

Phase transitions for a planar quadratic contact process

Mariya Bessonov¹

NYC College of Technology, Department of Mathematics, 300 Jay Street, Brooklyn, NY 11201

Richard Durrett¹

Duke University, Department of Mathematics, Box 90320, Durham, NC 27708

Abstract

We study a two dimensional version of Neuhauser's long range sexual reproduction model and prove results that give bounds on the critical values λ_f for the process to survive from a finite set and λ_e for the existence of a nontrivial stationary distribution. Our first result comes from a standard block construction, while the second involves a comparison with the "generic population model" of [Bramson and Gray \(1991\)](#). An interesting new feature of our work is the suggestion that, as in the one dimensional contact process, edge speeds characterize critical values. We are able to prove the following for our quadratic contact process when the range is large but suspect they are true for two dimensional finite range attractive particle systems that are symmetric with respect to reflection in each axis. There is a speed $c(\theta)$ for the expansion of the process in each direction. If $c(\theta) > 0$ in all directions, then $\lambda > \lambda_f$, while if at least one speed is positive, then $\lambda > \lambda_e$. It is a challenging open problem to show that if some speed is negative, then the system dies out from any finite set.

Keywords: quadratic contact process, population dynamics, stationary distribution

2010 MSC: primary 60J10, secondary 92D25, 92D40

1. Introduction

The contact process introduced by [Harris \(1974\)](#), is perhaps the simplest spatial model for the spread of a species. A site in \mathbb{Z}^d can be occupied ($\xi_t(x) = 1$) or vacant ($\xi_t(x) = 0$). Occupied sites become vacant at rate 1, while vacant sites become occupied at rate λ times the number of neighbors that are occupied, i.e., the death rate is constant and the birth rate is linear. All 0's is an absorbing state. Due to monotonicity, if we let ξ_t^1 be the state of the process at time t when we start with all sites occupied then ξ_t^1 converges to a limit ξ_∞^1 , called the upper invariant measure, which is the largest possible stationary distribution. For this and the other facts about the contact process, see ([Liggett, 1999](#)).

Let ξ_t^A be the contact process started with 1's on A and 0 otherwise. In principle, the contact process could have two critical values:

- $\lambda_e = \inf\{\lambda : \lim_{t \rightarrow \infty} P(\xi_t^1(x) = 1) > 0\}$, the critical value for the existence of a nontrivial stationary distribution.
- $\lambda_f = \inf\{\lambda : P(\xi_t^A \neq \emptyset \text{ for all } t) > 0 \text{ for some finite set } A\}$.

Results of [Bezuidenhout and Gray \(1994\)](#) imply that for finite range attractive processes, $\lambda_e \leq \lambda_f$. In the case of the contact process, self-duality

$$P(\xi_t^{(0)} \neq 0) = P(\xi_t^1(0) = 1)$$

implies that the critical values coincide: $\lambda_e = \lambda_f$, a value we will call λ_c . Let

$$\tau^A = \inf\{t : \xi_t^A \equiv 0\}$$

Email addresses: mbessonov@citytech.cuny.edu (Mariya Bessonov), rtd@math.duke.edu (Richard Durrett)

be the time at which the process dies out. When $\lambda = \lambda_c$, $P(\tau^A < \infty) = 1$ for all finite sets A and $\xi_\infty^1 = \delta_0$. From this, it follows fairly easily that the phase transition is continuous, i.e., $P(\xi_\infty^1(x) = 1)$ is a continuous function of λ . If $\lambda > \lambda_c$, the complete convergence theorem implies that as $t \rightarrow \infty$,

$$\xi_t^A \Rightarrow P(\tau^A < \infty) \delta_0 + P(\tau^A = \infty) \xi_\infty^1.$$

As a consequence of this result, all stationary distributions have the form

$$\theta \delta_0 + (1 - \theta) \xi_\infty^1.$$

The distribution δ_0 is stationary, but is unstable under perturbation. If we add spontaneous births at rate β , then there is a unique stationary distribution ξ_∞^β for the modified contact process, and as $\beta \downarrow 0$ we have

$$P(\xi_\infty^\beta(x) = 1) \downarrow P(\xi_\infty^1(x) = 1).$$

In this paper, we will consider a variant of the contact process that has sexual reproduction, i.e., two individuals are needed to produce a new one. Many processes of this type have been studied, but many open problems remain. To emphasize that these processes are an important and natural generalization of the contact process with linear birth rates, we will call them quadratic contact processes. We use this term somewhat loosely. In the models we discuss, the birth rate will not always be a quadratic function of the number of occupied neighbors.

The oldest “quadratic contact process” is *Toom’s NEC model*, see (Toom, 1974, 1980). However, in its initial formulation, it was a system in which each site could be in one of two states $\xi_n(x) \in \{1, -1\}$. Let $\zeta_n(x)$ be the majority opinion among $x, x + (0, 1), x + (1, 0)$ at time n . If $\zeta_n(x) = +1$, then $\xi_{n+1} = +1$ with probability $1 - p$ and $\xi_{n+1}(x) = -1$ with probability p . If $\zeta_n(x) = -1$, then $\xi_{n+1} = -1$ with probability $1 - q$ and $\xi_{n+1}(x) = 1$ with probability q . Toom proved that for small p and q , there are two nontrivial stationary distributions. Note that in contrast to the Ising model, there is nonergodicity even when the system is asymmetric. For more on this process, see (Bennett and Grinstein, 1985; He et al., 1990; Muñoz et al., 2005).

Durrett and Gray (Durrett and Gray (1985)) reformulated Toom’s process as a continuous time growth model in which

$$\begin{aligned} 1 &\rightarrow 0 && \text{rate } 1 \\ 0 &\rightarrow 1 && \text{rate } \lambda \text{ if } \xi_t(x + (1, 0)) = \xi_t(x + (0, 1)) = 1 \end{aligned}$$

If the initial configuration for this process has no 1’s outside a square, then there will never be any births outside the square, so $\lambda_f = \infty$. Using the contour method, they proved (in an unpublished work (Durrett and Gray, 1985), see an announcement of results in (Durrett, 1986))

(DG1) $\lambda_e \leq 110$.

(DG2) If $p < p^* = 1 - p_c$, where p_c is the critical value for oriented bond percolation in $d = 2$, then the process starting from product measure with density p dies out.

(DG3) If $\lambda > \lambda_e$ and β is such that $6\beta^{1/4}\lambda^{3/4} < 1$ then when we add spontaneous births at rate β there are two stationary distributions.

(DG2) is easy to prove. If there is an infinite path of 0’s starting from a site x in which each step is up or right, then these 0’s can never become 1’s. Many such paths exist when the density of 0’s exceeds p_c for oriented percolation. Indeed, the set of 0’s on the line $x + y = k$ will be a discrete time contact process. Starting from a single 0 then with positive probability the discrete time contact process does not die out and when it does not die out the left and right edges grow linearly. Thus the up-right paths will be so numerous that the 1’s are trapped in finite sets in between the permanent 0’s and they will die out. The interest in this result is that it shows that the complete convergence theorem is false. The fact that product measures with density $p < p^*$ converge to δ_0 for any $\lambda < \infty$ suggests that this model has a discontinuous phase transition, but proving this is a difficult problem.

Recently, Chatterjee and Durrett (2013) have shown that the discrete time threshold two contact process on a random r -regular graph or on a homogeneous tree in which each vertex has degree r has a discontinuous phase transition if the degree $r \geq 3$. Varghese and Durrett (2013) have used simulation and heuristic arguments to study

two versions of the quadratic contact process on random graphs generated by the configuration model. In their vertex centered case, which is the one relevant to this paper, they find a discontinuous transition for the Erdős-Renyi random graph, but on power law random graphs with degree distribution $p_k \sim Ck^{-\alpha}$ and $\alpha < 3$ the critical value is 0. When $\lambda_c = 0$, a simple argument implies that the transition is continuous.

Chen (1992, 1994) generalized Durrett and Gray's model on \mathbb{Z}^2 . Let e_1, e_2 be the two unit vectors and define:

$$\begin{array}{cccc} \text{pair 1} & \text{pair 2} & \text{pair 3} & \text{pair 4} \\ x - e_1, x - e_2 & x + e_1, x - e_2 & x + e_1, x + e_2 & x - e_1, x + e_2 \end{array}$$

Chen's models are numbered by the pairs that can give birth: Model I (pair 1 = SW corner rule), Model IIa (pairs 1 and 2), Model IIb (pairs 1 and 3), Model III (pairs 1–3), and Model IV (any pair).

Chen (1992) proved for Model IV that if $0 < p < p(\lambda)$, then

$$P(0 \in \xi_t^p) \leq t^{-c \log_{2\lambda}(1/p)}.$$

If we add spontaneous births at rate β and let $\xi_\infty^{0,\beta}$ be the limit as $t \rightarrow \infty$ for the system starting from all 0's (which exists by monotonicity), then for large λ ,

$$\lim_{\beta \rightarrow 0} P(0 \in \xi_\infty^{0,\beta}) > 0.$$

The second result says that the all 0's state is unstable to perturbation. When the limit is 0, the all 0's state is stable to perturbation. In this case, if $\lambda > \lambda_e$, then the perturbed system will have two stationary distributions for small β . This is how (DG3) was proved. Chen (1994) shows that this is true for Model III.

