Two evolving social network models

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Abstract

In our first model, individuals have opinions in $[0,1]^d$. Connections are broken at rate proportional to their length ℓ, and a randomly chosen end point x connects to an individual chosen at random. If version (i) the new edge is always accepted. In version (ii) a new connection of length ℓ' is accepted with probability $\min\{\ell/\ell', 1\}$. Our second model is a dynamic version of preferential attachment. Edges are chosen at random for deletion, then one endpoint (again chosen at random) connects to vertex z with probability proportional to $f(d(z))$ where $f(k) = \theta(k+1) + (1-\theta)(\bar{d}+1)$, and $\bar{d}$ is the average degree. In words, this is a mixture of degree-proportional and at random rewiring. The common feature of these models is that they have stationary distributions that satisfy the detailed balance condition and are given by explicit formulas. In addition, the first model is closely related to long range percolation, and the second to the configuration model of random graphs. As a result, we can obtain explicit results about the degree distribution, connectivity and diameter for both models.

1 Introduction

In this article we study two models of social networks that evolve stochastically in time: (a) an opinion-dependent rewiring model and (b) a degree-dependent rewiring model.

1.1 Opinion-dependent rewiring model

There have been a number of studies recently in which the structure of a network coevolves with the opinions of its members [1]–[9]. Here we will study the simpler case of stubborn individuals who do not change their opinions. The starting point for this investigation was a paper of Henry, Pralat, and Zhang (HPZ) [10] who considered a model in which N individuals have opinions chosen uniformly in $[-1,1]^d$ and at any time there are M edges connecting them. In their discrete time formulation, on each step an edge (x,y) is chosen at random and the edge is broken with probability $pd(x,y)$ where $d(x,y)$ is the dissimilarity of x and y and p is chosen small enough so that this probability cannot exceed 1. For simplicity, we will take $d(x, y)$ to be the usual Euclidean distance but to have more connection with long-range percolation in Section 2, one might want to take $d(x, y) = |x - y|^\beta$.

In this and all other models we consider, when an edge is broken, we pick an endpoint x of the edge at random and connect it to a new vertex $\neq x$ and not already a neighbor of x. In the HPZ model the choice of new vertex is made at random from all legal possibilities.

HPZ assumed $d(x, y) \in (0, 1)$ for all x, y, and discretized their model so that all edges had lengths in $\{1/K, 2/K, \ldots, K/K\}$ to conclude that in equilibrium the average number of edges of length i/K satisfied

$$N_i = \frac{M/i}{\sum_{j=1}^{K} 1/j}$$

where M is the number of edges.

In this paper, we use a continuous time formulation in which edges of length ℓ break at rate ℓ. For convenience, we switch the opinion space to $[0, 1]^d$, but retain the assumption that opinions are chosen independently and uniformly at random. As in the HPZ model, broken edges are given an orientation (x, y) and x connects to a new vertex that does not make a self loop or parallel edge. In addition to (i) the random rewiring version of HPZ, we also consider (ii) a Metropolis-Hastings (MH) dynamics in which a randomly chosen edge of length ℓ' is accepted with probability $\min\{\ell/\ell', 1\}$. Figure 1 shows a sample of each dynamic at equilibrium when $d = 2$.

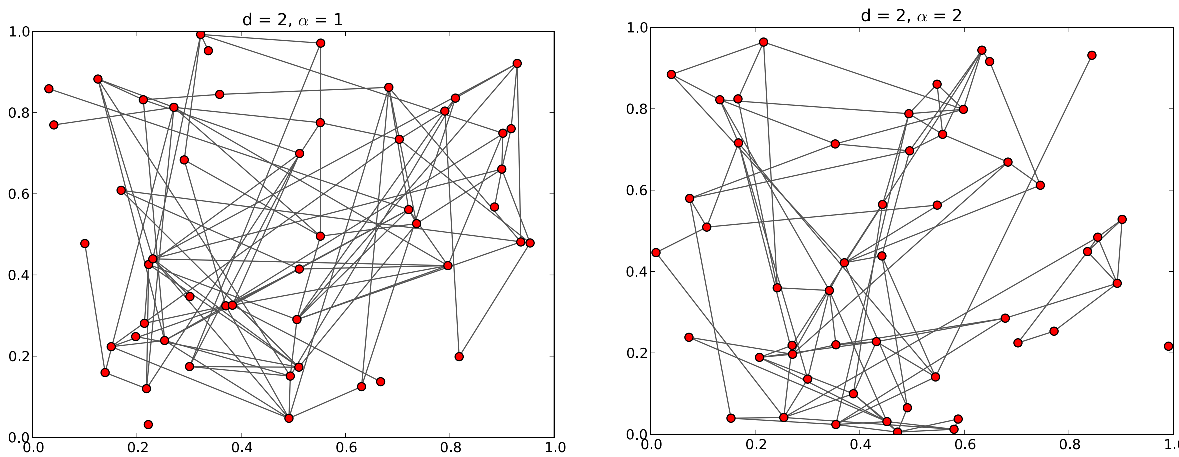

Figure 1: Sample graphs at equilibrium for the opinion-dependent rewiring model in dimension 2 with $N = 50$, $M = 100$. The random rewiring dynamics (a) yield a graph that appears more dense because edges tend to be longer than for the Metropolis-Hastings dynamics (b).

1.2 Degree-dependent rewiring model

In our second model, we begin with a graph that has N vertices and M edges, and at rate 1 edges are chosen to be broken. When an edge is chosen, we pick an endpoint x of the edge at random and connect it to a vertex z chosen with probability proportional to $f(d(z))$, where $d(z)$ denotes the degree of vertex z. Let

$$f(i) = \theta(i + 1) + (1 - \theta)(\bar{d} + 1),$$

where $\bar{d} = 2M/N$ is the average degree and $\theta \in [0, 1]$. We add 1 in the first factor so that vertices of degree 0 can be chosen. Otherwise, they will accumulate over time and there is no stationary distribution. We put $\bar{d} + 1$ in the second factor, so choosing a vertex with probability proportional to $f(d(z))$ is equivalent to flipping a coin with probability θ of heads and then choosing a vertex with probability proportional to $d(z) + 1$ if the coin is heads and uniformly at random otherwise. Thus, the parameter θ dictates people's preferences towards forming friendships with more popular people.

