Phase Transitions in the Quadratic Contact Process on Complex Networks

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Abstract

The quadratic contact process (QCP) is a natural extension of the well studied linear contact process where infected (1) individuals infect susceptible (0) neighbors at rate λ and infected individuals recover ($1 \longrightarrow 0$) at rate 1. In the QCP, a combination of two 1's is required to effect a $0 \longrightarrow 1$ change. We extend the study of the QCP, which so far has been limited to lattices, to complex networks. We define two versions of the QCP – vertex centered (VQCP) and edge centered (EQCP) with birth events $1-0-1 \longrightarrow 1-1-1$ and $1-1-0 \longrightarrow 1-1-1$ respectively, where '$-$' represents an edge. We investigate the effects of network topology by considering the QCP on random regular, Erdős-Rényi and power law random graphs. We perform mean field calculations as well as simulations to find the steady state fraction of occupied vertices as a function of the birth rate. We find that on the random regular and Erdős-Rényi graphs, there is a discontinuous phase transition with a region of bistability, whereas on the heavy tailed power law graph, the transition is continuous. The critical birth rate is found to be positive in the former but zero in the latter.

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I. INTRODUCTION

Inspired by technological and social networks, the study of complex networks has seen a surge in the past fifteen years [1–5]. Research has traditionally progressed in two distinct directions – dynamics *of* networks and dynamics *on* networks. The former is concerned with the formation of a network or change in its structure with time, whereas the latter deals with processes (deterministic or stochastic) taking place on a fixed network. Preferential attachment and its many generalizations [6, 7] are prototypical examples of the first type. Examples of the second are epidemics [8–11], the voter model for the spread of an opinion [12–14], cascades [15–17] that model spread of a technology, and evolutionary games [18]. The phase transitions [19, 20] associated with these models have been of particular interest.

In the mathematics community, spatial models are studied under the heading of *interacting particle systems* [21]. One of the simplest models of those models is the contact process [22–24] (equivalent to the SIS model in epidemiology). In the linear contact process each site can be in one of two states which we will call 1 and 0. 0's become 1 at a rate proportional to the number of 1 neighbors they have and 1's become 0 at a constant rate (here and in all following models, unless otherwise specified, the processes occur in continuous time).

A natural extension of the linear process is the quadratic contact process (QCP) where each $0 \to 1$ event will require two other sites in state 1. We will occasionally refer to 1 as being the "occupied" state and 0 as being "vacant", and the events $0 \to 1$ and $1 \to 0$ to be birth and death events respectively. At this stage, the model is quite general in that we do not specify where the two 1's that cause the $0 \to 1$ event must be located with respect to the 0. On the 2D lattice, specifying these locations leads to different realizations of the QCP. For example, Toom's North-East-Center model (originally defined in discrete time) allows a 0 at site x to be filled if its neighbors $x + (0, 1)$ and $x + (1, 0)$ are occupied [25]. Chen [26, 27] has studied versions of Toom's model in which two or three specified adjacent pairs or all four adjacent pairs are allowed to reproduce. Evans, Guo and Liu [28–32] have studied the QCP as a model for adsorption-desorption on a two dimensional square lattice. In the version of the model studied by Liu [32], 0 becomes 1 at rate proportion to the number of adjacent pairs of 1 neighbors. He found a discontinuous phase transition with a region of bistability, where the 1's die out starting from a small density. He also found that by introducing spontaneous births at a sufficiently high rate, the transition becomes continuous.

The QCP is similar to Schlögl's second model [33] of autocatalysis characterized by chemical reactions $2X \longrightarrow 3X, X \longrightarrow \o$ where X represents the reactant. Grassberger [34] studied a version of Schlögl's second model in which each site has a maximum occupancy of two and doubly occupied sites give birth to a neighboring vacant site. He found that the model shows a continuous phase transition in 2D.

Studies to date on the QCP have been limited to regular lattices in low dimensions. In this paper, we extend the study to complex networks. There are two ways to view the QCP on networks:

- as a model that replaces the linear birth rate of the contact process that has been extensively studied on networks [35, 36], by a quadratic birth rate.
- as an alternative model for the spread of rumors, fads and technologies such as smart phones in a social network. In sociology the requirement of more than a single 1 for the "birth" event is called complex contagion [37]. Also related are the threshold contact process [38] and models for the study of "cascades" [16]. The key difference here is that the QCP involves a death event that represents the loss of interest in the fad or technology and the rate for birth events is a function of the actual number and not the fraction of occupied neighbors.

The questions we are interested in are: How does network topology affect these phase transitions? What model and network features lead to discontinuous versus continuous phase transitions?

The paper is organized as follows. We define the specific QCP that we study in Section II and we do mean field calculations in Section III. In Section IV we present a few rigorous results about the QCP. Simulation results are presented in Section V, followed by some concluding remarks in Section VI.

II. MODEL DEFINITION

The birth event in the linear contact process can be formulated as each $1-0$ edge converts to a $1-1$ edge at a constant rate λ. Such a definition can be easily extended to the quadratic case by defining the birth event in terms of connected vertex triples. Two such definitions are possible: $1-0-1 \longrightarrow 1-1-1$, and $1-1-0 \longrightarrow 1-1-1$. We call the former version the vertex centered QCP (VQCP) because the central 0 *vertex* is getting filled by its two neighboring 1s, and the latter as the edge centered QCP (EQCP) as it can be viewed as a $1-1$ *edge* giving birth on to a neighboring vacant vertex. Note that the models can also be defined in terms of how a vacant vertex gets filled i.e., suppose that a 0 vertex has k 1 neighbors and j $1-1$ neighbors [39], then the 0 vertex will become 1 at rates $\binom{k}{2}\lambda$ and $j\lambda$ in the VQCP and EQCP respectively. Death events $1 \longrightarrow 0$ occur at rate 1 as in the linear process.