Durrett and Neuhauser (1994) considered a model with deaths at rate 1, and births at rate $\beta \binom{k}{2} / \binom{2d}{2}$ at vacant sites with k occupied nearest neighbors. The mean field equation, which is derived by assuming that all sites are independent, in this case is

$$\frac{du}{dt} = -u + \beta(1 - u)u^2$$

This ODE has $\beta_e = 4$ and $\beta_f = \infty$, i.e., there is a nontrivial fixed point for $\beta \geq 4$ but 0 is always locally attracting. They showed that in the limit of fast stirring both critical values converged to 4.5. This threshold is the point where the PDE

$$\frac{\partial u}{\partial t} = u'' - u + \beta u(1 - u)$$

has traveling wave solutions $u(t, x) = w(x - ct)$ with $c > 0$. The largest fixed point of the mean field differential equation is $2/3$ at 4.5, but based on simulations they conjectured that the phase transition was continuous.

Neuhauser (1994) considered the contact process with sexual reproduction in $d = 1$ with long-range interaction in continuous time. In her model, the spatial locations are $\epsilon\mathbb{Z}$ and $\xi_t^\epsilon : \epsilon\mathbb{Z} \rightarrow \{0, 1\}$ has the following dynamics:

- (i) Particles die at rate 1.
- (ii) A pair of adjacent particles at x and $x + \epsilon$ produces an offspring with rate λ , which is sent to a location y with probability $k_\epsilon(x - y)$. k_ϵ is the offspring distribution kernel, derived from an exponentially decaying, symmetric probability kernel k on \mathbb{R} .
- (iii) The birth at y is suppressed if y is occupied.

For this process, she showed that:

Theorem 1 ((Neuhauser, 1994, Theorem 1)). *In the limit as $\epsilon \rightarrow 0$, starting from product measure, the density of particles, u , evolves as a solution to the integro-differential equation*

$$\frac{\partial u}{\partial t} = -u + \lambda(1 - u)(k * u^2) \tag{1}$$

In addition, (1) admits traveling wave solutions, and there is a nondecreasing function $c_k : (0, \infty) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ giving the wave speed corresponding to λ and k .

Theorem 2 ((Neuhauser, 1994, Theorem 3)). *If $c_k(\lambda) > 0$, then for small enough ϵ , there is a nontrivial stationary distribution. Additionally, there is a constant λ^* above which the wave speed is indeed strictly positive.*

Neuhauser's result plays an important part in our first proof, but the main motivation for this work came from research done by Guo, Evans, and Liu (2008); Guo, Liu, and Evans (2007, 2009); Liu (2009); Liu, Guo, and Evans (2007). They considered a modification of Model IV in which particles hop according to the simple exclusion process at rate h . Birth rates at a site are $1/4$ times the number of adjacent pairs of occupied sites, while deaths occur at rate p . Having $h > 0$ means that $p_f(h) > 0$. When h is small, $p_f(h) < p_e(h)$, in which case “both the vacuum and active steady state are stable.” When $p_e(h) > p_f(h)$, this model has a discontinuous phase transition.

They defined a speed $V(p, h, S)$ using simulation of the process in a strip with slope S and argued that

$$\begin{aligned} p &< p_e(h) && \text{if some } V(p, h, S) > 0 \\ p &< p_f(h) && \text{if all } V(p, h, S) > 0 \end{aligned}$$

It is an interesting problem to define the speeds rigorously for a class of attractive finite range process and to prove the relationships in the last display. Here, we avoid that problem by taking a long range limit to get an integro-differential equation for which the existence of speeds can be proved using results of Weinberger (1982).

We consider the two-dimensional contact process with sexual reproduction in discrete time (the results generalize to all higher dimensions as well). Let $k : \mathbb{R}^2 \rightarrow [0, 1]$ be a probability density that is invariant under reflections in either axis: i.e., $k(x_1, x_2) = k(-x_1, x_2)$ and $k(x_1, x_2) = k(x_1, -x_2)$ and has compact support. At time $n \in \{0, 1, 2, \dots\}$ the state of a site x on the lattice \mathbb{Z}^2/L is given by $\xi_n^L(x)$, which can take on the values 1 (occupied) or 0 (vacant). Starting with an initial configuration of particles, ξ_0^L on \mathbb{Z}^2/L , the process evolves in the following manner:

- (i) At time n , given the configuration at the previous time $n - 1$, with probability β , a vacant site x on the lattice will generate a random variable U_x with density k , choose the site y closest to $x + U_x$, and then choose one of its four nearest neighbors z at random. If both of the chosen sites are occupied, x will also become occupied.
- (ii) After all births have occurred, with probability η , each particle is killed, independently of the others.

If we take the limit of the particle system as $L \rightarrow \infty$ and ignore technical details, the density of 1's at time n should satisfy

$$u_{n+1} = (1 - \eta) \left[u_n + \beta(1 - u_n) (k * u_n^2) \right]. \quad (2)$$

In the situation in which $u_n(x) \equiv v_n$

$$v_{n+1} = (1 - \eta) \left[v_n + \beta(1 - v_n) v_n^2 \right]. \quad (3)$$

When $\beta > 4\eta/(1 - \eta)$, there are three equilibrium solutions of (3): 0 and $\rho_u < 1/2 < \rho_s$ given by

$$\frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4\eta}{\beta(1 - \eta)}} \right)$$

Using results of Weinberger (1982), we can prove existence of wave speeds under the assumption $\beta > 4\eta/(1 - \eta)$. That is, if S^1 is the circle of radius one in the plane, then there exists a function $c^* : S^1 \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, which gives the speed of propagation of plane waves solutions of (2) in the direction θ .

Theorem 3. *If $c^*(\theta) > 0$ for all $\theta \in S^1$, then for large L , there is a nontrivial stationary distribution.*

The proof of this result is based on (Neuhauser, 1994). There are three ingredients:

- (i) a hydrodynamic limit, which shows that as $L \rightarrow \infty$ the particle system converges to a solution of (2),
- (ii) a convergence result of Weinberger (1982) for solutions of (2), and
- (iii) a block construction.

Our symmetry assumption implies that if the speed in some direction (θ_1, θ_2) is positive then there are a total of four directions obtained by reflection across the axes that also have positive speeds. From this, we see that if a large region of 0's develops then the process will be able to fill in this hole and hence there should be a stationary distribution. This intuition can be made rigorous by using a comparison with a “generic population process” of [Bramson and Gray \(1991\)](#).

Theorem 4. *If there exists a $(\theta_1, \theta_2) \in S^1$ and $c^*(\theta) > 0$, then for large L , there is a nontrivial stationary distribution.*

Our application is more complicated than their proof of Toom’s eroder theorem. In their proof the bad regions are areas that contain 0’s, while in our proof the bad regions are areas where the density of 1’s is not large enough in some small box. In [\(Bramson and Gray, 1991\)](#), the state outside a bad region is always the same, all 1’s. In this paper, we must use functions from Weinberger’s proof of the existence of wave speeds to control the decrease of the bad region.

By the same logic, if some speed is negative and the initial configuration is finite then it can be put inside a shrinking parallelogram.

Conjecture 1. *If there is a $(\theta_1, \theta_2) \in S^1$ with $c^*(\theta) < 0$, the system cannot survive starting from an initial configuration with only a finite number of particles.*

In a number of situations, for example see [\(Cox et al., 2013, Chapter 7\)](#), block constructions have been used to show that a process dies out. However, in that situation, one shows that a large enough dead region will expand. Since our process is good at filling in holes, that conclusion is false here, and one will need a much different method to prove our conjecture.

2. Hydrodynamic Limit

In this section, we will show that as $L \rightarrow \infty$, the particle system converges to the solution of the integro-differential equation (2). The first thing to do is to explain what it means for a sequence of configurations $\xi_n : \mathbb{Z}^2/L \rightarrow \{0, 1\}$ to converge to a function $u_n(x)$. To do this, we will let $\gamma \in (0, 1)$ and partition the plane into squares with side $L^{-\gamma}$ whose corners are at the points $L^{-\gamma}(m, n)$ with $m, n \in \mathbb{Z}$. For any $x \in \mathbb{Z}^2/L$, let x^* be the bottom-left corner of the box containing x , and let

$$B^L(x) = x^* + [0, L^{-\gamma})^2. \quad (4)$$

Each such box contains $\sim L^{2-2\gamma}$ points.