This is a variant of the original preferential attachment model of Barabási and Albert [12], which has been widely studied for randomly grown graphs, where it leads to a power-law degree distribution. See [13]–[16]. In contrast, here we use it to define a dynamic random graph with a fixed number of vertices and edges.

2 Results: Opinion-dependent rewiring

2.1 Stationary distribution

Let $v(G)$ and $e(G)$ be the number of vertices and edges of a graph G, and let $|e|$ be the length of the edge e. We assume throughout that $2M/N \to \lambda > 0$ as $N \to \infty$.

Theorem 1. *Conditional on the locations of the N vertices, the equilibrium distribution for the opinion-dependent rewiring model is given by*

$$\pi_1(G) = \begin{cases} C(\alpha, N, M) \prod_{e \in G} |e|^{-\alpha} & \text{if } v(G) = N,\ e(G) = M \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = 1$ for random rewiring, and $\alpha = 2$ for the MH dynamics.

To prove this result, we check in Section 4 that the detailed balance condition is satisfied. While it is nice to have an explicit formula for $\pi_1(G)$, if one wants to generate graphs with this distribution one must simulate the chain, which can be time consuming. To avoid this problem, we give another construction of π_1. Consider a percolation model on N vertices in which edge e is present with probability $g(|e|)$, independent of the other edges. Letting μ denote the probability measure on graphs in the percolation model, conditional on the locations of the vertices,

$$\mu(G) = \prod_{e \in G} \frac{g(|e|)}{1 - g(|e|)} \prod_{e} (1 - g(|e|)) \quad \text{if } v(G) = N.$$

The second product depends only on the set of vertices, so it can be absorbed into the normalizing constant. If we let

$$g(k) = \frac{b}{b + k^\alpha}, \quad \text{then} \quad \frac{g(|e|)}{1 - g(|e|)} = \frac{b}{|e|^\alpha}, \tag{1}$$

and $\pi_1(G)$ can be viewed as the probability of G under μ conditioned on the number of edges being M. To account for the fact that the number of edges under μ is random, we can choose

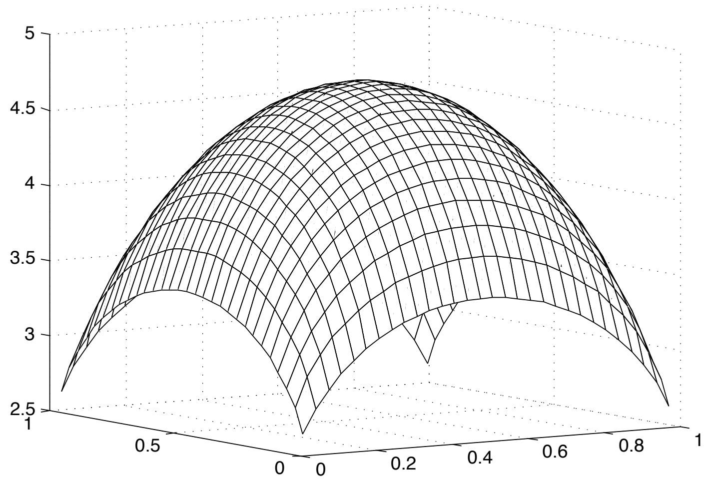

Figure 2: Expected degree function, $\lambda(x)$, for $d = 2$ and $\alpha = 1$ in the opinion-dependent rewiring model with mean degree $\lambda = 4$.

b' and b'' so that if $G' \sim \mu_{b'}$ and $G'' \sim \mu_{b''}$ then $\mathbb{E}e(G') = M(1 - M^{-1/3})$ and $\mathbb{E}e(G'') = M(1 + M^{-1/3})$ (note that b', b'' depend on the locations of the vertices, and the expectations are conditional on the locations). Then with high probability $e(G') < M < e(G'')$, so we can couple G, G' and G'' such that $G \sim \pi_1$ has exactly M edges, and every edge in G' is an edge in G and every edge in G is an edge in G''. By this coupling, it is sufficient to study the behavior of the degree distribution, giant component size and diameter for μ, as these features vary continuously as functions of the mean degree, and the mean degrees of G, G', and G'' will all approach the same limit.

We now choose $b(N, M)$ so that the expected value under μ of $e(G)$ is M and $2M/N \to \lambda$ as $N \to \infty$. Writing Q as short-hand for $[0, 1]^d$ we want

$$N^2 \int_Q \int_Q \frac{b}{b + |x - y|^\alpha}\, dy\, dx = 2M,$$

so $b(N, M) \to 0$. There are two cases with different behavior. Changing to polar coordinates, we see that

$$\int_{|z|<1} |z|^{-\alpha} dz \begin{cases} < \infty & \alpha < d \\ = \infty & \alpha \geq d. \end{cases}$$

In the first case $b(N, M) \sim c_\lambda / N$ and the expected degree of a vertex at x,

$$\lambda(x) = \lim_{N \to \infty} N \int_Q \frac{b}{b + |x - y|^\alpha}\, dy = c_\lambda \int_Q |x - y|^{-\alpha}\, dy, \tag{2}$$

is not constant. Figure 2 shows the function $\lambda(x)$ for $\alpha = 1$ and $d = 2$, where we numerically evaluated $c_1 \approx 0.336$, and $c_\lambda = \lambda c_1$ by integrating equation 2. If $\alpha \geq d$ we have $Nb(N, M) \to 0$, so most connections are to vertices at distance $o(1)$ from x and $\lambda(x) \equiv \lambda$ (on the interior of Q, where all vertices lie with probability 1).