If the death rate is changed to zero, the VQCP reduces to bootstrap percolation [40] where vertices that are occupied remain occupied forever and vacant vertices that have at least two occupied neighbors become occupied. While bootstrap percolation is typically defined in discrete time, the final configuration of the network is independent of whether the dynamics happens in discrete or continuous (as in our model) time.

We will use random graphs as models for complex networks on which the QCP is taking place. We will denote by d the degree of a randomly chosen vertex in the network and the degree distribution by $p_k = \mathbb{P}(d = k)$. We are interested in networks with size $n \to \infty$ and where the vertex degrees are uncorrelated. The specific random graphs that we will consider are

- *Random regular graphs* $\mathrm{RR}(\mu)$ in which each vertex has degree μ. Since everyone has exactly μ friends, this graph is not a good model of a social network. However, the fact that it looks locally like a tree will facilitate proving results.

- *Erdős-Rényi random graphs* $\mathrm{ER}(\mu)$ where each pair of vertices is connected with probability μ/n. In the $n \to \infty$ limit, the degree distribution of the limiting graph is Poisson with mean μ. This is a prototypical model for the situation in which the degree distribution has a rapidly decaying tail.

- *Power law random graphs* $\mathrm{PL}(\alpha)$ with degree distribution $p_k = ck^{-\alpha}$. We are particularly interested in graphs where the exponent α lies between 2 and 3, which has been found to be the case for many real world networks [41]. We construct our graphs using the configuration model, so the degrees are uncorrelated.

We will occasionally refer to RR and ER as homogeneous networks as their degree distributions are peaked around the mean, in contrast to PL where the distribution has a heavy tail.

III. MEAN FIELD CALCULATIONS

We can attempt an analytical study of the dynamics by writing the equations for the various *moments* of the network. Let g be a small graph labeled with 1's and 0's. We define the $g-$moment, written as $\langle g\rangle$, of a $\{0,1\}$ valued process on a graph G as the expected number of copies of g that exist in the set of all subgraphs of G. For example if $g = 1-0-1$, we look at all the connected vertex triples in the network and count the ones where the center vertex is in state 0 and the other two vertices are in state 1. We will write $\rho(\lambda, \rho^{(0)}; t)$ as the density $\langle 1\rangle/n$ at time t with a birth rate of λ and an initial configuration where each vertex is independently occupied with a probability $\rho^{(0)}$. The order parameter for our phase transitions is the steady state density

$$\rho_*(\lambda, \rho^{(0)}) = \lim_{t\to\infty} \rho(\lambda, \rho^{(0)}; t)\,. \tag{1}$$

We define the critical birth rate λ_c as the birthrate above which there exists a stable steady state density that is greater than zero, i.e.,

$$\lambda_c = \inf\{\lambda : \rho_*(\lambda, 1) > 0\}\,. \tag{2}$$

In the definition above, we chose $\rho^{(0)} = 1$ since it has the best chance of having a positive limit. We also define a critical initial density ρ_c as the minimum initial density required to reach a positive steady state density when the birth rate is infinite, i.e.,

$$\rho_c = \inf\{\rho^{(0)} : \lim_{\lambda\to\infty} \rho_*(\lambda, \rho^{(0)}) > 0\}\,. \tag{3}$$

From their definitions, it is straight forward to write the dynamical equations of $\langle 1\rangle$ for the VQCP and the EQCP,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle 1\rangle = -\langle 1\rangle + \lambda \begin{cases} \langle 1-0-1\rangle & \text{for the VQCP} \\ \langle 1-1-0\rangle & \text{for the EQCP} \end{cases}\,. \tag{4}$$

If we were to write the equations for the third order moments that appear on the RHS of (4), those equation would involve still higher order moments. Continuing this way, we end

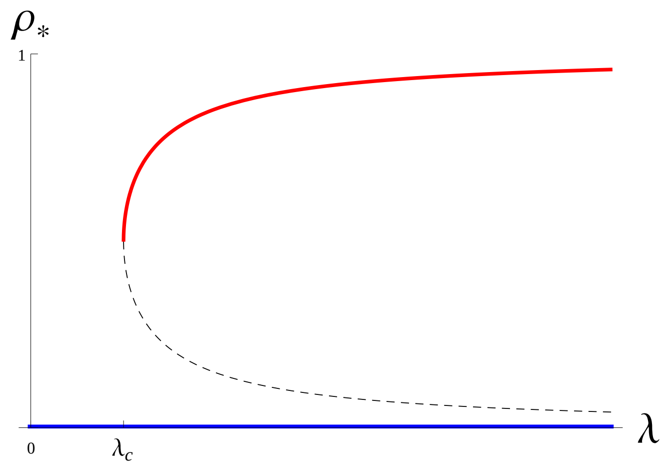

FIG. 1. The blue, dashed and red curves correspond to $\rho_* = 0$, ρ_- and ρ_+ respectively obtained from the mean field calculation for both QCP types on homogeneous networks.

up with an infinite series of equations that are not closed. Therefore we resort to a mean field approximation by assuming the states of neighbors of a vertex to be independent at all times.

A. Homogeneous networks

In the following we do a naive calculation that ignores the correlation between degree and occupancy, which should be reasonable for homogeneous networks. With these assumptions, $\langle 1-0-1 \rangle$ will be $n\rho^2(1-\rho)\langle\binom{d}{2}\rangle$. Plugging this value into (4) we get

$$\dot{\rho} = -\rho + \lambda\rho^2(1-\rho)\left\langle\binom{d}{2}\right\rangle . \tag{5}$$

Setting the RHS of (5) to zero gives a cubic equation whose roots are the possible steady state densities ρ_*. Clearly, zero is a trivial root of (5). The other two roots are

$$\rho_\pm = \frac{1}{2}\left[1 \pm \sqrt{1-\frac{\lambda_c}{\lambda}}\right] . \tag{6}$$