Let $S_n^L(x)$ be the number of particles in $B^L(x)$ at time n :

$$S_n^L(x) = \sum_{y \in B^L(x)} \xi_n^L(y).$$

Define $\xi_n^L \sim u_n$ to mean that for all K , as $L \rightarrow \infty$

$$\sup_{x \in [-K, K]^2 \cap \mathbb{Z}^2/L} \left| \frac{S_n^L(x)}{L^{2-2\gamma}} - u_n(x^*) \right| \rightarrow 0$$

Let $R_n^L(x)$ = number of pairs of adjacent particles in $B^L(x)$ at time n :

$$R_n^L(x) = \sum_{y \in B^L(x)} \zeta_n^L(y) \quad \text{where} \quad \zeta_n^L(y) = \xi_n^L(y) \cdot \frac{1}{2} [\xi_n^L(y + e_1) + \xi_n^L(y + e_2)],$$

where e_1 and e_2 are unit vectors of \mathbb{Z}^2/L . Define $\zeta_n^L \sim u_n^2$ to mean that for all $K < \infty$, as $L \rightarrow \infty$

$$\sup_{x \in [-K, K]^2 \cap \mathbb{Z}^2/L} \left| \frac{R_n^L(x)}{L^{2-2\gamma}} - u_n(x^*)^2 \right| \rightarrow 0$$

Theorem 5. Suppose that $u_0(x) : \mathbb{R}^2 \rightarrow [0, 1]$ is continuous and that the sequence of initial configurations $\xi_0^L \sim u_0$ and $\zeta_0^L \sim u_0^2$. Then as $L \rightarrow \infty$, $\xi_n^L \sim u_n$ and $\zeta_n^L \sim u_n^2$, where u_n is the solution of

$$u_{n+1} = (1 - \eta) \left[u_n + \beta(1 - u_n) (k * u_n^2) \right].$$

Proof. By induction, it suffices to prove the result for $n = 1$. Let T_x be the translation by x . In order to simplify the computation of expectations and variances, we will modify the birth step so that the first parent y will be chosen according to the probability kernel $T_{x^*}k_L$, instead of T_xk_L . As before, the second parent is chosen at random, with equal probability, from the nearest neighbors of the first parent. The next lemma shows that at time 1 the two processes are equal with high probability.

Lemma 2.1. If $\xi_0^L(x) = \hat{\xi}_0^L(x)$ for each $x \in \mathbb{Z}^2/L$, then

$$P(\xi_1^L(x) \neq \hat{\xi}_1^L(x)) \rightarrow 0 \text{ as } L \rightarrow \infty.$$

Proof. For $y \in \mathbb{Z}^2/L$, let $\alpha_L(y) = k_L(y - x)$, $\beta_L(y) = k_L(y - x^*)$, and let

$$p_L = \sum_{y \in \mathbb{Z}^2/L} \min\{\alpha_L(y), \beta_L(y)\}$$

A standard argument from analysis shows that if $f \in L^1(\mathbb{R}^2)$ and $\delta = (\delta_1, \delta_2) \rightarrow (0, 0)$, then $\|T_\delta f - f\|_1 \rightarrow 0$ (approximate f by a continuous function g). This implies that as $L \rightarrow \infty$, $p_L \rightarrow 1$ uniformly for $x \in \mathbb{Z}^2/L$.

To couple the two processes, use the same coin flips to see if births should occur. If a birth event occurs at site $x \in \mathbb{Z}^2/L$ at time 1, flip a coin with probability p_L of heads. If heads comes up, then with probability

$$\min\{\alpha_L(y), \beta_L(y)\}/p_L$$

choose y as the first parent for x in both processes, and then make the same choice of second parent z . Otherwise, the particle in ξ_L chooses its first parent y with probability

$$(\alpha_L(y) - \beta_L(y))^+ / (1 - p_L),$$

the particle in $\hat{\xi}_L$ chooses its first parent y with probability

$$(\beta_L(y) - \alpha_L(y))^+ / (1 - p_L),$$

and the choice of second parents are made independently. Once the births have been done, we use the same coin flips to decide the deaths and the proof is complete. \square

Let P_0 denote the probability law for the process $\hat{\xi}_n^L$ with initial configuration $\hat{\xi}_0^L$. In order to write out the computations more compactly, introduce the notations $X_i = \hat{\xi}_i^L(x)$, $Y_i = \hat{\xi}_i^L(y)$, and $Z_i = \hat{\xi}_i^L(z)$ for $i = 0, 1$ and $x, y, z \in \mathbb{Z}^2/L$, and set

$$K_L(x) = \sum_{y \in \mathbb{Z}^2/L} k_L(y - x^*) \hat{\xi}_0^L(y) \cdot \frac{1}{4} \sum_{z \sim y} \hat{\xi}_0^L(z).$$

so that

$$P_0(X_1 = 1) = (1 - \eta) [X_0 + \beta(1 - X_0) \cdot K_L(x)] := p_x.$$

Since X_1 is Bernoulli, $\text{Var}_0(X_1) = p_x - p_x^2$.

The total number of particles alive in $B^L(x)$ at time 1 is

$$\hat{S}_1^L(x) = \sum_{y \in B^L(x)} \hat{\xi}_1^L(y),$$

so the expected proportion of occupied sites in $B^L(x)$ at time 1 is

$$\mathbb{E}_0 \left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} \right) = (1 - \eta) \left[\frac{\hat{S}_0^L(x)}{L^{2-2\gamma}} + \beta \left(1 - \frac{\hat{S}_0^L(x)}{L^{2-2\gamma}} \right) K_L(x) \right]. \quad (5)$$

For $y \neq z$ in $B^L(x)$,

$$\begin{aligned} \text{Cov}_0(Y_1, Z_1) &= \mathbb{E}_0(Y_1 Z_1) - \mathbb{E}_0(Y_1) \mathbb{E}_0(Z_1) \\ &= (1 - \eta)^2 \left[Y_0 Z_0 + \beta \cdot K_L(x) ((1 - Y_0) Z_0 + Y_0 (1 - Z_0)) + (\beta \cdot K_L(x))^2 (1 - Y_0)(1 - Z_0) \right] - p_y p_z = 0. \end{aligned}$$

It is for this reason we modified our process so that all sites in $B_L(x)$ use the kernel $k_l(\cdot - x^*)$.

Since the covariance is 0,

$$\text{Var}_0 \left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} \right) = \frac{1}{L^{4-4\gamma}} \sum_{y \in B^L(x)} (p_y - p_y^2) \leq \frac{1}{L^{4-4\gamma}} \sum_{y \in B^L(x)} p_y$$

The variance of $\hat{S}_1^L(x)/L^{2-2\gamma}$ is

$$\leq \frac{1}{L^{4-4\gamma}} \left[\hat{S}_0^L(x) + \beta (L^{2-2\gamma} - \hat{S}_0^L(x)) \right] \leq C \cdot \frac{1}{L^{2-2\gamma}}.$$

where C does not depend on L . By Chebyshev's inequality,

$$P_0 \left(\left| \frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} - \mathbb{E}_0 \left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} \right) \right| \geq \delta \right) \leq \delta^{-2} \text{Var} \left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} \right) \leq \frac{C}{\delta^2 L^{2-2\gamma}}.$$

There are $4L^{2\gamma} \cdot K^2$ boxes of side length $L^{-\gamma}$ in each $[-K, K]^2$ box, so

$$P_0 \left(\sup_{x \in [-K, K]^2 \cap \mathbb{Z}^2/L} \left| \frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} - \mathbb{E}_0 \left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} \right) \right| \geq \delta \right) \leq \frac{CL^{2\gamma}}{\delta^2 L^{2-2\gamma}}. \quad (6)$$

For $\gamma \in (0, \frac{1}{2})$, this probability tends to 0 as $L \rightarrow \infty$. For $x \in \mathbb{R}^2$ and $x_L \in \mathbb{Z}^2/L$ such that $x_L \rightarrow x$ as $L \rightarrow \infty$,

$$K_L(x_L) \rightarrow (k * u_0^2)(x).$$

By the assumptions of Theorem 5,

$$\hat{S}_0^L(x)/L^{2-2\gamma} = S_0^L(x)/L^{2-2\gamma} \rightarrow u_0(x).$$

Together with (5), this implies that

$$\mathbb{E}_0 \left(\frac{\hat{S}_1^L(x_L)}{L^{2-2\gamma}} \right) \rightarrow u_1(x) \quad \text{as } L \rightarrow \infty.$$

Then by (6), when $L \rightarrow \infty$,

$$\hat{S}_1^L(x_L)/L^{2-2\gamma} \rightarrow u_1(x)$$

and thus $\hat{\xi}_1^L \sim u_1$, and from Lemma 2.1 it follows that $\xi_1^L \sim u_1$.