Theorem 2. *If N is large then the degree distribution of the opinion dependent model is approximately Poisson with mean λ when $\alpha \geq d$, but a mixture of Poissons when $\alpha < d$.*

Figure 3 shows the degree distribution in $d = 2$ when $\alpha = 1$ and $\alpha = 2$. In the second case the observed degree distribution is close to Poisson as expected. This is also true in the first case even though Theorem 2 predicts a mixture of Poissons, which by Figure 2 involves means from 2.5 to 4.6. This is not a contradiction since the mixture turns out to be close to a Poisson. Computations for a simplified version of our situation in which $U \sim$ Uniform$(2.5, 4.6)$ and $(X|U = u) \sim$ Poisson(u), show a total variation distance of 0.024 between X and $Y \sim$ Poisson(3.54).

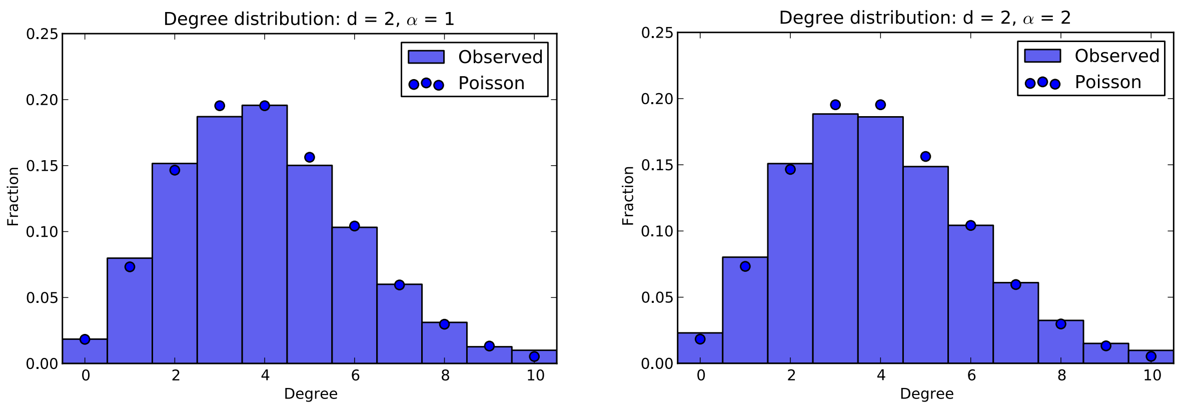

Figure 3: Degree distributions at equilibrium for the opinion-dependent rewiring model with (a) random rewiring and (b) Metropolis-Hastings rewiring in dimension 2 with $N = 5000$, $M = 10000$, averaged over 100 times.

2.2 Connectivity and diameter

The stationary distribution of the opinion-dependent rewiring model resembles long-range percolation, so we can derive results about connectivity and distances in the network at equilibrium from analogous statements about the percolation model. A vertex at x will connect to an average of $\lambda(x)$ other vertices with the ones chosen being distributed according to

$$\frac{c_\lambda}{\lambda(x)} \frac{1}{|x-y|^\alpha}.$$

When N is large the first stages of growth of the component of x are a multitype branching process in which the spatial location gives the type.

If we divide space into cubes of side $1/k$ and declare that each point in one of the small cubes Q_i gives birth like its midpoint x_i then we get a multitype branching process with a finite set of types. Letting M_{ij} be the mean number of children of type j from a parent of type i, the mean matrix for the nth generation is then given by the nth power of the matrix $M^n_{i,j}$. $M_{i,j}$ is a positive symmetric matrix so the entries grow exponentially at a rate ν^n given by the maximum eigenvalue $\nu = \max\{\|Mv\|_2 : \|v\|_2 = 1\}$ where $\|v\|_2 = (\sum_i v_i^2)^{1/2}$ is the usual measure of a vector's length.

One can analyze the original branching process with types in $Q = [0, 1]^d$ by discretizing and passing to the limit. Fortunately for us, Bollobás, Janson, and Riordan (BJR) [26] have already worked out the details in exactly the form we need. The vertices of their graph are

$x_1, x_2, \ldots x_n$, which in our case are chosen at random from Q. To make the connection with their notation let $\kappa(x,y) = c_\lambda/|x-y|^\alpha$ and make a connection from x_i to x_j with probability

$$p_{i,j} = \min\{1, \kappa(x_i, x_j)/N\}.$$

In the case $\alpha < d$,

$$\lambda(x) = \int_{[0,1]^d} \kappa(x,y)\, dy < \infty \quad \text{has} \quad \sup_x \lambda(x) < \infty$$

and the conditions of their Definition 2.7 are satisfied.

To determine conditions for the existence of a giant component, BJR introduce the operator

$$(T_\kappa f)(x) = \int_Q \kappa(x,y) f(y)\, dy$$

The kernel $\kappa(x,y)$ is the continuous analogue of the matrix $M_{i,j}$. Again we are interested in its maximal eigenvalue ν_κ:

$$\nu_\kappa = \max\{\|T_\kappa f\|_2 : \|f\|_2 = 1\}.$$

This quantity can also be described by a variational problem:

$$\nu_\kappa = \max\{\int_{Q\times Q} f(x)\kappa(x,y)f(y)\, dx\, dy : \|f\|_2 = 1\}.$$

Taking $f \equiv 1$ we see that $\nu_\kappa \geq \int_Q \lambda(x)\, dx = \lambda$, the average degree.