These solutions are real only when $\lambda > \lambda_c = 4/\langle\binom{d}{2}\rangle$. In the language of nonlinear dynamics, (5) exhibits a saddle node bifurcation at λ_c. It is easy to see that zero and ρ_+ are stable fixed points whereas ρ_- is an unstable fixed point. This can be seen in Fig. 1. The limiting critical initial density is

$$\rho_c = \lim_{\lambda \to \infty} \rho_- = 0\,. \tag{7}$$

For ER(μ) we have $\langle \binom{d}{2} \rangle = \mu^2/2$ which gives $\lambda_c = 8/\mu^2$. For PL($\alpha \le 3$) we have $\langle \binom{d}{2} \rangle = \infty$ so $\lambda_c = 0$ while PL($\alpha > 3$) has finite $\langle \binom{d}{2} \rangle$ leading to a non-zero value for λ_c. The mean field calculation for the EQCP is essentially the same as done above and predicts the same qualitative features. Thus, for networks with finite $\langle \binom{d}{2} \rangle$, the simple mean field calculation predicts a discontinuous phase transition at $\lambda = \lambda_c$ and a region of bistability for $\lambda > \lambda_c$, for both QCP types.

B. Heavy tailed degree distributions

The mean field calculation of Section III A is simplistic since it ignores the fact that the occupancy probability depends on the degree. Pastor-Satorras and Vespignani [8] improved the mean field approach for the linear contact process by defining ρ_k, the fraction of vertices of degree k that are occupied, and θ, the probability that a given edge points to an occupied vertex. These variables can be related through the size biased degree distribution $q_k = kp_k/\langle d \rangle$ which is the distribution of the degree of a vertex at the end of a randomly chosen edge.

$$\theta = \sum_k q_k \rho_k \tag{8}$$

Note that for homogeneous networks we assumed $\theta = \rho$. As before, the state of the neighbors of a vacant vertex are assumed to be independent. So the number of occupied neighbors of a vertex of degree k follow a distribution Binomial(k, θ). This enables us to apply this approach to the VQCP. We write equations for ρ_k,

$$\dot{\rho}_k = -\rho_k + \lambda(1 - \rho_k)\binom{k}{2}\theta^2\,. \tag{9}$$

So in steady state

$$\rho_{k*} = \frac{\lambda \binom{k}{2} \theta_*^2}{1 + \lambda \binom{k}{2} \theta_*^2}\,. \tag{10}$$

Combining (8) and (10) leads us to a self-consistent equation for θ_*.

$$\theta_* = \theta_* I(\lambda, \theta_*)\,, \tag{11}$$

where

$$I(\lambda,\theta) = \sum_k^\infty \frac{kp_k}{\langle d\rangle}\left[\frac{\lambda\binom{k}{2}\theta}{1+\lambda\binom{k}{2}\theta^2}\right]\,. \tag{12}$$

Clearly, $\theta_* = 0$ is a solution of (11). Finding a non-trivial solution involves solving

$$I(\lambda,\theta_*) = 1 \quad , \quad \theta_* \in (0,1)\,. \tag{13}$$

For power law graphs PL(α), the mean field calculation predicts

- If $\alpha > 3$, $\lambda_c > 0$ and the transition is discontinuous.
- If $\alpha = 3$, $\lambda_c > 0$ and the transition is continuous.
- If $2 < \alpha < 3$, $\lambda_c = 0$, the transition is continuous, and $\rho_*(\lambda) \sim C\lambda^{\gamma(\alpha)}$

We put the details in the appendix.

A second way to determine the nature of the phase transition is to adapt the argument of Gleeson and Cahalane [16], which can be applied if we use a discrete time version of the model in which a vertex with k neighboring pairs will be occupied at the next step with probability $1-(1-p)^k$. The computation in their formulas (1)–(3) supposes that the vertices at a distance n from x are independently occupied with probability ρ_0. The function $G(\rho)$ defined in their (3) gives the occupancy probabilities at distance $k-1$ assuming that the probabilities at distance k are ρ. Iterating G n times and letting $n \to \infty$ gives a prediction about the limiting density in the cascade. if one repeats the calculation for our system then 0 is an unstable fixed point when $\alpha < 3$, while it is locally attracting for $\alpha > 3$. This agrees with the mean-field prediction of $\lambda_c = 0$ in the former case and a discontinuous transition with $\lambda_c > 0$ in the second.

IV. SOME RIGOROUS RESULTS

We have not been able to extend the mean field calculation to the EQCP on power law graphs, but by generalizing an argument of Chatterjee and Durrett [35] we can prove that

$\lambda_c = 0$ for $\alpha \in (2, \infty)$. The details are somewhat lengthy, so we only explain the main idea. Consider a tree in which the vertex 0 has k neighbors and each of its neighbors has l neighbors and l is chosen so that $l\lambda \geq 10$. One can show that if k is large then with high probability the infection will persist on this graph for time $\geq \exp(c(\lambda)k)$. In a power law graph one can find such trees with $k = n^{1/(\alpha-1)}$. Using the prolonged persistence on these trees as a building block one can easily show that if we start with all vertices occupied the infection persists for time $\geq \exp(n^{1-\epsilon})$ with a positive fraction of the vertices occupied. With more work (see [42, 43]) on can prove persistence for time $\exp(c(\lambda)n)$.