Our next step is to prove a result for pairs. Let

$$\hat{R}_n^L(x) = \sum_{y \in B_*^L(x)} \hat{\xi}_n^L(y) \quad \text{where} \quad \hat{\xi}_n^L(y) = \hat{\xi}_n^L(y) \cdot \frac{1}{2} \left[\hat{\xi}_n^L(y + e_1) + \hat{\xi}_n^L(y + e_2) \right]$$

and $B_*^L(x)$ is the set of points y so that $y + e_1$ and $y + e_2$ are also in the box. As argued in the first part of the proof, given the initial configuration the $\hat{\xi}_1^L(y)$ are independent so when $y, z \in B^L(x)$ we have

$$E \left(\hat{\xi}_n^L(y) \hat{\xi}_n^L(z) \right) = p_x^2.$$

The pairs $(\hat{\xi}_1^L(y), \hat{\xi}_1^L(z))$ for $y \in B_*^L(x)$ and $z = y + e_1$ or $y + e_2$ are not independent when $\{y, z\} \cap \{y', z'\} \neq \emptyset$ but one still has the estimate

$$\text{Var}_0 \left(\frac{\hat{R}_1^L(x)}{L^{2-2\gamma}} \right) \leq \frac{C}{L^{2-2\gamma}}$$

so the previous proof can be repeated almost word for word to conclude that $\hat{\xi}_1^L \sim u_1^2$ and by Lemma 2.1 $\xi_1^L \sim u_1^2$. \square

With the hydrodynamic limit established, the proof of Theorem 3 is routine. Let $\delta > 0$ be small enough so that $\rho_s - 2\delta > \rho_u + 2\delta$. (Weinberger, 1982, Theorem 6.2) implies that

Lemma 2.2. *If K is large enough and $N \geq N_K$ then $u_0(x) > \rho_u + \delta$ on $[-K, K]^2$ implies that $u_N(x) > \rho_s - \delta$ on $[-4K, 4K] \times [-K, K]$.*

Let

$$I_n = 2nK + [-K, K]^2.$$

We say that I_n is good at time t if we have $S_t^L(x) \geq \rho_u + 2\delta$ for all $x \in I_n$. Combining Theorem 5 with Lemma 2.2 gives

Lemma 2.3. *Let $\epsilon > 0$. If L is large enough and I_0 is good at time 0, then with probability $\geq 1 - \epsilon$, I_1 and I_{-1} are good at time N .*

Since the process cannot move by more than distance 1 in one time step, the events involved in the last lemma have a finite range of interaction that only depends on N . It follows from this and results in (Durrett, 1995) that if $\epsilon < \epsilon_N$, the good sites dominate a supercritical oriented percolation and hence there is a nontrivial stationary distribution.

3. Proof of Theorem 4

The result will be proved using the comparison model of Bramson and Gray (1991). The goal is to show that our quadratic contact process and their comparison process can be coupled so that regions in the quadratic contact process with a low density of particles are entirely contained in the vacant region of the comparison model. Then the existence of a nontrivial stationary distribution for the comparison model will imply the existence of a nontrivial stationary distribution for the quadratic contact model.

In the comparison process of Bramson and Gray, at each time $t > 0$, the plane \mathbb{R}^2 is divided into two regions: a vacant and a nonvacant region. Let A_t denote the vacant region of the comparison process at time t and take $A_0 = \emptyset$. Fix unit orientation vectors $n_1, n_2, n_3 \in S^1$. A_t will be a union of triangles, each triangle having edges perpendicular to the orientation vectors. For there to exist triangles with edges perpendicular to the n_i , it must be the case that the n_i , $i = 1, 2, 3$ span \mathbb{R}^2 and that

$$n_1 = -\alpha_2 n_2 - \alpha_3 n_3, \tag{7}$$

for some $\alpha_2, \alpha_3 > 0$. The edges of the triangles move inward at linear rates, to be specified below. $T(x, s; t) \subset A_t$ will denote the triangle that first appears at time s and with incenter at $x \in \mathbb{R}^2$.

New triangles that are part of the vacant region are created in two ways:

I. Let \mathcal{P} be a Poisson process on $\mathbb{R}^2 \times [0, \infty)$ with intensity ϵ . If $(x, s) \in \mathcal{P}$, a triangular vacant region with edges perpendicular to the orientation vectors and with incenter at $x \in \mathbb{R}^2$ and having radius r , is created at time s . The i^{th} edge of this triangle, corresponding to the orientation vector n_i will move inward with linear rate a_i , $i = 1, 2, 3$ (in general a_i 's may be negative, and in this case the corresponding edges move outward). The collection of all such triangles at time t is what Bramson and Gray call the *noninteractive region* at t and the individual triangles, they call *death regions*.

II. The second way that new triangles can appear in the vacant region takes into account possible overlaps and collisions of the shrinking and/or expanding triangles that make up the vacant region. If $(x, s) \in \mathcal{P}$, and there exist $T_k(x_k, s_k; s) \in A_s$ such that

$$T(x, s; s) \cap \left(\bigcap_k T_k(x_k, s_k; s) \right) \neq \emptyset,$$

taking the maximal such intersection, this intersection forms an *overlap region* that is created at time s . This region will also be triangular with edges perpendicular to n_i , $i = 1, 2, 3$ (Bramson and Gray, 1991, Proposition 1). A single death region $T(x, s; s)$ can give rise to more than one overlap region. The edges of the overlap region move outward at positive rate b , the *interaction rate*, until each edge catches up to the corresponding edges of all of the regions that produced the original overlap. At that point, the edges of the overlap or collision region will begin to move inward

with rates a_i . In addition to overlaps from newly formed death regions that overlap already existing regions, triangles with outward moving edges can collide with each other: if there exist $T_k(x_k, s_k; s) \in A_s$ such that

$$T(x, s; s) \cap \left(\bigcap_k T_k(x_k, s_k; t) \right) = \emptyset,$$

for $t < s$, but

$$T(x, s; s) \cap \left(\bigcap_k T_k(x_k, s_k; s) \right) = \{x\}.$$

In this case, a collision region is formed, starting out as a triangle with inradius 0 at the single point $\{x\}$, and with edges perpendicular to the n_i , $i = 1, 2, 3$, moving outward at rate b until they have caught up with all of the corresponding edges of the triangles that initiated the collision.

If $a_i > 0$, for $i = 1, 2, 3$, the process has a nontrivial stationary distribution when the error rate ϵ is small enough (Bramson and Gray, 1991, Theorem 1).

Coming back to the quadratic contact process, take ξ_1 to be a direction with positive wave speed $\alpha_1 > 0$, since we have assumed that there is at least one positive speed. Because the offspring distribution kernel has \mathbb{Z}^2 -symmetry, and because the wave speed $c^*(\xi)$ is a lower semi-continuous function in ξ (Weinberger, 1982, Proposition 5.1), the existence of a positive speed in one direction implies the existence of a positive speed in three directions ξ_1, ξ_2 , and ξ_3 , that span \mathbb{R}^2 and satisfy (7). Let α_2 and α_3 be the wave speeds corresponding to ξ_2 and ξ_3 , respectively.

We shall say that the box $B^L(x)$ has *low density* at time n (see (4) for the definition of $B^L(x)$) if the density of the particles inside $B^L(x)$ falls below an appropriate threshold $\alpha \in (\rho_u, \rho_s)$, to be specified below. We will refer to boxes that do not have low density as good boxes.

Recall that $0 < \rho_u < \rho_s < 1$ are the nonzero fixed points of the operator

$$Q[u] = (1 - \eta) \left[u + \beta(1 - u)(k * u^2) \right],$$

which exist when $\beta > 4\eta/(1 - \eta)$.

For what follows, we need that

Lemma 3.1. *Q is monotone, i.e. if $u \leq v$, then $Q[u] \leq Q[v]$; furthermore, if $u < v$, then $Q[u] < Q[v]$.*

Proof.

$$\begin{aligned} Q[v] - Q[u] &= (1 - \eta) \left[(v - u) + \beta \left((1 - v)k * v^2 - (1 - u)k * u^2 \right) \right] \\ &\geq (1 - \eta) \left[(v - u) + \beta \left((1 - v)k * v^2 - (1 - u)k * v^2 \right) \right] \\ &= (1 - \eta) (v - u) \left(1 - \beta(k * v^2) \right) \\ &\geq 0. \end{aligned}$$

The first inequality holds since $0 \leq u \leq v$, and the second inequality holds because $v \geq u$ and $0 \leq k * v^2 \leq 1$ since k is a probability kernel and $0 \leq v \leq 1$. If $u < v$, the last inequality is strict. \square

The collection of low density boxes at time n will be contained in the vacant region of the comparison process. Let \tilde{A}_n be the collection of all low density boxes of the contact process on \mathbb{Z}^2/L at time n . The process initially has all sites occupied, so that $\tilde{A}_0 = \emptyset$.

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a function with the following properties (see Figure 1 for an example):

- i) ψ is continuous,
- ii) ψ is nonincreasing,
- iii) $\psi(-\infty) \in (\rho_u, \rho_s)$ and $\psi(s) = 0$ for $s \geq 0$.