Theorem 3. *If $\nu_\kappa > 1$, then with high probability a giant component will exist, which contains a positive fraction of the vertices (see [26] Theorem 3.9), and the expected pairwise distance will be $\sim \log N/\log \nu_\kappa$ (see [26] Theorem 3.14).*

The results above take care of our two special cases $\alpha = 1, 2$ in $d \geq 3$ and $\alpha = 1$ in $d = 2$. Figure 4 shows the sizes of the giant component in a simulation of the case $\alpha = 1$, $d = 2$. Let λ_c be the critical value for percolation, such that a giant component of linear size exists with high probability when $\lambda > \lambda_c$, and with high probability all components are of size $o(n)$ when $\lambda < \lambda_c$. Our simulations suggest that $\lambda_c \approx 1$. Since $\nu_\kappa \geq \lambda$, it follows that $\lambda_c \leq 1$. Numerical computation of the eigenvalue ν_κ in this case suggests $\lambda_c \approx 0.98$, though this is difficult to discern from the simulations of the dynamic graph model. Figure 5 plots the diameter versus $\log N$ in the cases $d = 2$, $\alpha = 1$ and $d = 3$, $\alpha = 2$. In all cases $M = 2N$ so the average degree is 4. The dependence on $\log N$ is linear and is close to the slope of $1/\log(4) = 0.7213$ one would have if ν_κ was equal to the average degree.

To see what happens when $\alpha \geq d$, we fix $x \in (0,1)^d$. The number of vertices within distance $N^{-1/d}$ of x converges to a Poisson random variable, and is therefore $O(1)$. Thus, by truncating the integral and changing to polar coordinates, the expected number of edges

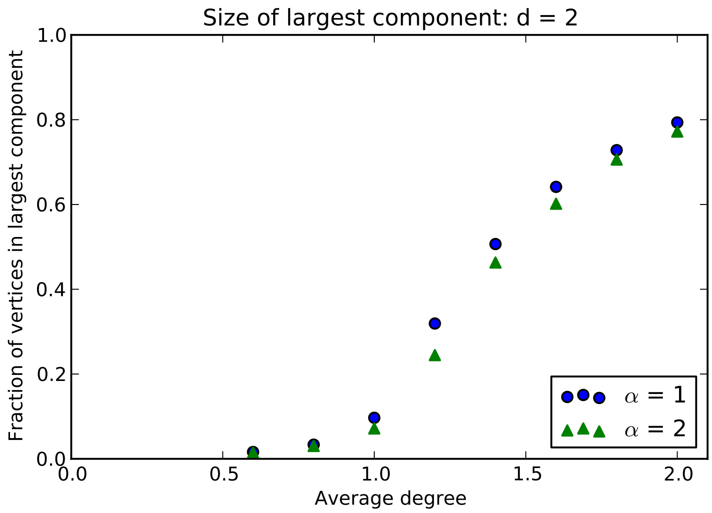

Figure 4: Fraction of vertices in the largest component at equilibrium for the opinion-dependent rewiring model with random rewiring ($\alpha = 1$) and Metropolis-Hastings rewiring ($\alpha = 2$) in dimension 2 with $N = 1000$, averaged over 100 times.

to the vertex at x in the percolation model is

$$\mathbb{E} \sum_{y \in G_N} \frac{b}{b + |x-y|^\alpha} \sim bN \int_{N^{-1/d} \le |z| \le 1} |z|^{-\alpha}\, dz$$
$$= bN \int_{N^{-1/d} \le r \le 1} r^{d-\alpha-1}\, C_d dr$$
$$\sim \begin{cases} bC_d N \log N & \alpha = d \\ bC_d N^{\alpha/d} & \alpha > d, \end{cases}$$

where the expectation is taken over the locations of the vertices in G_N, $C_2 = 2\pi$ and $C_1 = 2$. So to have mean degree λ we will take $b_\lambda = \lambda/(C_d N \log N)$ when $\alpha = d$ and $b_\lambda = \lambda/(C_d N^{\alpha/d})$ when $\alpha > d$.

Conjecture 1. *When $\alpha = d$, $\lambda_c = 1$. If $\lambda > 1$ the expected pairwise distance $\sim \log N/\log\lambda$.*

This conjecture is supported by our simulations. Figure 4 plots the size of the giant component when $\alpha = d = 2$ and suggests that $\lambda_c = 1$. Figure 5 plots the diameter versus $\log N$ in the case $d = 2$, $\alpha = 2$, and $M = 2N$. The dependence on $\log N$ is linear and is close to the predicted slope of $1/\log(4) = 0.7213$.

When $\alpha > d$, the probability of a connection from a vertex at x to a vertex at $x + kN^{-1/d}$ is

$$\frac{b}{b + (|k|\, N^{-1/d})^\alpha} = \frac{c_\lambda}{c_\lambda + |k|^\alpha}.$$

If we restrict our attention to $\alpha \in \{1, 2\}$, then the only example of this situation is $d = 1$, $\alpha = 2$. This is closely related to a model Aizenman and Newman [21] have studied on the integers. To be precise, they study a model in which the probability of an edge from x to

