,\ |I| + |J| = n\}$ is a local framing of $F^p(\bigwedge^n T^*M)$. Let

$$\mathcal{E}^n(M) \ = \ F^0\mathcal{E}^n(M) \ \subset \ F^1\mathcal{E}^n(M) \ \subset \cdots \subset \ F^n\mathcal{E}^n(M) \ \subset \ F^{n+1}\mathcal{E}^n(M) \ = \ 0$$

be the induced filtration on the space of smooth n-forms on M. We have $\mathrm{d} : F^p\mathcal{E}^n(M) \to F^p(\mathcal{E}^{n+1}(M))$. So we have a filtration of the de Rham complex. This yields a spectral sequence [GH94, p. 464].

The E_0 page is

$$E_0^{p,q} \ = \ \frac{F^p(\mathcal{E}^{p+q}(M))}{F^{p+1}(\mathcal{E}^{p+q}(M))}\,.$$

Locally an element of $E_0^{p,q}$ may be represented by $\eta = \sum_{|I|=p} \eta_I \wedge \mathrm{d}u_I$, with $\eta_I = \eta_I(u, v, \mathrm{d}v)$ a q–form involving only the $\mathrm{d}v_j$. And $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ maps $d_0\eta = \sum_{|I|=p}(\mathrm{d}_v\eta_I) \wedge \mathrm{d}u_I$. Then elements of $E_1^{p,q} = \ker d_0 / \mathrm{im}\, d_0$ are represented by p-forms on N that take value in the bundle $H^q_{\mathrm{dR}}(\pi\text{-fibre})$ with fibre over $x \in N$ give by $H^q(\pi^{-1}(x))$. Given one such form, $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ acts by $d_1\eta \sum_{|I|=p}(\mathrm{d}_u\eta_I) \wedge \mathrm{d}u_I$. So

$$E_2^{p,q} \;=\; H^p_{\mathrm{dR}}(N, H^q_{\mathrm{dR}}(\pi\text{-fibre})) \;=\; H^p(N, R^q\pi_*\mathbb{C})\,.$$

In general $E_2^{p,q} \neq E_\infty^{p,q}$. However, ...

Leray for a family of compact Kähler manifolds

Theorem A.4.15 ([Del68])**.** *Let $f : \mathcal{X} \to S$ be a smooth surjective holomorphic mapping $f : \mathcal{X} \to S$ of Kähler manifolds with compact fibres (as in §3.1). Then $H^n(\mathcal{X}, \mathbb{Q}) \simeq \bigoplus_{p+q=n} H^p(S, R^q f_*\mathbb{Q}_{\mathcal{X}})$. And if S is simply connected, then $H^n(\mathcal{X}, \mathbb{Q}) \simeq \bigoplus_{p+q=n} H^p(S, \mathbb{Q}) \otimes H^q(X_{s_0}, \mathbb{Q})$.*

Idea of the proof. Use the Hard Lefschetz Theorem 2.2.24 to show that the Leray Spectral Sequence (§A.4.5) collapses at the second page, cf. [GH94, p. 466]. □

Corollary A.4.16. *The cohomology $H^n(\mathcal{X}, \mathbb{Q})$ surjects onto $\mathrm{Gr}^0 H^n(\mathcal{X}, \mathbb{Q}) = H^0(S, R^n f_*\mathbb{Q}_{\mathcal{X}})$.*

As a special case of the Grothendieck spectral sequence

If $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$ are left-exact covariant functors of abelian categories with enough injectives, it is natural to ask if there is a relationship between the right derived functors of FG and those of F and G. If $G(I)$ is F-acyclic for every injective object $I \in \mathcal{A}$, then for every $A \in \mathcal{A}$, there exists a spectral sequence $E_2^{p,q} = (R^pF)(R^qG)(A) \implies R^{p+q}(FG)(A)$.

Example A.4.17 (Leray spectral sequence)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism, and fix $\mathcal{S} \in \mathrm{Sh}(X, \mathcal{O}_X)$. We obtain the Leray spectral sequence by taking $\mathrm{Sh}(X, \mathcal{O}_X) \xrightarrow{f_*} \mathrm{Sh}(Y, \mathcal{O}_Y) \xrightarrow{\Gamma(Y,\cdot)} \mathrm{Ab}$, and noting that $\Gamma(Y, \cdot) \circ f_* = \Gamma(X, \cdot)$. One uses the fact that f_* is right adjoint to the exact f^{-1} to show that f_* maps injectives to injectives.

One may show that

$$0 \to H^1(Y, f_*\mathcal{S}) \to H^1(X,\mathcal{S}) \to H^0(Y, R^1 f_*\mathcal{S}) \to H^2(Y, f_*\mathcal{S}) \to H^2(X,\mathcal{S})$$

is exact.