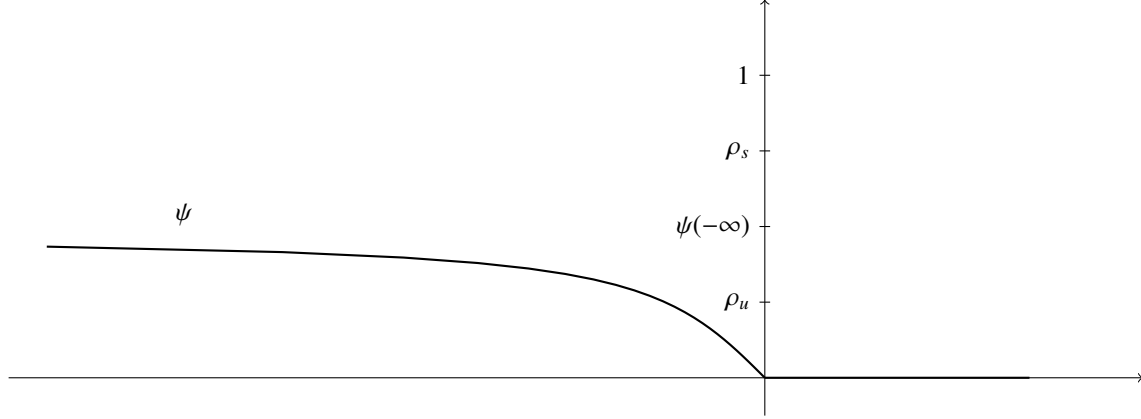


Figure 1: ψ must be continuous, nonincreasing, and $\psi(-\infty) \in (\rho_u, \rho_s)$. With these conditions satisfied, $c^*(\xi)$ is independent of the choice of ψ (Weinberger, 1982, Section 5).

(We consider $s \in \mathbb{R}$ as just a real variable here, with no connection to the time $s \in [0, \infty)$ in the previous section.)

As in (Weinberger, 1982, Section 5), for $c \in \mathbb{R}$ and $\xi \in S^1$, the unit circle in \mathbb{R}^2 , we consider the operator on continuous functions from \mathbb{R} to \mathbb{R}

$$R_{c,\xi}[f](s) := \max\{\psi(s), Q[f(x \cdot \xi + s + c)](0, 0)\}, \quad s \in \mathbb{R},$$

where c , s , and ξ are held fixed and f is considered as a function of $x \in \mathbb{R}^2$. Let $f_0 = \psi$, and

$$f_{n+1} = R_{c,\xi}[f_n], \quad n = 0, 1, \dots$$

By (Weinberger, 1982, Lemma 5.1), for each $s \in \mathbb{R}$, the sequence $\{f_n(s)\}$ is nondecreasing in n , and, for each $n = 0, 1, \dots$, the function f_n is nonincreasing in its argument. Moreover, $\{f_n\}$ increases to a limiting function f with $f(-\infty) = \rho_s$. If $c < c^*$, where c^* is the wave speed in the direction ξ , then $f(\infty) = \rho_s$. Indeed, c^* is independent of the initial choice of ψ satisfying the above conditions. Thus, if $c < c^*$,

$$f \equiv \rho_s$$

(see (Weinberger, 1982, Section 5)). Let

$$c \in (0, \min\{\alpha_i/2 : i = 1, 2, 3\}),$$

and note that $c > 0$. For each of the directions ξ_i with positive wave speed α_i , $i = 1, 2, 3$, consider the sequence $\{f_{n,i}\}$ where $f_{0,i} = \psi$ and

$$f_{n+1,i} = R_{c,\xi_i}[f_{n,i}], \quad n = 0, 1, \dots$$

For the coupling with the comparison process, the following will play an important role

Lemma 3.2. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for each $i = 1, 2, 3$,*

$$f_{n,i}(s - c) \leq Q[f_{n,i}(x \cdot \xi_i + s)](0, 0) \quad (8)$$

Proof. By definition,

$$f_{n,i}(s - c) = R[f_{n-1,i}(s - c)] = \max\{\psi(s - c), Q[f_{n-1,i}(x \cdot \xi_i + s)](0, 0)\}$$

Since $c < c_i^*$, by definition of c_i^* (see (Weinberger, 1982, Section 5)), for each $s \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} f_{n,i}(s) = \rho_s.$$

since ψ is nonincreasing in s and $\psi(-\infty) < \rho_s$, for some large enough $n_i \in \mathbb{N}$, we must have that

$$\psi(s - c) \leq Q[f_{n-1,i}(x \cdot \xi_i + s)](0, 0).$$

Thus for $n \geq n_i$,

$$f_{n,i}(s - c) = Q[f_{n-1,i}(x \cdot \xi_i + s)](0, 0) \leq Q[f_{n,i}(x \cdot \xi_i + s)](0, 0),$$

the inequality due to the fact that $f_{n,i}$ is nondecreasing in n and Q is monotone. We take $n_0 = \max_{i=1,2,3} n_i$. \square

Lemma 3.3. *There exists $n_0 \in \mathbb{N}$ and $c > 0$ such that for all $n \geq n_0$ and for each $i = 1, 2, 3$, and there exists $\psi : \mathbb{R} \rightarrow \mathbb{R}$, continuous, nonincreasing, with $\psi(-\infty) \in (\rho_u, \rho_s)$ and $\psi(s) = 0$ for $s \geq 0$, such that*

$$f_{n,i}(s - c) < Q[f_{n,i}(x \cdot \xi_i + s)](0, 0) \quad (9)$$

for $s \in \mathbb{R}$ with $f_{n,i}(s - c) > 0$.

Proof. Take $\psi : \mathbb{R} \rightarrow \mathbb{R}$ to be a continuous, strictly decreasing function with $\psi(-\infty) \in (\rho_u, \rho_s)$ and $\psi(s) = 0$ for $s \geq 0$. By induction, we will show that $f_{k,i}(s) = R[f_{k-1,i}(s)]$, with $f_{0,i}(s) = \psi(s)$ is also strictly decreasing for s for which $f_{k,i}(s) > 0$ and for each $k = 0, 1, 2, \dots$. Suppose that $f_{k,i}(s)$ is strictly decreasing for s for which $f_{k,i}(s) > 0$.

$$f_{k+1,i}(s) = \max\{\psi(s), Q[f_{k,i}(x \cdot \xi_i + s + c)](0, 0)\},$$

and $Q[f_{k,i}(x \cdot \xi_i + s + c)](0, 0)$ is strictly decreasing in s by (3.1) since $f_{k,i}$ is strictly decreasing in s . Since $f_{k+1,i}$ is the maximum of two strictly decreasing functions, it is also strictly decreasing. Since (8) holds for any $c \in (0, \min\{\alpha_i/2 : i = 1, 2, 3\})$ and both functions are strictly decreasing in s as long as they are non-zero, for $c = \min\{\alpha_i/2 : i = 1, 2, 3\}$, strict inequality must hold. \square

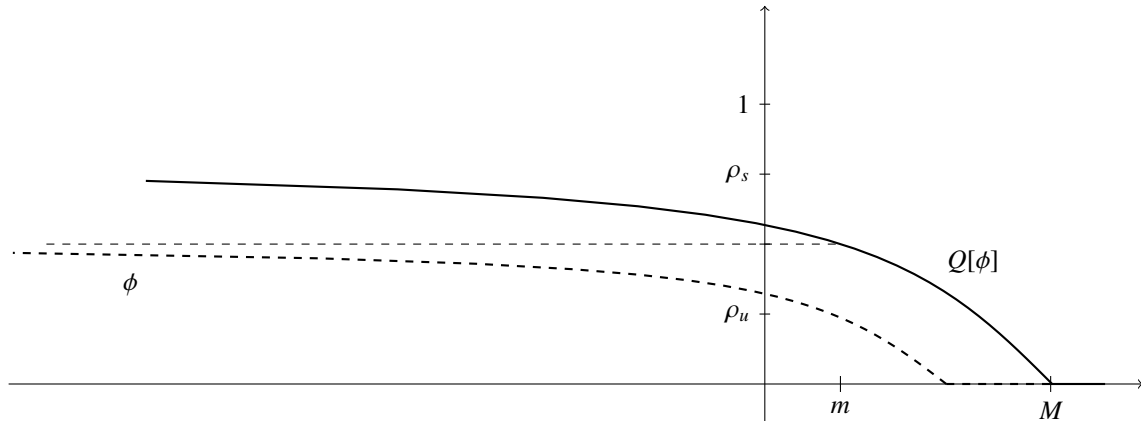
Suppose $f_{n_0,i}$ satisfies (9) for $s \in \mathbb{R}$ with $f_{n_0,i}(s - c) > 0$. Now, set $\alpha = \min_i \{f_{n_0,i}(-\infty)\}$ and let

$$m = \min_i \sup \{s : Q[f_{n_0,i}(x \cdot \xi_i + s)](0, 0) = \alpha\} \quad \text{and} \quad M = \max_i \inf \{s : Q[f_{n_0,i}(x \cdot \xi_i + s)](0, 0) = 0\}.$$

and set $l := M - m$.