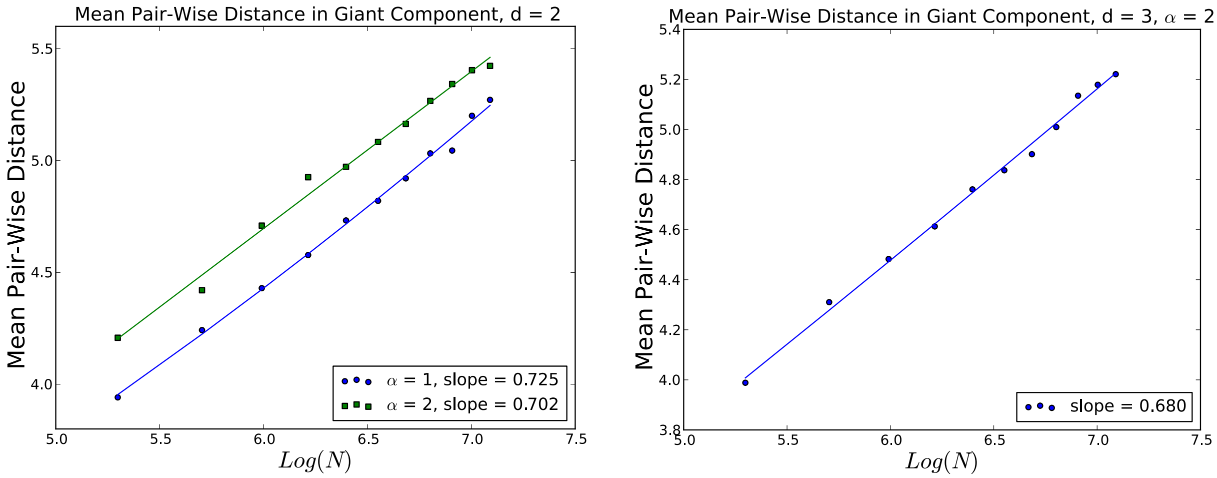

Figure 5: Diameter at equilibrium for the opinion-dependent rewiring model (a) in dimension 2 with random rewiring ($\alpha = 1$) and Metropolis-Hastings rewiring ($\alpha = 2$), and (b) in dimension with 3 with Metropolis-Hastings rewiring. In all cases $M = 2N$, and data points are averaged over 100 times.

y is $p < 1$ if $|x - y| < M$ and is $b/|x - y|^2$ if $|x - y| \geq M$. They show that if $b \leq 1$ there is no infinite component for any value of $p < 1$, while if there is an infinite component with density ρ then $b\rho^2 \geq 1$.

Benjamini and Berger [22] were first to study long-range percolation on the circle $\mathbb{Z}$ mod N. On this object the natural distance is $d(x, y) = \min\{|x - y|, N - |x - y|\}$. To avoid probabilities > 1 they supposed that the probability of an edge from x to y was $1 - \exp(-\beta d(x, y)^{-\alpha})$. Combining their results with later work we have the following results for the diameter of the model on the d-dimensional cube $\{1, 2, \ldots N\}^d$.

- [24, 25] If $d < \alpha < 2d$ then the diameter is $(\log N)^{\Delta + o(1)}$ where $\Delta = 1/\log_2(2d/\alpha)$.
- [23] If $\alpha = 2d$ then the distance is $\leq N^\eta$ where η depends on the constant β.

Note that since our α is an integer, the only overlap with the systems considered in this case occurs for $\alpha = 2$, $d = 1$, where the behavior of the diameter is only conjectured.

Figure 6 plots the distance versus N in the model with $d = 1$, $\alpha = 2$. The fitted curve is N^η with $\eta = 0.402$.

3 Results: Degree-dependent rewiring

3.1 Stationary distribution

Let $F(k) = \prod_{i=0}^{k-1} f(i)$ for $k \geq 1$ and let $F(0) = 1$. In Section 5 we prove the following result by checking that π_2 satisfies the detailed balance condition.

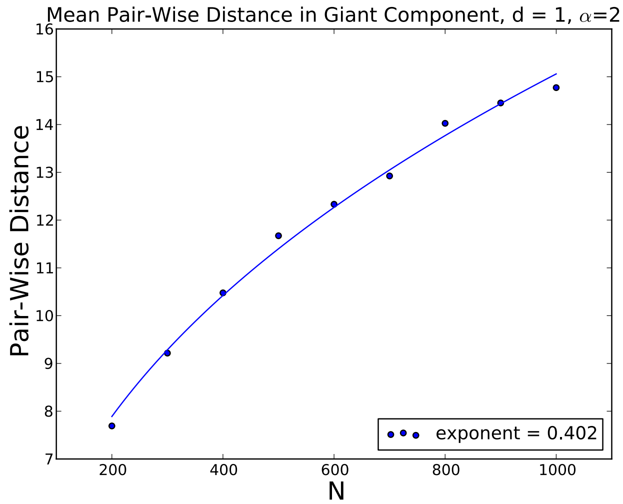

Figure 6: Distance versus N in the opinion-dependent rewiring model with $d = 1$, $\alpha = 2$. The fit curve is Distance $= N^{0.402}$.

Theorem 4. *Let d_i denote the degree of vertex i in the graph G. The stationary distribution for the degree-dependent rewiring model is given by*

$$\pi_2(G) = \begin{cases} c(\theta, N, M) \prod_{i=1}^{N} F(d_i) & \text{if } \sum_{i=1}^{N} d_i = 2M \\ 0 & \text{otherwise.} \end{cases}$$

The stationary distribution $\pi_2(G)$ only depends on the sequence of degrees, and is uniform over all graphs with the same degree sequence. This is also true for the configuration model, which has i.i.d. degrees $D_1, D_2, \ldots, D_N$. To build the graph from the degrees, one conditions on the sum $D_1 + D_2 + \cdots + D_N$ being even, attaches D_i half-edges to vertex i, and pairs the half-edges at random.

A graph generated by the configuration model can have self-loops or parallel edges, but if the degree distribution has finite second moment, there is positive probability that it does not [17]. A second difficulty in comparing with $\pi_2(G)$ is that we may not have $\sum_i D_i = 2M$. To avoid these problems we will consider a conditioned version of the configuration model.

In the configuration model, if $g(k)$ is the probability that a vertex has degree k and $\sum_k kg(k) = 2M/N$ then

$$P(D_1 = d_1, D_2 = d_2, \ldots, D_N = d_N) \sim c_1(M) \prod_{i=1}^{N} g(d_i),$$

where $c_1(M) \sim c/\sqrt{M}$. If G is a simple graph with the given degree sequence

$$P(G|D_1 = d_1, \ldots, D_N = d_N) = c_2(M) \prod_{i=1}^{N} d_i!,$$

where $1/c_2(M) = (2M)!/M!2^M$ is the number of ways of pairing the $2M$ half-edges. To see this note that the adjacency matrix of G will tell us the vertices that are neighbors of i, and then we have $d_i!$ ways of assigning the neighbors to the half-edges at i.