For both types of QCP it is easy to show that it is impossible to have a discontinuous transition with $\lambda_c = 0$. The proof for VQCP is as follows. Let $\langle 1_k \rangle$ be the expected number of occupied sites of degree k and $\langle 10_k 1 \rangle$ be the expected number of 1-0-1 triples when the 0 vertex has degree k. We can write an equation similar to (4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle 1_k \rangle = -\langle 1_k \rangle + \lambda \langle 1 - 0_k - 1 \rangle \tag{14}$$

which means at steady state

$$\begin{aligned} \langle 1_k \rangle_* = \lambda \langle 1 - 0_k - 1 \rangle_* \leq \lambda \langle 0_k \rangle_* \binom{k}{2} \\ \Rightarrow \rho_{k*} \leq \lambda (1 - \rho_{k*}) \binom{k}{2} \Rightarrow \rho_{k*} \leq \frac{\lambda \binom{k}{2}}{1 + \lambda \binom{k}{2}}\,. \end{aligned} \tag{15}$$

So, as $\lambda \to 0$, $\rho_{k*} \to 0$ and $\rho_* = \sum_k \rho_{k*} p_k \to 0$. Thus the transition will be continuous. The proof for EQCP is similar. In that case the subscript k stands for the secondary degree $d^{(2)}$ which is defined as the number of neighbors of neighbors of a given vertex (not including itself), i.e., $d^{(2)}(x) = |\{z : z \sim y, y \sim x, z \neq x\}|$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle 1_k \rangle = -\langle 1_k \rangle + \lambda \langle 1 - 1 - 0_k \rangle \tag{16}$$

So at steady state

$$\langle 1_k \rangle_* = \lambda \langle 1 - 1 - 0_k \rangle_* \leq \lambda \langle 0_k \rangle_* k \Rightarrow \rho_{k*} \leq \frac{\lambda k}{1 + \lambda k} \tag{17}$$

Thus for both QCP types we find that if $\lambda_c = 0$ then the phase transition is continuous.

For both QCP types on random r-regular graphs we can show that the critical birth rate is positive as follows. In the EQCP let there be m occupied vertices. Each of these m

vertices can have at most r neighbors that are vacant and can give birth on to them at rate $\leq (r-1)\lambda$ or die at rate 1. So the total birth rate in the network is $\leq (r-1)\lambda rm$ against a death rate of m, and it follows that $\lambda_c > 1/r(r-1)$. Similarly, for the VQCP the total birth rate is $\leq \lambda\binom{r}{2}rm$ and it follows that $\lambda_c > 1/r\binom{r}{2}$. These arguments depend on the degree being bounded, so they do not work for Erdős-Rényi and power law graphs.

V. SIMULATION RESULTS

We perform simulations of the QCP on RR(4), ER(4) and PL(2.5). We generate the random regular and power law random graphs using the recipe called *configuration model* [44]. We draw samples d_x from the degree distribution and attach that many "half-edges" to vertex x. We pair all the half edges in the network at random. We then delete all self loops and multiple edges. When $\alpha > 2$ this does not significantly modify the degree distribution. If $\sum_x d_x$ turns out to be odd (an event with probability $\approx \frac{1}{2}$), we ignore the last remaining unpaired half-edge. Furthermore, for PL(2.5) we start the degree distribution at 3 as, in the VQCP, the vertices of degree 1 and 2 are impossible or difficult to get occupied.

To deal with finite size effects, we observe how the plot of the steady state density $\rho_*(\lambda, 1)$ versus λ starting with all vertices occupied changes when size n of the network ranging from 10^3 to 10^5. Fig. 2 shows the results of both QCP types on RR(4) and ER(4). Here the curves seem to converge to a positive value implying a positive λ_c. The results for PL(2.5) are shown in Fig. 3. We observe that the transition happens close to zero and moves towards zero with increasing n indicating that the critical birth rate is zero. As explained earlier, if $\lambda_c = 0$ then the transition is continuous. This is consistent with the the mean field predictions for the VQCP and rigorous result for EQCP. In addition, in Fig. 3(a), the critical exponent for the $n = 10^5$ curve can be measured to be approximately 1.45 which is close to the mean field value of 1.5 (obtained by setting $\alpha = 2.5$ in (A.4)).

In order to further investigate the phase transitions in random regular and Erdős-Rényi graphs, we look at the steady state density attained by starting from two different initial densities for the same network size $n = 10^5$. Fig. 4 again shows a similar pattern across both QCP and both network types. In Fig. 4(b) we see that for birth rates between 0.9 and 2.3, the VQCP survives when the starting configuration had all vertices occupied but dies out when starting with only one-tenth of the vertices occupied. Thus we see bistability

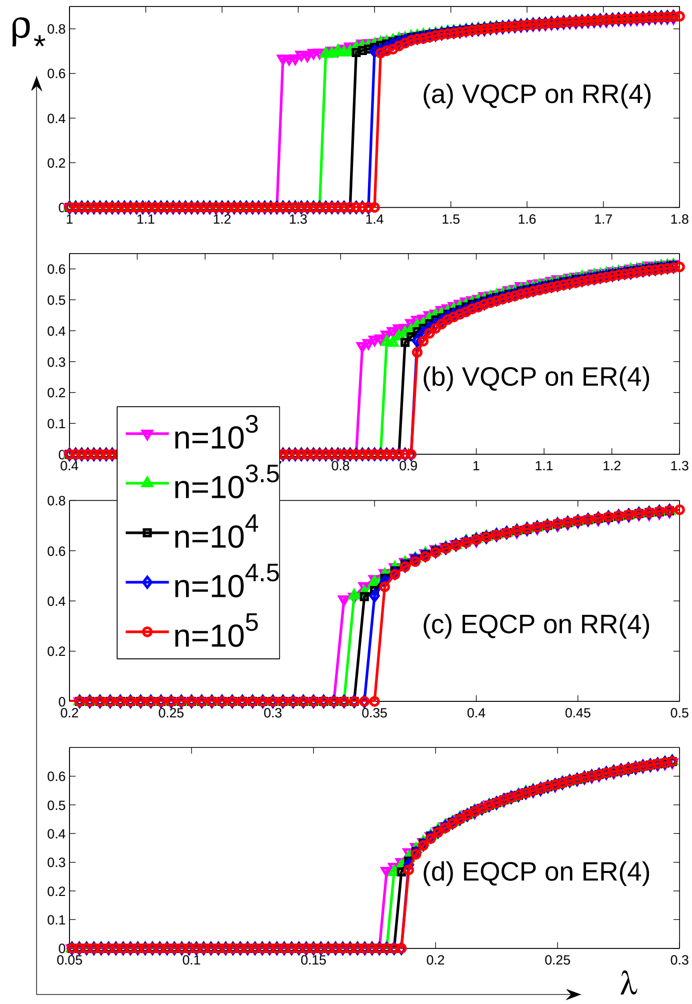

FIG. 2. Steady state density reached, starting from all vertices occupied, for QCP on homogeneous networks of various sizes n.

in the region $\lambda \in (0.9, 2.3)$ implying a discontinuous transition and consequently that λ_c is positive and close to 0.9. This is qualitatively in agreement with the mean field prediction seen in Fig. 1, although the critical birth rate of 0.9 shows a deviation from the mean field value of $8/4^2 = 0.5$.