The coupling will take place in the following way. If an *error* occurs in the contact process (the detailed description of what constitutes an error is below) such as a box $B^L(x)$ that is good at time $n - 1$ becomes a low density box at time n , a triangular vacant region $\tilde{T}(x, n; n)$ will appear in the contact process, and in the comparison process $B^L(x)$ is also covered by a corresponding triangular region, $T(x_*, n_*, n)$, $n_* \in [n, n + 1)$ chosen uniformly at random, and $x_* \in \mathbb{R}^2$ chosen uniformly at random from $B^L(x) \subset \mathbb{R}^2$, of an appropriate size, so that the center of the box is at the center of the triangle in the contact process with inscribed circle of radius

$$r = \lceil l + 2d(B) + 2c + d(k) \rceil,$$



initially at time n , where $d(B) = L^{-\gamma} \sqrt{2}$ is the diameter of the box $B_L(x)$ and

$$d(k) = \sup \{|x - y| : x, y \in \mathbb{R}^2, k(x) \neq 0, k(y) \neq 0\}$$

is the diameter of the kernel. In the comparison process, the corresponding triangle has the same center but inscribed circle of radius

$$r_* := r - (n + 1 - n_*).$$

Once such a region is created, the edges of $T(x_*, n_*; n)$ in the comparison process move inward, each with linear rate c . While the edges of $\tilde{T}(x, n; n)$ will have moved inward by c units from time n to $n + 1$, since the contact process is in discrete time. The region in the comparison process at time $t, t \geq n_*$ will be denoted by $T(x_*, n_*; t)$.

Set the *interaction rate* b to be r . In the comparison process, should two or more triangles collide or overlap, a new *collision* or *overlap* region is formed at that time, which is the intersection of the maximal collection of the colliding or overlapping regions that has nonempty intersection. The edges of the new collision or overlap region then move outward at rate b in each direction until they have caught up to all of the respective edges of the regions that initiated the overlap or collision in each respective direction, and after that the edge of the overlap or collision region will again move inward with rate c . By construction, for integer times $m > n$, $\tilde{T}(x, n; m) = T(x_*, n_*; m)$.

Bramson and Gray use a Poisson point process in their comparison model. We will consider a point process of errors, \mathcal{P} , derived from the quadratic contact process, which will be described below. Although \mathcal{P} is not quite a Poisson point process, it shares two key properties, (11) and (12), with a Poisson point process. In fact, the results of Bramson and Gray still hold with such an underlying point process as \mathcal{P} , satisfying (11) and (12), instead of the Poisson process.

3.1. Type I and type II errors

To describe \mathcal{P} , we will consider two different types of errors that are possible, type I and type II errors. Define

$$h_i(s) = \begin{cases} \alpha & \text{for } s \leq -(r - d(k) - c) \\ f_{n_0, i}(s) & \text{for } s > -(r - d(k) - c) \end{cases} \quad (10)$$

When either error occurs at some $x \in \mathbb{R}^2$, a triangular region $\tilde{T}^e(x, n; n)$ is created with inscribed circle of radius r . The edges of $\tilde{T}^e(x, n; n)$ move inward with linear speed c until the triangle closes and no longer exists. If $B(y; d(k)) \notin \tilde{A}_n$, for each $k = 0, 1, \dots, m$, where $B(y; r)$ represents the closed ball of radius r centered at y , the expected density of $B^L(y)$ is greater than $Q^k(\alpha) > \alpha$, where Q^k represents k iterations of Q . As long as no other error regions of \tilde{A}_{n+k} overlap $\tilde{T}^e(x, n; n + k)$, due to the low density box at x , the expected density of boxes inside $\tilde{T}^e(x, n; n + m)$ is at least greater than $h_i(\xi_i \cdot (x - y) - mc)$ for each $i = 1, 2, 3$, from (9) and thus it is above

$$\max_{i=1,2,3} h_i(\xi_i \cdot (x - y) - mc)$$

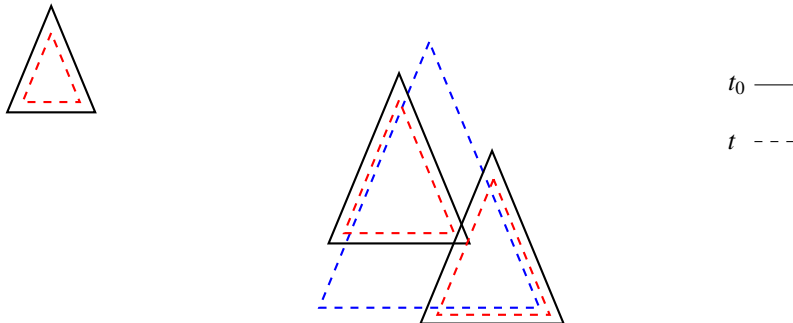


Figure 2: An overlap region is formed at some time t_0 . The edges of the overlap region move outward, and by time $t > t_0$, they have caught up with the edges of the triangles initiating the overlap, so all edges are now moving in. The overlap region at time t is blue. The vacant triangular regions at time t have become smaller and are in red.

for $y \in \tilde{T}^e(x, n; n + m)$. Note that each such triangle \tilde{T}^e has a type of buffer region of good boxes with density $\geq \alpha$ inside of \tilde{T}^e that is distance $d(k)$ away from the edges of the triangle, as long as there are no overlapping triangles. With inward moving edges, once the triangle has inscribed circle of radius $d(k)$ or less, all boxes in the triangle are good. Any box that intersects both \tilde{T}^e and \tilde{A}_{n-1}^c is expected to have density of particles $> \alpha$ at time $n - 1$. We now define the errors.

Type I Error: Such an error occurs if the density of a box in \tilde{A}_n^c suddenly falls below α at time n , but the density in that box and all boxes within $d(k)$ of that box was $\geq \alpha$ at time $n - 1$. Suppose that

$$B^L(x) \cap \tilde{A}_{n-1}^c = \emptyset.$$

Thus, $B^L(x)$ is good at time $n - 1$ and all other boxes within distance $d(k)$ are also good at $n - 1$. If $B^L(x)$ becomes a low-density box at time n , this spontaneous error will be considered a *Type I* error. Note that if two (or more) boxes become low-density boxes at time n that are within distance $d(k)$, two (or more) Type I errors occur and the two (or more) triangles that are created at time n will certainly overlap, thus giving rise to overlap regions.

Type II Error: Such errors may occur when there is (are) an already low-density box(es) in a region of the plane but with density above appropriate translates of h_i 's. That is, assume that

$$B^L(x) \cap \tilde{A}_{n-1}^c \neq \emptyset.$$

To motivate the definition of a type II error that will follow and explain (10), note that if a type I error occurs at $x \in \mathbb{R}^2$ at time k , a triangular region $\tilde{T}^e(x, k; k)$ is formed. If $B^L(x)$ is the only low-density box in $\tilde{T}^e(x, k; k)$, then for each $y \in \tilde{T}^e(x, k; n)$, the density of $B^L(y)$ is greater than

$$h(y, k, n) := \max_{i=1,2,3} h_i(\xi_i \cdot (x - y) - (n - k)c).$$

In the above case, when no overlap regions are present, a type II error occurs if $y \in \tilde{T}^e(x, k; n)$ and the density of $B^L(y)$ at time n falls below $h(y, k, n)$.

For what follows, a triangular overlap region centered at $x \in \mathbb{R}^2$ created at time k will be denoted $\tilde{T}^o(x, k; k)$. If a type II error occurs, an overlap region(s) will certainly be created. If $B^L(y)$ is contained in more than one triangular region at time n , consider the maximal collection of triangular regions such that

$$B^L(y) \subset \bigcap_{j \in J} \tilde{T}_j(y_j, n_j; n),$$

where J is a countable index set and $\tilde{T}_j(y_j, n_j; n)$ is a triangular region centered at $y_j \in \mathbb{Z}^2/L$ that was created at time $n_j \in \mathbb{N}$ and can either be of the type \tilde{T}^e or of the type \tilde{T}^o .

Then the results of Bramson and Gray imply that there exists a region $\tilde{T}(y, m; n)$ centered at y and created at time m such that

$$\bigcap_{j \in J} \tilde{T}_j(y_j, n_j; t) \subset \tilde{T}(y, m; t) \text{ for all times } t \in [m, n],$$

and so $B^L(y) \subset \tilde{T}(y, m; n)$.

Case i: If $\bigcap_{j \in J} \tilde{T}_j(y_j, n_j; n)$ contains only regions of the type \tilde{T}^o and no regions of the type \tilde{T}^e , no type II error occurs at x . This is because no “recent” error has occurred within $d(k)$ of y , and all boxes have density $> \alpha$. So, if an error were to occur at y , it would be of type I.

Case ii: Suppose $\bigcap_{j \in J} \tilde{T}_j(y_j, n_j; n)$ contains at least one region of the type \tilde{T}^e . In this case, a type II error may occur at x .

For $i = 1, 2, 3$, let $H_{i,j}(n)$ be the halfspace in \mathbb{R}^2 containing $\tilde{T}(y_j, n_j; n)$ and with boundary containing the side of the triangle perpendicular to ξ_i .