To make the connection between the degree-dependent rewiring model and the configuration model, we note that if we put γ^{d_i} inside the product and change the normalizing constant in $\pi_2(G)$ then we want

$$c'(\theta, N, M) \prod_{i=1}^{N} \frac{F(d_i)\gamma^{d_i}}{d_i!} = c_1(M)c_2(M) \prod_{i=1}^{N} g(d_i).$$

Thus, the degree-dependent rewiring model will look like the configuration model with

$$g(k) = c_\gamma F(k)\gamma^k/k!, \tag{3}$$

where the constants γ and c_γ are chosen to make the probabilities sum to one and the average degree $\bar{d} = 2M/N$. There is a unique solution because the distribution $g(k)$ is stochastically increasing in γ. At this point, the reader might worry that the conditioning will keep us from using the body of results that have been developed for the configuration model. Molloy and Reed [18] developed their results for the configuration model under the mild assumptions that, in the graph of size n, the degree sequence, $v_i(n) \geq 0$, had $\sum_i v_i(n) = 1$, $\sum_i iv_i(n)$ even, $i(i-2)v_i(n) \to i(i-2)p_i$ uniformly, and $\sum_i i(i-2)v_i(n) \to \sum_i i(i-2)p_i$ with the sum converging uniformly. Our model satisfies these conditions.

To understand the nature of g we begin with the extreme cases. When $\theta = 0$ our process reduces to random rewiring, so the degree distribution will be Poisson. To get this from the formulas above, note that $F(k) = (\bar{d}+1)^k$ so $c_\gamma = e^{-\gamma(\bar{d}+1)}$ and we take $\gamma = \bar{d}/(\bar{d}+1)$ to have the right mean degree. When $\theta = 1$, $F(k) = k!$, which cancels with the $k!$ in the denominator. This means we should take $c_\gamma = (1-\gamma)$ so that we have the shifted geometric distribution that takes values $k \in \{0, 1, 2 \ldots\}$. To have the right mean we set

$$\frac{1}{1-\gamma} - 1 = \frac{2M}{N} \quad \text{or} \quad \gamma = \frac{2M}{2M+N}.$$

The distributions for $0 < \theta < 1$ interpolate between the Poisson and geometric. Writing

$$f(i) = \theta \left(i + \frac{(1-\theta)\bar{d}+1}{\theta} \right) \equiv \theta(i+\kappa),$$

where the second equation defines κ, we have

$$F(k) = \theta^k \Gamma(k+\kappa)/\Gamma(\kappa),$$

where $\Gamma(r) = \int_0^\infty x^{r-1}e^{-r}\,dr$ is the usual gamma function that has $\Gamma(r) = (r-1)\Gamma(r-1)$. From this it follows that

$$g(k) = c_\beta \beta^k \Gamma(k+\kappa)/k!, \tag{4}$$

where $c_\beta = (1-\beta)^\kappa/\Gamma(\kappa)$ makes the probabilities sum to one and

$$\beta = \frac{2M}{2M+N}\theta$$

is chosen to give us the correct mean degree. Figure 7 shows the degree distribution for $\theta = 1/2$ is in agreement with the theoretical prediction. See Section 6 for the computation of c_β and β.

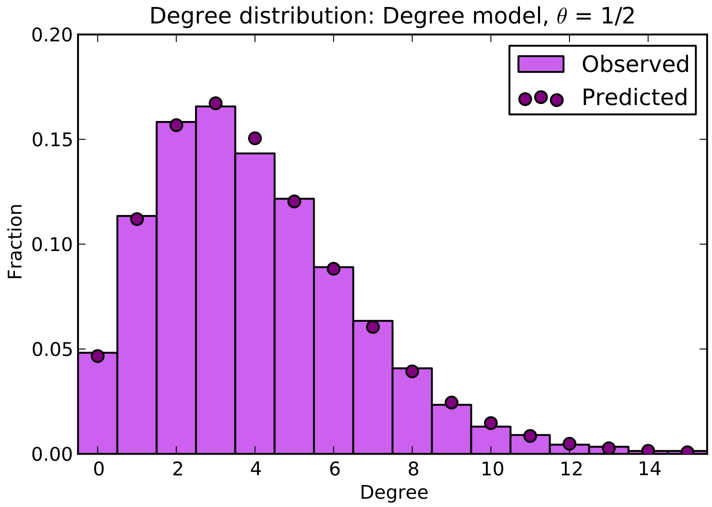

Figure 7: Observed and predicted equilibrium degree distribution for the degree-dependent rewiring model with $\theta = 1/2$, $N = 5000, M = 10000$.

3.2 Connectivity and diameter

To determine properties of the degree-dependent rewiring models, we consider properties of the configuration model, for which the condition for the existence of a giant component are simple and explicit. Let p_k be the degree distribution. Let $\mu = \sum_k kp_k$ be the mean degree, let $q_{k-1} = kp_k/\mu$ be the size-biased degree distribution, and let $\nu = \sum_j jq_j$ be its mean. Let $\phi(x) = \sum_x p_k x^k$, and $\psi(x) = \sum_j q_j x^j$ be the generating functions for the degree distributions, and let ρ be the smallest solution of $\psi(\rho) = \rho$ in $[0, 1]$.

Theorem 5. *If $\nu > 1$ then there is a giant component which contains a fraction $1 - \phi(\rho)$ vertices. The giant component has diameter $\sim \log_\nu(N)$.*

For a complete proof see Theorems 3.1.3 and 3.4.1 in [11]. To explain this result we recall the reasoning behind it. To see if there is a giant component we begin by examining the component containing 1. Vertex 1 will have j neighbors with probability p_j, but one of its neighbors will have degree k with probability kp_k/μ, since vertex 1 has k chances to connect to a vertex of degree k. In the early stages of examining the component containing 1, the number of vertices at distance m, Z_m, will be a branching process in which the average number of children in all generations after the first is ν. If $\nu > 1$ there is positive probability that the branching process does not die out, which corresponds to having a giant component. To compute the diameter we note that $\mathbb{E}Z_m = \mu\nu^{m-1}$ and $\mathbb{E}Z_m \approx N$ when $m = \log_\nu N = \log N / \log \nu$.