Fontes and Schonmann [45] have shown that for bootstrap percolation on the tree there is a critical density p_c so that if the initial density is $< p_c$ then the final bootstrap percolation configuration has no giant component of occupied sites. In this situation having deaths at a positive rate in the VQCP will lead to an empty configuration. The last argument is for the tree, but results of Balogh and Pittel [46] show that similar conclusions hold on the random regular graph. While this argument is not completely rigorous, the reader should note that since all of the VQCP are dominated by bootstrap percolation, it follows that the limiting

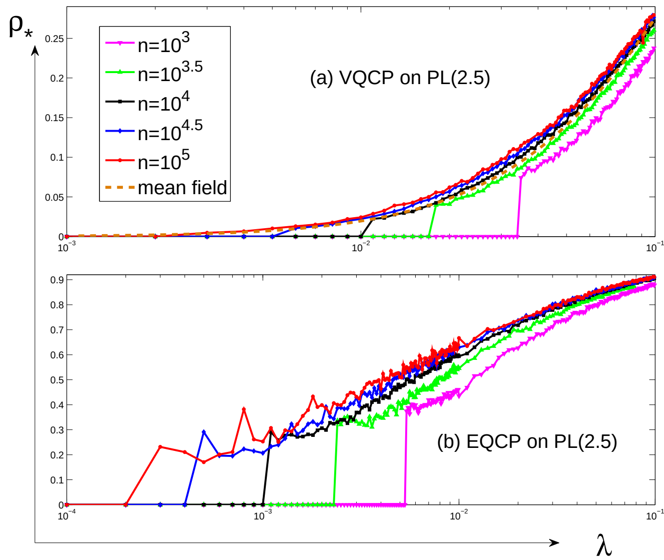

FIG. 3. Steady state density reached, starting from all vertices occupied, for QCP on power law networks of various sizes n. Note that the λ axis is in the log scale.

critical initial density defined in (3) has $\rho_c > 0$ in contrast to the mean field prediction in (7). Fig. 5 shows the final density attained as a function of the initial density when the birth rate is infinite (and death rate is positive). We see that ρ_c in the VQCP is positive for the random regular and Erdős-Rényi graphs whereas it is zero for the power law graph. The corresponding results (not shown here) in the case of the EQCP indicate that $\rho_c = 0$ for random regular, Erdős-Rényi and power law random graphs.

VI. CONCLUSION

In this paper we have investigated the properties of two versions of the quadratic contact process on three types of random graphs. The mean field calculations we performed agree qualitatively with the simulation results. This may be due to the fact that complex networks have exponential volume growth and therefore are like infinite dimensional lattices where mean field is exact.

Table I summarizes what is known about the phase transitions of contact processes in 1 and 2 dimensional lattices and on the random graphs RR, ER and PL. The positivity of the critical birth rate for 1D, 2D and RR follows trivially from the boundedness of their degrees. For VQCP on a 1D lattice, two consecutive 0's can never get filled and it follows that $\lambda_c = \infty$. The results for the linear process on RR are inferred from the rigorous results

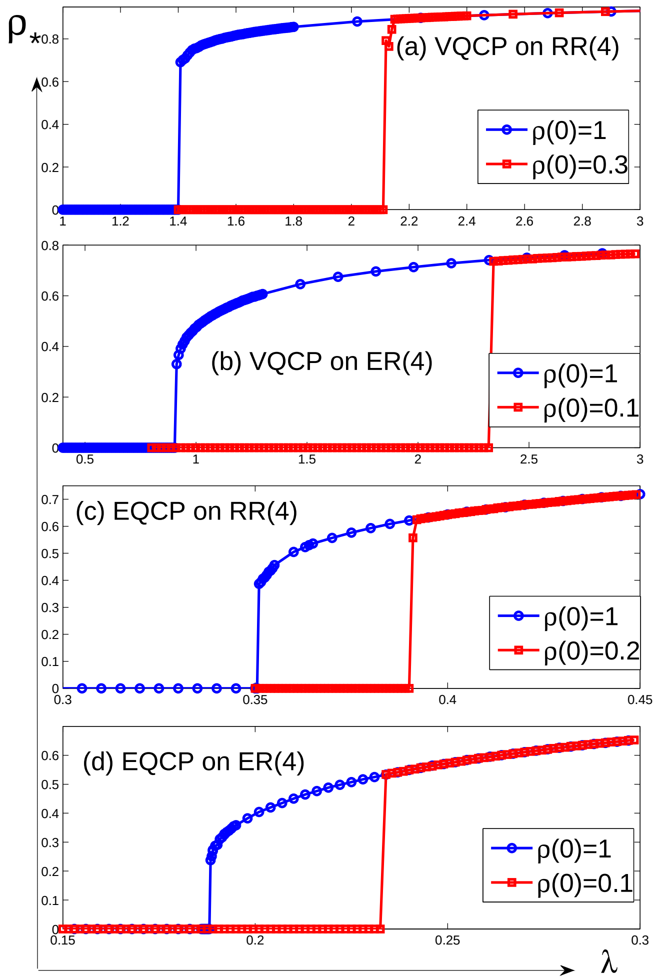

FIG. 4. Steady state density reached, starting from two different initial densities $\rho^{(0)}$, for QCP on homogeneous networks of size $n = 10^5$. Notice the similarity with the mean field prediction on Fig. 1.

for trees and the fact that RR is locally tree like.