Furthermore, the i^{th} edge and corresponding halfspace containing \tilde{T} and its i^{th} edge at time t , $H_i(t)$ either: (i) moves outward with rate b , or (ii) moves inward with rate c and is such that

$$\bigcup_{j \in J} H_{i,j}(t) \subset H_i(t).$$

If $H_i(n)$ is moving outward with rate b for each $i = 1, 2, 3$, no type II error occurs at x . If $H_i(n)$ is *not* moving outward with rate b for at least one of $i = 1, 2, 3$, let $E \subset \{1, 2, 3\}$ be the set of directions i for which $H_i(n)$ is *not* moving outward with rate b . Let

$$h_{E,x}(n) := \max_{e \in E} \inf_{j \in J} h_e(\xi_e \cdot (x - y_j) - (n - m_{j,e})c),$$

where we define $m_{j,e}$ as follows:

- i) n_j , the time that the region $\tilde{T}(y_j, n_j; n)$ is created, if $\tilde{T}(y_j, n_j; n)$ is a region of the form \tilde{T}^e ,
- ii) the least integer greater than or equal to the time when $H_e(y_j, n_j, n)$ is not moving outward, if $\tilde{T}(y_j, n_j; n)$ is a region of the form \tilde{T}^o .

A type II error occurs at x at time n if the density of $B^L(x)$ falls below $h_{E,x}(n)$. Note that in a given box $B^L(x)$, only one error may occur at a particular time n , type I or type II, not both.

3.2. Probability of an error

First consider an upper bound on the probability that a type I error occurs. Let $x \in \mathbb{Z}^2/L$ and $B^L(x)$ the box containing x . Then if the box is good, the total number of particles in it is

$$S_{n-1}^L(x) > \alpha \cdot m,$$

where $m = L^{2-2\gamma}$ is the number of sites in $B(x)$. From previous calculations,

$$\mathbb{E}(S_n^L(x)) = \sum_{y \in B^L(x)} Q[u_{n-1}^L](y)$$

and

$$\mathbb{E}[S_n^L(x) \mid B(x) \text{ is good at time } n-1] = \sum_{y \in B^L(x)} Q[u_{n-1}^L](y) \geq Q(\alpha) \cdot m > \alpha \cdot m$$

We also have

$$\text{Var}(S_n^L(x)) \leq C \cdot m.$$

Now fix any $\delta_1 \in (0, Q(\alpha) - \alpha)$. By Chebyshev's inequality,

$$P_{\xi_{n-1}}^L \left(\left| S_n^L(x) - \sum_{y \in B^L(x)} Q[u_{n-1}^L](y) \right| \geq \delta_1 m \right) \leq \frac{Cm}{\delta_1^2 m^2} = \frac{C_1}{m} = C_1 L^{2\gamma-2} := \epsilon_1$$

This gives an upper bound on the probability of a good box spontaneously going bad, for a good box has density of particles $> \alpha$ and is therefore expected at the next time step to have density of particles $> Q(\alpha)$, so

$$Q(\alpha) - \delta_1 > \alpha.$$

As $L \rightarrow \infty$, $\epsilon_1 \rightarrow 0$.

For an upper bound on the probability of a type II error, choose $\delta_2 = \min_{i=1,2,3} \delta_{2,i} > 0$, with

$$\delta_{2,i} \in (0, \inf_{s \in [r,0]} \{Q[f_{n,i}(x \cdot \xi_i + s + c)](0,0) - h_i(s)\}).$$

By (9),

$$\inf_{s \in [r,0]} \{Q[f_{n,i}(x \cdot \xi_i + s + c)](0,0) - h_i(s)\} > 0.$$

By Chebyshev's inequality,

$$P_{\xi_{n-1}^L} \left(\left| S_n^L(x) - \sum_{y \in B^L(x)} Q(u_{n-1}^L)(y) \right| \geq \delta_2 m \right) \leq \frac{Cm}{\delta_2^2 m^2} = \frac{C_2}{m} = C_2 L^{2\gamma-2} := \epsilon_2$$

Thus, given the configuration at time $n-1$, the probability of an error occurring at time n , type I or type II, is bounded above by $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, with $\epsilon \rightarrow 0$ as $L \rightarrow \infty$.

3.3. Point process of errors

Next, we describe the point process, \mathcal{P} on $\mathbb{R}^2 \times [0, \infty)$ used in the comparison process. It is derived from the quadratic contact process on \mathbb{Z}^2/L . For each type I or type II error that occurs in the contact process, there is a single corresponding point in \mathcal{P} . If the error occurs in $B^L(x)$ at time n , let $(y, t) \in \mathcal{P}$ where $y \in x^* + [0, L^{-\gamma})^2$ is a single point in the box, chosen uniformly and at random from $x^* + [0, L^{-\gamma})^2$ and t is chosen uniformly and at random from $[n, n+1)$.

Although \mathcal{P} is not quite a Poisson point process, it shares two key properties with the Poisson process, the only two used in Bramson and Gray's proof, that are sufficient to demonstrate a nontrivial stationary distribution for large enough L . Namely (see (Bramson and Gray, 1991, 2-1 and 2-2)),

$$P(|B \cap \mathcal{P}| \geq 2) = O(\lambda(B)^2) \quad (11)$$

as $\lambda(B) \rightarrow 0$, where λ is Lebesgue measure on $\mathbb{R}^2 \times [0, \infty)$, for Borel sets B . (11) is satisfied by the quadratic contact error process \mathcal{P} , since at most one error can occur in $B^L(x)$ for any $x \in \mathbb{Z}^2/L$.

The second property is that for all small enough disjoint cubes B_1, B_2, \dots, B_m in $\mathbb{R}^2 \times [0, \infty)$,

$$P\left(\bigcap_{j=1}^m \{B_j \cap \mathcal{P} \neq \emptyset\}\right) \leq \prod_{j=1}^m (2\epsilon L^{2\gamma} \lambda(B_j)). \quad (12)$$

If $B_j = b_j \times [s_j, t_j]$, where b_j is a cube in \mathbb{R}^2 , it is sufficient to assume that

$$\lambda(b_j) < L^{-2\gamma}$$

for each j and $t_j - s_j < 1$ for $j = 1, 2, \dots, m$ (the time interval may also be open or half open) and that $b_j \subset \mathbb{R}^2$ overlaps at only one box $B^L(x)$. If b_j overlaps more than one box $B^L(x)$, it can be split into a disjoint union of cubes each overlapping just one box. To see that (12) is then satisfied, first consider the case when $n \in (t_j - s_j)$ for all $j = 1, \dots, m$, and some $n \in \mathbb{N}$.

Given the configuration of the quadratic contact process at time $n-1$, births at different sites at time n are independent of each other. The same holds for deaths. For each j ,

$$P(B_j \cap \mathcal{P} \neq \emptyset) \leq \epsilon L^{2\gamma} \lambda(B_j).$$

Since the densities of particles in different boxes are independent, (12) holds.

We still assume that

$$B_j = b_j \times [s_j, t_j],$$

where b_j is a cube in \mathbb{R}^2 , $\lambda(b_j) < L^{-2\gamma}$ for each j and $t_j - s_j < 1$. If $n_j \in (t_j - s_j)$ for some $n_j \in \mathbb{N}$, split B_j into two cubes:

$$B_{j,1} = b_j \times [s_j, n_j] \quad \text{and} \quad B_{j,2} = b_j \times [n_j, t_j].$$

We note that

$$P(\{B_{j,1} \cap \mathcal{P} \neq \emptyset\} \cap \{B_{j,2} \cap \mathcal{P} \neq \emptyset\}) \leq 2\epsilon L^{2\gamma} \max\{\lambda(B_{j,1}), \lambda(B_{j,2})\}$$

So we can suppose this does not occur and let $n_j = \lfloor s_j \rfloor$. Also suppose that the cubes are ordered in a way that

$$n_1 \leq n_2 \leq \dots \leq n_m.$$

(12) can be shown by induction on m . It clearly holds for $m = 1$. Let

$$A_j = \{B_j \cap \mathcal{P} \neq \emptyset\}.$$

Then

$$P\left(\bigcap_{j=1}^m A_j\right) = P\left(A_m \mid \bigcap_{j=1}^{m-1} A_j\right) P\left(\bigcap_{j=1}^{m-1} A_j\right) \leq \epsilon L^{2\gamma} \lambda(B_m) \prod_{j=1}^{m-1} (2\epsilon L^{2\gamma} \lambda(B_j)).$$

The second factor comes from the induction assumption and the first factor is from the upper bound on the error probability.

3.4. Comparison Result

Theorem 6. *Let \tilde{A}_n be the bad region of the contact process at time n with death probability η , birth probability β , and finite offspring distribution kernel k such that there is at least one direction with a positive wave speed.*

Let A_t be the vacant region of the comparison process with point process \mathcal{P} , orientation vectors $n_i = \xi_i$, speeds $a_i = c$, and interaction rate $b = r$. Then the processes A_n and \tilde{A}_n can be jointly coupled so that

$$\tilde{A}_n \subset A_n$$

for all $n \in \{0, 1, 2, \dots\}$. Hence, for large enough L , the quadratic contact process has a nontrivial stationary distribution.