Figure 8 shows the sizes of the largest component in simulations for $\theta = 0, 1/2, 1$ as the mean degree is varied. Given the small size of the graph there is considerable run to run variability but there is good agreement with the theoretical calculations.

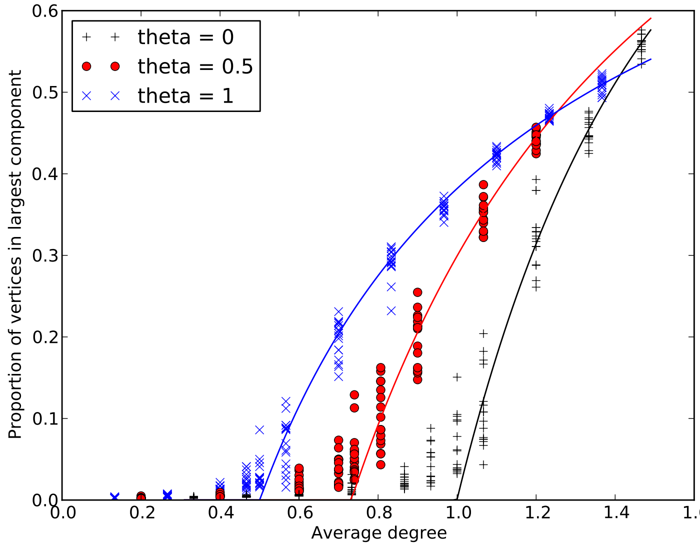

Figure 8: Fraction of vertices in the largest component at equilibrium for the degree-dependent rewiring model, $N = 3000$. Lines indicate the theoretical limiting curves for $\theta = 1, 0.5, 0$ from left to right. There are 15 independent data points shown for each set of parameter values.

4 Opinion-dependent rewiring model

For convenience, in this section we assume that each *oriented* edge (x, y) is chosen at a rate equal to its length. This speeds up the dynamics by a factor of 2 but has no effect on the stationary distribution. Consider graphs G and H that differ by one edge, such that $\{x, y\}$ is in G but not H, and $\{x, z\}$ is in H but not G. For a transition from G to H, the following must occur.

1. The oriented edge (x, y) is selected. This occurs at rate $d(x, y)$.

2. Vertex z is selected. This occurs with probability $1/(N - d(x) - 1)$ since $z \neq x$ and cannot be a neighbor of x.

3. The rewiring is accepted. This occurs with probability 1 for the random rewiring dynamics or $\min\{1, d(x, y)/d(x, z)\}$ for the Metropolis-Hastings dynamics.

Therefore the transition rate for the MH dynamics is

$$P(G, H) = \frac{d(x, y)}{N - d(x) - 1} \cdot \min\left(1, \frac{d(x, y)}{d(x, z)}\right), \tag{5}$$

with the second factor omitted in the random rewiring case.

Proof of Theorem 1. Consider first the random rewiring dynamics. To have detailed balance, we want to have

$$\left(\prod_{e \in E_G} \frac{1}{|e|}\right) \frac{d(x, y)}{N - d(x) - 1} = \left(\prod_{e \in E_H} \frac{1}{|e|}\right) \frac{d(x, z)}{N - d(x) - 1},$$

which holds since each side is equal to

$$\left(\prod_{e\in E_G\cap E_H}\frac{1}{|e|}\right)\frac{1}{N-d(x)-1}.$$

For the MH dynamics, suppose without loss of generality that $d(x,y)<d(x,z)$. To have detailed balance, we want to have

$$\left(\prod_{e\in E_G}\frac{1}{|e|^2}\right)\frac{d(x,y)^2/d(x,z)}{N-d(x)-1}=\left(\prod_{e\in E_H}\frac{1}{|e|^2}\right)\frac{d(x,z)}{N-d(x)-1}$$

which holds since multiplying by $d(x,z)$ makes each side is equal to

$$\left(\prod_{e\in E_G\cap E_H}\frac{1}{|e|}\right)\frac{1}{(N-d(x)-1)}.$$

□

Proof of Theorem 2. Assume first that the opinions are chosen in $[0,1]^d$ according to a Poisson process with intensity N, and consider the degree of the vertex at x. It is sufficient to consider μ and infer the result for π_1 by the aforementioned coupling between the edges conditioned on the locations of the vertices. In the percolation model, μ, x is initially attached to all other vertices, then these links are independently kept or removed according to $g(|e|)$ as in (1). This is an inhomogeneous thinning of the original Poisson process, so the distribution of the number of edges to x is Poisson with mean $\lambda(x)$ given by (2) if $\alpha<d$ and $\lambda(x)=2M/N$ otherwise. In our model, the N vertices are chosen uniformly in $[0,1]^d$, which can be viewed as a Poisson process conditioned to have exactly N points. To account for this, we consider two coupled Poisson processes with intensities $N(1\pm N^{-1/3})$, such that with high probability (as $N\to\infty$) the number of vertices in the two processes straddles N. For each Poisson process the degree of x converges to Poisson$(\lambda(x))$ as $N\to\infty$. □

5 Degree-dependent rewiring model

Recall that $f(i)=\theta\,(i+1)+(1-\theta)\,(\bar{d}+1)$, where i is the degree of a vertex, and $\bar{d}$ is the mean degree of the graph. The first step is to note:

$$\begin{aligned}\sum_x f(d(x))&=\theta\left(\sum_x d(x)+N\right)+(1-\theta)(N\,\bar{d}+N)\\&=\theta\,(2M+N)+(1-\theta)(2M+N)=2M+N,\end{aligned}\qquad(6)$$

since the sum of the degrees $=2M=N\bar{d}$.

Again consider graphs G and H that differ by one edge, with $\{x,y\}$ in G but not H, and $\{x,z\}$ in H but not G. For a transition from G to H, the following must occur.