The results indicate that the EQCP is qualitatively not very different from the linear contact process on low dimensional lattices and power law graphs, in contrast to the VQCP, which differs from its low dimensional analogue. In view of the fact that they are very different in how they fill vacant vertices on a network, the similarity between VQCP and EQCP in their phase transitions on complex networks is a little perplexing.

The EQCP can easily propagate on a chain and "cross bridges" connecting communities, compared to the VQCP which always requires two occupied neighbors. In the EQCP vertices with a large number of neighbors of large degree are the key to its survival. However, in the VQCP it is impossible for the central vertices to repopulate the leaves, so these structures

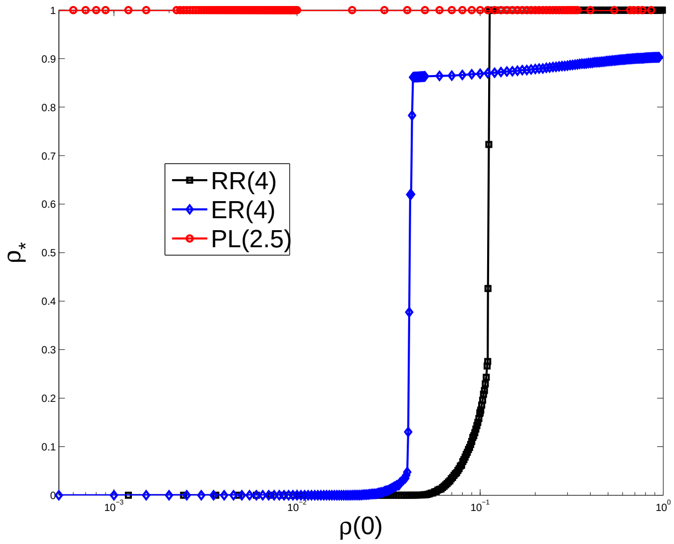

FIG. 5. Steady state density when the birth rate is infinite in the VQCP on various networks of size $n = 10^5$. Note that the $\rho^{(0)}$ axis is in the log scale.

	Linear CP	Vertex QCP	Edge QCP
1D	cont. , + [47, 48]	NA , ∞	cont. , + [49]
2D	cont. , + [48, 50]	discon. , + [32]	cont. , + [34]
RR	cont. [51] , + [52]	discon. , +	discon. , +
ER	cont. , + [36]	discon. , +	discon. , +
PL((2,3))	cont. , 0 [35]	cont. , 0	cont. , 0
PL(3)	cont. , 0 [35]	cont. , +	cont. , 0
PL((3,∞))	cont. , 0 [35]	discon. , +	cont. , 0

TABLE I. Nature of phase transitions of contact processes on various networks. Note that '0','+' and '∞' stand for zero, positive and infinite values respectively of λ_c.

are not long lasting. In contrast, the Gleeson-Cahalane calculation suggests that survival is due to the fact that as waves of particles move through the system the densities increase.

Appendix

Our task is to find the solutions of $I(\lambda, \theta) = 1$, with $\theta \in (0, 1)$ where

$$I(\lambda, \theta) = \sum_{k=1}^{\infty} \frac{kp_k}{\langle d \rangle} \left[\frac{\lambda \binom{k}{2} \theta}{1 + \lambda \binom{k}{2} \theta^2} \right]$$

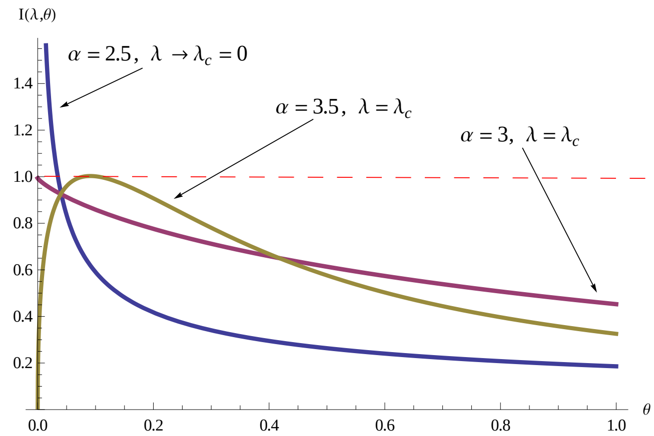

FIG. 6. $I(\lambda,\theta)$ versus θ near $\lambda=\lambda_c$ for various power law graphs.

To simplify the computation, C will denote a positive finite constant whose value is not important and that may change from line to line. In what follows $a \sim b$ means $a/b \to 1$

To begin, we note that if $p_k \sim Ak^{-\alpha}$ with $\alpha > 4$ then using the fact that the denominator $1+\lambda\binom{k}{2}\theta^2 \geq 1$,

$$I(\lambda,\theta) \leq \frac{\theta\lambda}{2\langle d\rangle}\sum_k Ck^{3-\alpha} \to 0 \tag{A.1}$$

as $\theta \to 0$. When $3 < \alpha \leq 4$ we break the sum at $k = \lfloor 1/\theta \rfloor$ where $\lfloor x \rfloor$ is the largest integer $\leq x$. Lower bounding the denominator by 1 in the first sum and by $\lambda\binom{k}{2}\theta^2$ in the second

$$I(\lambda,\theta) \leq \frac{\theta\lambda}{2\langle d\rangle}\sum_{k=1}^{\lfloor 1/\theta \rfloor} Ck^{3-\alpha} + \frac{1}{\theta\langle d\rangle}\sum_{k=\lfloor 1/\theta\rfloor+1}^{\infty} Ck^{1-\alpha} \to 0 \tag{A.2}$$

as $\theta \to 0$, since $\sum_k k^{1-\alpha} < \infty$ and $\sum_{k=1}^{\lfloor 1/\theta\rfloor} k^{3-\alpha} \sim Ck^{4-\alpha}$ and $4-\alpha < 1$. Since $I(\lambda,1) < 1$ for all λ, the curve will have $\sup_{\theta\in[0,1]} I(\lambda_c,\theta) = 1$ at some λ_c. For $\lambda > \lambda_c$ there will be two roots, the larger of which is the relevant solution since we must have $\theta(\lambda) > \theta(\lambda_c)$.