Grothendieck's spectral sequence is constructed as follows. Fix an injective resolution $A \hookrightarrow I^\bullet$. As a complex in a category with enough injectives $G(I^\bullet)$ admits a *fully injective* resolution $G(I^\bullet) \hookrightarrow J^{\bullet,\bullet}$. This means that $G(I^a) \hookrightarrow J^{a,\bullet}$ is an injective resolution *and* each of

$$\begin{aligned} \ker\{G(I^a) \to G(I^{a+1})\} &\hookrightarrow \ker\{J^{a,\bullet} \to J^{a+1,\bullet}\} \\ \mathrm{im}\{G(I^{a-1}) \to G(I^a)\} &\hookrightarrow \mathrm{im}\{J^{a-1,\bullet} \to J^{a,\bullet}\} \\ H^a(G(I^\bullet)) &\hookrightarrow H^a(J^{\bullet,0}) \to H^a(J^{\bullet,1}) \to \cdots \end{aligned}$$

is an injective resolution. Then one considers the spectral sequences associated to the double complex $F(J^{\bullet,\bullet})$.

Via Čech cohomology

Let $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism, and fix $\mathcal{S} \in \mathrm{Sh}(X,\mathcal{O}_X)$.

Fix a cover $\{V_i\}$ of Y, and a cover $\{U_{ia}\}$ of $f^{-1}(V_i)$ so that $\{U_{ia}\}$ is a Leray cover of X with respect to $\mathcal{S}$; that is, $\check{H}^q(U_I,\mathcal{S}) = 0$ for all $q > 0$ and I. Let $\mathcal{C}^\bullet(\{U_{ia}\},\mathcal{S}) \in \mathrm{Sh}(X,\mathcal{O}_X)$ be the acyclic Čech resolution of $\mathcal{S}$ (Example A.4.14). Then $\mathcal{K}^\bullet = f_*\mathcal{C}^\bullet(\{U_{ia}\},\mathcal{S}) \in \mathrm{Sh}(Y,\mathcal{O}_Y)$ is a complex. The associated cohomology sheaves are $\mathcal{H}^q(\mathcal{K}^\bullet) = R^q f_*\mathcal{S}$. Following Definition 5.4.16 we consider the double complex $\mathcal{C}^p(\{V_i\},\mathcal{K}^q) = \mathcal{C}^p(\{V_i\}\,,\, f_*\mathcal{C}^q(\{U_{ia}\},\mathcal{S}))$. Then $\mathbb{H}^k(Y,\mathcal{K}^\bullet) = \mathbb{H}^k(Y, f_*\mathcal{C}^\bullet(\{U_{ia}\},\mathcal{S})) = H^k(X,\mathcal{S})$. We conclude that

$${}'E_2^{p,q} = \check{H}^p(Y, R^q f_*\mathcal{S}) \implies H^{p+q}(X,\mathcal{S})\,.$$

A.4.6 Hypercohomology

Hypercohomology is a generalization of sheaf cohomology that takes as its input not a single sheaf, but a complex of sheaves. In particular, a SES

$$0 \;\to\; \mathcal{A}^\bullet \;\to\; \mathcal{B}^\bullet \;\to\; \mathcal{C}^\bullet \;\to\; 0$$

of complexes of sheaves will induce a LES

$$0 \;\to\; \mathbb{H}^0(\mathcal{A}^\bullet) \;\to\; \mathbb{H}^0(\mathcal{B}^\bullet) \;\to\; \mathbb{H}^0(\mathcal{C}^\bullet) \;\to\; \mathbb{H}^1(\mathcal{A}^\bullet) \;\to\; \mathbb{H}^1(\mathcal{B}^\bullet) \;\to\; \cdots$$

in hypercohomology. Suppose that $(\mathcal{K}^\bullet, d)$ is a complex of sheaves. Given $\mathcal{K}^\bullet$, there exists a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ with each $\mathcal{I}^k$ an injective object, and each $\mathcal{K}^k \to \mathcal{I}^k$ an injective map [Voi07]. Given a left-exact functor F, we define

$$R^kF(\mathcal{K}^\bullet) \;\stackrel{\text{dfn}}{=\!=}\; H^k(F(\mathcal{I}^\bullet))\,.$$

A quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ induces a canonical isomorphism $R^kF(\mathcal{K}_1^\bullet) \simeq R^kF(\mathcal{K}_2^\bullet)$. For a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{A}^\bullet$, with each $\mathcal{A}^k$ F-acyclic, we have $R^kF(\mathcal{K}^\bullet) = H^k(F(\mathcal{A}^\bullet))$.

We obtain *hypercohomology* by taking the global sections functor

$$\mathbb{H}^k(X, \mathcal{K}^\bullet) \;=\; R^k\Gamma(\mathcal{K}^\bullet) \;=\; H^k(\Gamma(\mathcal{A}^\bullet))\,.$$

See §5.4.2 for further discussion.

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