Proof. The two processes are coupled in the following way: if in the contact process, an error occurs at time n in some box, place a single point in the cube

$$Q(x) = B^L(x) \times [n, n+1) \in \mathbb{R}^2 \times [0, \infty)$$

uniformly at random. If no error occurs, leave the corresponding box empty in \mathcal{P} . The rates have been set up so that all boxes with density $< \alpha$ in the contact process are covered by some triangular region. A triangular region automatically covers type I and type II errors. All other low density boxes can only be in the vicinity of the type I and II errors: within distance $d(K)$ times the age of the error. The interaction rate b ensures that if there is a large cluster of errors, the overlap/collision region grows faster than the surrounding bad boxes can spread (only by $d(k)$ units per single time step), and from the perimeter of the bad regions, the positive wave speeds “propagate” the high density boxes into the former bad regions.

We use an induction argument. Run the contact process from an initial configuration with every site occupied: $\tilde{A}_0 \subset A_0$ since both processes begin with empty vacant region. Now, assuming $\tilde{A}_{n-1} \subset A_{n-1}$, we show that after one time step, we still have $\tilde{A}_n \subset A_n$, which will follow from our earlier definitions of the errors and parameters.

Suppose that $x \in \tilde{A}_n$. This means that the density of particles in $B^L(x)$ at time n is less than α . If also $x \notin \tilde{A}_{n-1}$, all boxes within $d(k)$ of x are good at time $n-1$, since any triangular region $\tilde{T}(y, m; n-1)$ has a “buffer” region within $d(k)$ of each edge in which the density of particles is expected to be $> \alpha$, by construction. Thus, for an error to occur at time n in $B^L(x)$, it must be a type I error, and $B^L(x)$ is covered by a triangular vacant region centered at a uniformly selected point in $Q(x)$ in the comparison process, so $B^L(x) \in A_n$.

If $x \in \tilde{A}_n$ and $x \in \tilde{A}_{n-1}$, there are two possibilities:

- (i) either the density in $B^L(x)$ at time $n-1$ was less than α , or
- (ii) the density in $B^L(x)$ at time $n-1$ was greater than or equal to α .

If (i) holds, then $B(x; d(k) + c) \subset A_{n-1}$, since by definition of the errors, if the density of a box is below α , it is contained in a triangular vacant region and so is the ball around it of radius at least $d(k) + c$. Thus, since a triangular vacant region shrinks by at most c units in any direction in one time unit, $B^L(x) \subset \tilde{A}_n \subset A_n$.

If (ii) holds and if x is within $d(k)$ of the nonvacant region at time $n-1$, at this distance by construction the expected density of $B^L(x)$ is $> \alpha$. Then a type II error occurs at x at time n , so $B^L(x) \subset \tilde{A}_n \subset A_n$. If x is not within $d(k)$ of the nonvacant region at time $n-1$, the density of $B^L(x)$ is either above $h_{E,x}(n-1)$ and above $h_{E,x}(n)$, in which case $x \in A_n$, or it is above $h_{E,x}(n-1)$ and below $h_{E,x}(n)$, in which case a type II error occurs at x .

The comparison process A_n has a nontrivial stationary distribution. Hence, so does the contact process. \square

Acknowledgements

M. Bessonov was partially supported by the NSF grant DMS-1515800 and PSC-CUNY grants 67095-00 45 and 68170-00 46. R. Durrett was partially supported by the NSF grant DMS-1005470.

References

- Bennett, C., Grinstein, G., 1985. Role of irreversibility in stabilizing complex and nonergodic behavior in locally interacting systems. *Phys. Rev. Lett.* 55 (7).
URL <http://dx.doi.org/10.1103/PhysRevLett.55.657>
- Bezuidenhout, C., Gray, L., 1994. Critical attractive spin systems. *Ann. Probab.* 22 (3).
URL <http://dx.doi.org/10.1214/aop/1176988599>
- Bramson, M., Gray, L., 1991. A useful renormalization argument. In: Durrett, R., Kesten, H. (Eds.), *Random walks, Brownian motion, and interacting particle systems. Festschrift in Honor of Frank Spitzer*. Birkhäuser, Boston, pp. 113–152.
URL http://dx.doi.org/10.1007/978-1-4612-0459-6_6
- Chatterjee, S., Durrett, R., 2013. A first order phase transition in the threshold $\theta \geq 2$ contact process on random r -regular graphs and r -trees. *Stochastic Process. Appl.* 123 (2), 561–578.
URL <http://dx.doi.org/10.1016/j.spa.2012.10.001>
- Chen, H.-N., 1992. On the stability of a population growth model with sexual reproduction on \mathbb{Z}^2 . *Ann. Probab.* 20, 232–285.
URL <http://www.jstor.org/stable/2244555>
- Chen, H.-N., 1994. On the stability of a population growth model with sexual reproduction on \mathbb{Z}^d , $d \geq 2$. *Ann. Probab.* 22, 1195–1226.
URL <http://www.jstor.org/stable/2245021>
- Cox, J., Durrett, R., Perkins, E., 2013. Voter model perturbations and reaction diffusion equations. Vol. 349 of *Astérisque*.
URL <http://arxiv.org/abs/1103.1676>
- Durrett, R., 1986. Some peculiar properties of a particle system with sexual reproduction. In: Tautu, P. (Ed.), *Stochastic Spatial Processes*. Vol. 1212 of *Lecture Notes in Math*. Springer, pp. 106–111.
URL <http://dx.doi.org/10.1007/BFb0076241>
- Durrett, R., 1995. Ten lectures on particle systems. In: *St. Flour Lecture Notes*. Vol. 1608 of *Lecture Notes in Math*. Springer-Verlag, New York, pp. 97–201.
URL <http://dx.doi.org/10.1007/BFb0095747>
- Durrett, R., Gray, L., 1985. Some peculiar properties of a particle system with sexual reproduction, unpublished manuscript.
- Durrett, R., Neuhauser, C., 1994. Particle systems and reaction diffusion equations. *Ann. Probab.* 22 (1), 289–333.
URL <http://dx.doi.org/10.1214/aop/1176988861>
- Guo, X., Evans, J., Liu, D., 2008. Generic two-phase coexistence, relaxation kinetics, and interface propagation in the quadratic contact process: Analytic studies. *Phys. A: Statistical Mechanics and its Applications* 387 (1), 177–201.
URL <http://dx.doi.org/10.1016/j.physa.2007.09.002>
- Guo, X., Liu, D., Evans, J., 2007. Generic two-phase coexistence, relaxation kinetics, and interface propagation in the quadratic contact process: Simulation studies. *Phys. Rev. E* 75 (6), 061129.
URL <http://dx.doi.org/10.1103/PhysRevE.75.061129>
- Guo, X., Liu, D., Evans, J., 2009. Schlögl’s second model for autocatalysis with particle diffusion: lattice-gas realization exhibiting generic two-phase coexistence. *J. of Chem. Phys.* 130, 074106.
URL <http://dx.doi.org/10.1063/1.3074308>
- Harris, T. E., 1974. Contact interactions on a lattice. *Ann. Probab.* 2, 969–988.
URL <http://dx.doi.org/10.1214/aop/1176996493>
- He, Y., Jayaprakash, C., Grinstein, G., 1990. Generic nonergodic behavior in locally interacting continuous systems. *Phys. Rev. A* 42, 3348–3355.
URL <http://dx.doi.org/10.1103/PhysRevA.42.3348>
- Liggett, T. M., 1999. *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes*. Springer.
URL <http://dx.doi.org/10.1007/978-3-662-03990-8>
- Liu, D., 2009. Generic two-phase coexistence and nonequilibrium criticality in a lattice version of Schlögl’s second model for autocatalysis. *J. Stat. Phys.* 135 (1), 77–85.
URL <http://dx.doi.org/10.1007/s10955-009-9708-2>
- Liu, D., Guo, X., Evans, J., 2007. Quadratic contact process: Phase separation with interface-orientation-dependent equestability. *Phys. Rev. Lett.* 98, 050601.
URL <http://dx.doi.org/10.1103/PhysRevLett.98.050601>
- Muñoz, M., de los Santos, F., Telo da Gama, M., 2005. Generic two-phase coexistence in nonequilibrium systems. *Eur. Phys. J. B – Condensed Matter and Complex Systems* 43, 73–79.
URL <http://dx.doi.org/10.1140/epjb/e2005-00029-3>
- Neuhauser, C., 1994. A long range sexual reproduction process. *Stochastic Process. Appl.* 53 (2), 193–220.
URL [http://dx.doi.org/10.1016/0304-4149\(94\)90063-9](http://dx.doi.org/10.1016/0304-4149(94)90063-9)
- Toom, A., 1974. Nonergodic systems of automata. *Probl. Inform. Transm.* 10 (3), 239–246.
- Toom, A., 1980. Stable and attractive trajectories in multicomponent systems. In: *Multicomponent Random Systems*. Vol. 6 of *Advances in Probability and Related Topics*. Dekker, New York, pp. 549–575.
- Varghese, C., Durrett, R., 2013. Phase transition in the quadratic contact process on complex networks. *Phys. Rev. E* 87, 062819.
URL <http://dx.doi.org/10.1103/PhysRevE.87.062819>

Weinberger, H., 1982. Long-time behavior of a class of biological models. *SIAM J. Math. Anal.* 13 (3), 353–396.
URL <http://dx.doi.org/10.1137/0513028>