When $2 < \alpha \leq 3$, changing variables $k = x/\theta$ and then approximating the sum by an integral we have that as $\theta \to 0$

$$\begin{aligned} I(\lambda,\theta) &\sim C\sum_{x\in\theta\mathbb{Z}^+} (x/\theta)^{1-\alpha}\frac{\lambda\binom{x/\theta}{2}\theta}{1+\lambda\binom{x/\theta}{2}\theta^2} \\ &\sim C\theta^{\alpha-3}\int_0^\infty x^{1-\alpha}\frac{\lambda x^2}{1+\lambda x^2}\,dx\,. \end{aligned} \tag{A.3}$$

From this we see that if $2 < \alpha < 3$ then as $\theta \to 0$, $I(\lambda,\theta) \to \infty$, so there is a solution to $I(\lambda,\theta_\lambda)$ for any $\lambda > 0$, and $\theta_\lambda \to 0$ as $\lambda \to 0$. To get an approximate formula for θ_λ we change variables $x = y/\sqrt{\lambda}$ to get

$$I(\lambda, \theta) \sim C\theta^{\alpha-3}\lambda^{(\alpha-2)/2} \int_0^\infty y^{1-\alpha} \frac{y^2}{1+y^2}\, dy\,.$$

For small values of λ, solving $I(\lambda, \theta_\lambda) = 1$ gives

$$\theta_\lambda \approx \lambda^{(\alpha-2)/2(3-\alpha)}. \tag{A.4}$$

The steady state density ρ_* can be calculated from θ_* using (10).

$$\rho_* = \sum_k p_k \rho_{k*} = C \sum_k k^{-\alpha} \frac{\lambda\binom{k}{2}\theta_*^2}{1+\lambda\binom{k}{2}\theta_*^2}$$

Approximating the sum by an integral as before, we get

$$\rho_* \sim C\theta_*^{\alpha-1}\lambda^{(\alpha-1)/2} \int_0^\infty y^{-\alpha} \frac{y^2}{1+y^2}\, dy \sim C\lambda^{\gamma(\alpha)} \tag{A.5}$$

where the critical exponent is

$$\gamma(\alpha) = (\alpha-1)\left[\frac{\alpha-2}{2(3-\alpha)}\right] + \frac{\alpha-1}{2} = \frac{1}{3-\alpha} - \frac{1}{2}.$$

In the borderline case $\alpha = 3$ the limit as $\theta \to 0$ is finite. Since $\theta \to I(\lambda, \theta)$ is decreasing, there is a critical value λ_c so that $I(\lambda_c, 0) = 1$ and for $\lambda > \lambda_c$ we have one solution $I(\lambda, \theta_\lambda) = 1$, which has $\theta_\lambda \to 0$ as $\lambda \to \lambda_c$.

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[1] Mark E. J. Newman, "The Structure and Function of Complex Networks," SIAM Review **45**, 167–256 (2003).

[2] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: Structure and dynamics," Physics Reports **424**, 175 – 308 (2006).

[3] Guido Caldarelli, *Scale-Free Networks: Complex Webs in Nature and Technology*, OUP Catalogue No. 9780199211517 (Oxford University Press, 2007).

[4] Reuven Cohen and Shlomo Havlin, *Complex networks: structure, robustness and function* (Cambridge University Press, 2010).

[5] Mark E. J. Newman, *Networks : An Introduction* (Oxford University Press, Oxford New York, 2010).

[6] Réka Albert and Albert-László Barabási, "Statistical mechanics of complex networks," Rev. Mod. Phys. **74**, 47–97 (2002).

[7] Colin Cooper and Alan Frieze, "A general model of web graphs," Random Struct. Algorithms **22**, 311–335 (2003).

[8] Romualdo Pastor-Satorras and Alessandro Vespignani, "Epidemic dynamics and endemic states in complex networks," Phys. Rev. E **63**, 066117 (2001).

[9] Mark E. J. Newman, "Spread of epidemic disease on networks," Phys. Rev. E **66**, 016128 (2002).

[10] Noam Berger, Christian Borgs, Jennifer T. Chayes, and Amin Saberi, "On the spread of viruses on the internet," in *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, SODA '05 (Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005) pp. 301–310.

[11] Erik Volz, "SIR dynamics in random networks with heterogeneous connectivity," Journal of Mathematical Biology **56**, 293–310 (2008).

[12] Vishal Sood and Sidney Redner, "Voter model on heterogeneous graphs," Phys. Rev. Lett. **94**, 178701 (2005).

[13] Rick Durrett, *Random Graph Dynamics* (Cambridge University Press, New York, NY, USA, 2006).

[14] Claudio Castellano, Santo Fortunato, and Vittorio Loreto, "Statistical physics of social dynamics," Rev. Mod. Phys. **81**, 591–646 (2009).

[15] Duncan J. Watts, "A simple model of global cascades on random networks," Proceedings of the National Academy of Sciences **99**, 5766–5771 (2002).

[16] James P. Gleeson and Diarmuid J. Cahalane, "Seed size strongly affects cascades on random networks," Phys. Rev. E **75**, 056103 (2007).

[17] Andrea Montanari and Amin Saberi, "The spread of innovations in social networks," Proceedings of the National Academy of Sciences **107**, 20196–20201 (2010).

[18] Hisashi Ohtsuki, Christoph Hauert, Erez Lieberman, and Martin A. Nowak, "A simple rule for the evolution of cooperation on graphs and social networks," Nature **441**, 502–505 (2006).