1. The oriented edge (x,y) is selected. This occurs at rate 1.

2. Vertex z is selected. This occurs with probability $f(d(z))/(2M+N)$ by (6).

Let the degree of y in G be j and the degree of z in H be k. Thus, the degree of y in H is $j-1$, and the degree of z in G is $k-1$. Therefore the transition rate is

$$P(G,H) = \frac{f(k-1)}{2M+N}. \tag{7}$$

Let $F(k) = \prod_{i=0}^{k-1} f(i)$ for $k \geq 1$ and $F(0) = 1$.

Proof of Theorem 4. To have detailed balance we want

$$\frac{f(k-1)}{2M+N} \prod_{w \in G} F(d_G(w)) = \frac{f(j-1)}{2M+N} \prod_{w \in H} F(d_H(w)), \tag{8}$$

where d_G and d_H denote the degrees in the respective graphs. This holds since each side is equal to

$$\frac{F(j)F(k)}{2M+N} \prod_{w \neq y,z} F(d(w)).$$

To see this note that $F(i) = f(i-1)F(i-1)$. □

6 Equilibrium degree distribution

Proposition 1. *Let $0 < \theta < 1$, $\bar{d} = 2M/N$, $\kappa = [(1-\theta)\bar{d} + 1]/\theta$ and*

$$p_k = c_\beta \beta^k \Gamma(k+\kappa)/k! \qquad \text{where} \quad \beta = \frac{2M}{2M+N}\theta$$

then $c_\beta = (1-\beta)^\kappa/\Gamma(\kappa)$ makes the probabilities sum to 1, and $\beta = 2M/(2M+N)$ makes the mean $\bar{d}$.

Proof. Recalling $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx$

$$\sum_{k=0}^\infty \frac{\beta^k}{k!} \int_0^\infty e^{-x} x^{k+\kappa-1}\,dx = \int_0^\infty e^{-x(1-\beta)} x^{\kappa-1}\,dx = (1-\beta)^{-\kappa}\Gamma(\kappa)$$

where in the last step we have changed variables $y = x(1-\beta)$. From this we see that

$$c_\beta = (1-\beta)^\kappa/\Gamma(\kappa)$$

Using the same trick again, we have

$$\sum_{k=0}^\infty k\frac{\beta^k}{k!} \int_0^\infty e^{-x} x^{k+\kappa-1}\,dx = \int_0^\infty e^{-x(1-\beta)} x^{\kappa-1} \sum_{k=0}^\infty e^{-\beta x} k \frac{(\beta x)^k}{k!}\,dx$$
$$= \beta \int_0^\infty e^{-x(1-\beta)} x^\kappa\,dx = \beta(1-\beta)^{-(\kappa+1)}\Gamma(\kappa+1)$$

where in the second equality we have recalled the mean of the Poisson distribution. Combining the last two computations we see that the mean is

$$\frac{\beta}{1-\beta}\kappa = \frac{\beta}{1-\beta}\frac{(1-\theta)\bar{d}+1}{\theta}$$

Setting this $= \bar{d}$ we have

$$\frac{\beta}{1-\beta} = \frac{\theta\bar{d}}{(1-\theta)\bar{d}+1}$$

Cross-multiplying

$$\beta[(1-\theta)\bar{d}+1] = (1-\beta)\theta\bar{d}$$

so we want $\beta[\bar{d}+1] = \theta\bar{d}$ which means

$$\beta = \theta\frac{\bar{d}}{\bar{d}+1} = \theta\frac{2M}{2M+N}$$

which completes the proof. □

Using Proposition 1 we can explicitly compute the generating functions for the degree distribution, $\phi(x) = \sum_k p_k x^k$, and the size-biased degree distribution, $\psi(x) = \sum_k q_k x^k$, where $q_k = (k+1)p_{k+1}/\mu$ and $\mu = \sum_k kp_k = \bar{d}$.

Proposition 2. *If* $0 < \theta \leq 1$, $\kappa = [(1-\theta)\bar{d}+1]/\theta$, *and* $\beta = \frac{2M}{2M+N}\theta$ *then*

$$\phi(x) = \left[\frac{1-\beta}{1-\beta x}\right]^{\kappa}, \qquad \psi(x) = \left[\frac{1-\beta}{1-\beta x}\right]^{\kappa+1}.$$

Therefore,

$$\nu = \sum_{k=1}^{\infty} kq_k = \frac{(\kappa+1)\beta}{1-\beta}, \quad \textit{and}$$

$$\bar{d}_{crit} = \frac{-\theta+\sqrt{\theta^2-\theta+1}}{1-\theta},$$

where $\bar{d}_{crit}$ *is the critical value for the mean degree such that a giant component exists for* $\bar{d} > \bar{d}_{crit}$ *but not for* $\bar{d} < \bar{d}_{crit}$.

Proof. Using the expression for p_k from Proposition 1,

$$\begin{aligned}
\psi'(x) &= \frac{1}{\mu}\sum_{k=0}^{\infty}(k+1)k\,p_{k+1}x^{k-1} \\
&= \frac{1}{\mu}\sum_{k=1}^{\infty}\frac{(k+1)k\,c_\beta\beta^{k+1}\Gamma(k+\kappa+1)}{(k+1)!}x^{k-1} \\
&= \frac{1}{\mu}\sum_{k=1}^{\infty}\frac{k(k-1)\,c_\beta\beta^{k+1}\Gamma(k+\kappa)}{k!}x^{k-1} \\
&\quad + \frac{\kappa+1}{\mu}\sum_{k=1}^{\infty}\frac{k\,c_\beta\beta^{k+1}\Gamma(k+\kappa)}{k!}x^{k-1} \\
&= \beta x\,\psi'(x) + (\kappa+1)\beta\psi(x).
\end{aligned}$$

Solving this differential equation with boundary condition $\psi(1) = 1$ yields the desired expression for $\psi(x)$.

The derivation for $\phi(x)$ is analogous. Then by evaluating $\psi'(1)$ we obtain the expression for $\nu = \psi'(1)$, and setting $\nu = 1$ and solving gives the expression for $\bar{d}_{\text{crit}}$. □

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