[19] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, "Critical phenomena in complex networks," Rev. Mod. Phys. **80**, 1275–1335 (2008).

[20] Alain Barrat, Marc Barthlemy, and Alessandro Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, New York, NY, USA, 2008).

[21] Thomas M. Liggett, *Interacting particle systems* (Springer-Verlag, New York, 1985).

[22] T. E. Harris, "Contact interactions on a lattice," The Annals of Probability **2**, 969–988 (1974).

[23] Malte Henkel, Haye Hinrichsen, and Sven Lübeck, *Non-Equilibrium Phase Transitions: Volume 1: Absorbing Phase Transitions*, Vol. 1 (Springer, 2009).

[24] Joaquín Marro and Ronald Dickman, *Nonequilibrium phase transitions in lattice models* (Cambridge University Press, 2005).

[25] Andrei L. Toom, "Stable and attractive trajectories in multicomponent systems," Multicomponent Systems **6**, 549–575 (1980).

[26] Hwa-Nien Chen, "On the stability of a population growth model with sexual reproduction on $\mathbb{Z}^2$," The Annals of Probability **20**, 232–285 (1992).

[27] Hwa-Nien Chen, "On the stability of a population growth model with sexual reproduction on $\mathbb{Z}^d$, $d \geq 2$," The Annals of Probability **22**, 1195–1226 (1994).

[28] Xiaofang Guo, Da-Jiang Liu, and James W. Evans, "Generic two-phase coexistence, relaxation kinetics, and interface propagation in the quadratic contact process: Simulation studies," Phys. Rev. E **75**, 061129 (2007).

[29] Da-Jiang Liu, Xiaofang Guo, and James W. Evans, "Quadratic contact process: Phase separation with interface-orientation-dependent equistability," Phys. Rev. Lett. **98**, 050601 (2007).

[30] Xiaofang Guo, James W. Evans, and Da-Jiang Liu, "Generic two-phase coexistence, relaxation kinetics, and interface propagation in the quadratic contact process: Analytic studies," Physica A: Statistical Mechanics and its Applications **387**, 177 – 201 (2008).

[31] Xiaofang Guo, Da-Jiang Liu, and James W. Evans, "Schloegl's second model for autocatalysis with particle diffusion: Lattice-gas realization exhibiting generic two-phase coexistence," The Journal of Chemical Physics **130**, 074106 (2009).

[32] Da-Jiang Liu, "Generic two-phase coexistence and nonequilibrium criticality in a lattice version of schlgls second model for autocatalysis," Journal of Statistical Physics **135**, 77–85 (2009).

[33] Friedrich Schlögl, "Chemical reaction models for non-equilibrium phase transitions," Zeitschrift für Physik A Hadrons and Nuclei **253**, 147–161 (1972).

[34] Peter Grassberger, "On phase transitions in Schlögl's second model," Zeitschrift für Physik B Condensed Matter **47**, 365–374 (1982).

[35] Shirshendu Chatterjee and Rick Durrett, "Contact processes on random graphs with power law degree distributions have critical value 0." Ann. Probab. **37**, 2332–2356 (2009).

[36] Roni Parshani, Shai Carmi, and Shlomo Havlin, “Epidemic threshold for the susceptible-infectious-susceptible model on random networks,” Phys. Rev. Lett. **104**, 258701 (2010).

[37] Damon Centola and Michael Macy, “Complex contagions and the weakness of long ties,” American Journal of Sociology **113**, 702–734 (2007).

[38] Shirshendu Chatterjee and Rick Durrett, “A first order phase transition in the threshold contact process on random r-regular graphs and r-trees,” Stochastic Processes and their Applications **123**, 561 – 578 (2013).

[39] The number of $1-1$ neighbors of a vertex x is $|\{(y,z) : x \sim y, y \sim z, z \neq x, \text{state of } y = \text{state of } z = 1\}|$.

[40] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, “Bootstrap percolation on complex networks,” Phys. Rev. E **82**, 011103 (2010).

[41] Albert-László Barabási and Eric Bonabeau, “Scale-Free Networks,” Scientific American **288**, 60–69 (2003).

[42] T. Mountford, D. Valesin, and Q. Yao, “Metastable densities for contact processes on power law random graphs,” ArXiv e-prints (2011), arXiv:1106.4336 [math.PR].

[43] T. Mountford, J.-C. Mourrat, D. Valesin, and Q. Yao, “Exponential extinction time of the contact process on finite graphs,” ArXiv e-prints (2012), arXiv:1203.2972 [math.PR].

[44] M. Molloy and B. Reed, “A critical point for random graphs with a given degree sequence,” Random Structures & Algorithms **6**, 161–180 (1995).

[45] L. R. G. Fontes and R. H. Schonmann, “Bootstrap Percolation on Homogeneous Trees Has 2 Phase Transitions,” Journal of Statistical Physics **132**, 839–861 (2008).

[46] József Balogh and Boris G. Pittel, “Bootstrap percolation on the random regular graph,” Random Structures & Algorithms **30**, 257–286 (2007).

[47] Richard Durrett, “On the growth of one dimensional contact processes,” The Annals of Probability **8**, 890–907 (1980).

[48] Thomas M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes*, Vol. 324 (Springer, 1999).

[49] S. Prakash and G. Nicolis, “Dynamics of the Schlögl models on lattices of low spatial dimension,” Journal of Statistical Physics **86**, 1289–1311 (1997).

[50] Carol Bezuidenhout and Geoffrey Grimmett, “The critical contact process dies out,” The Annals of Probability **18**, 1462–1482 (1990).

[51] Gregory J. Morrow, Rinaldo B. Schinazi, and Yu Zhang, “The critical contact process on a homogeneous tree,” Journal of Applied Probability **31**, 250–255 (1994).

[52] Robin Pemantle, “The contact process on trees,” The Annals of Probability **20**, 2089–2116 (1992).