Contact processes on random graphs with power law degree distributions have critical value 0.

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Abstract

If we consider the contact process with infection rate λ on a random graph on n vertices with power law degree distributions, mean field calculations suggest that the critical value λ_c of the infection rate is positive if the power $\alpha > 3$. Physicists seem to regard this as an established fact, since the result has recently been generalized to bipartite graphs by Gómez-Gardeñes et al (2008). Here, we show that the critical value λ_c is zero for any value of $\alpha > 3$, and the contact process, starting from all vertices infected, maintains a positive density of infected sites for time at least $\exp(n^{1-\delta})$ for any $\delta > 0$. Using the last result, together with the contact process duality, we can establish the existence of a quasi-stationary distribution in which a randomly chosen vertex is occupied with probability $\rho(\lambda)$. It is expected that $\rho(\lambda) \sim C\lambda^\beta$ as $\lambda \to 0$. Here we show that $\alpha - 1 \le \beta \le 2\alpha - 3$, and so $\beta > 2$ for $\alpha > 3$. Thus even though the graph is locally tree-like, β does not take the mean field critical value $\beta = 1$.

1 Introduction

In this paper we will study the contact process on random graphs with a power-law degree distribution, i.e., for some constant α, the degree of a typical vertex is k with probability $p_k \sim Ck^{-\alpha}$ as $k \to \infty$. Following Newman, Strogatz and Watts (2001, 2002), we construct the random graph G_n on the vertex set $\{1, 2, \ldots, n\}$ having degree distribution $\mathbf{p} = \{p_k : k \ge 0\}$ as follows. Let $d_1, \ldots, d_n$ be independent and have the distribution $P(d_i = k) = p_k$. We condition on the event $E_n = \{d_1 + \cdots + d_n \text{ is even}\}$ to have a valid degree sequence. As $P(E_n) \to 1/2$ as $n \to \infty$, the conditioning will have a little effect on the distribution of d_i's. Having chosen the degree sequence $(d_1, d_2, \ldots, d_n)$, we allocate d_i many half-edges to the vertex i, and then pair those half-edges at random. We also condition on the event that the graph is simple, i.e., it neither contains any self-loop at some vertex, nor contains multiple edges between two vertices. It can be shown (see e.g. Theorem 3.1.2 of Durrett (2007)) that if the degree distribution $\mathbf{p}$ has finite second moment, i.e., when $\alpha > 3$, the probability of the event that G_n is simple has a positive limit as $n \to \infty$, and hence the conditioning on this event will not have much effect on the distribution of d_i's.

We will be concerned with epidemics that take place on these random graphs. First consider the SIR (susceptible-infected-removed) model, in which sites begin as susceptible, and after being infected they get removed, i.e., become immune to further infection. In the simplest discrete-time formulation, an infected site x at time n will always be removed at time $n+1$ and for each susceptible neighbor y at time n x will cause y to become infected at time $n+1$ with probability p, with all of the infection events being independent.

In this case the spreading of the epidemic is equivalent to percolation. To compute the threshold, one notes that for a randomly chosen vertex x, the number of vertices at distance m from x, Z_m, is approximately a two-phase branching process in which the number of first generation children has distribution $\mathbf{p}$, but in the second and subsequent generations the offspring distribution is the size biased distribution $\mathbf{q} = \{q_k : k \geq 0\}$ satisfying

$$q_{k-1} = \frac{kp_k}{\mu} \quad \text{where } \mu = \textstyle\sum_k kp_k. \tag{1.1}$$

This occurs because vertices with degree k are k times as likely to be chosen for connections, and the edge that brings us to the new vertex uses up one of its degrees. For more details on this and the facts that we will quote in the next paragraph, see Chapter 3 of Durrett (2007).

With the above observation in hand, it is easy to compute the critical threshold for the SIR model. Let ν be the mean of the size biased distribution,

$$\nu = \sum_k kq_k. \tag{1.2}$$

Suppose we start the infection at a randomly chosen vertex x. Now if Y_m is the number of sites at distance m from x that become infected, then $EY_m = p\mu(p\nu)^{m-1}$. So the epidemic is supercritical if and only if $p > 1/\nu$. In particular, if $p_k \sim Ck^{-\alpha}$ as $k \to \infty$ and $\alpha \leq 3$, then $\nu = \infty$ and $p_c = 0$. Conversely if $\alpha > 3$ then $\nu < \infty$ and $p_c = 1/\nu > 0$. Hence for the SIR epidemic model on the random graph G_n with power-law degree distribution, there is a positive threshold for the infection to survive if and only if the power $\alpha > 3$.

Here, we will study the continuous-time SIS (susceptible-infected-susceptible) model and show that its behavior differs from that of the SIR model. In the SIS model, at any time t each site x is either infected or healthy (but susceptible). We often refer to the infected sites as occupied, and the healthy sites as vacant. We define the functions $\{\zeta_t : t \geq 0\}$ on the vertex set so that $\zeta_t(x)$ equals 0 or 1 depending on whether the site x is healthy or infected at time t. An infected site becomes healthy at rate 1 independent of other sites and is again susceptible to the disease, while a susceptible site becomes infected at a rate λ times the number of its infected neighbors. Harris (1974) introduced this model on the d-dimensional integer lattice and named it the *contact process*. See Liggett (1999) for an account of most of the known results. We will make extensive use of the *self-duality property* property of this process. If we let $\xi_t \equiv \{x : \zeta_t(x) = 1\}$ to be the set of infected sites at time t, we obtain a set-valued process. If we write ξ_t^A to denote the process with $\xi_0^A = A$, then the self-duality property says that

$$P(\xi_t^A \cap B \neq \emptyset) = P(\xi_t^B \cap A \neq \emptyset) \tag{1.3}$$

for any two subsets A and B of vertices.

Pastor-Satorras and Vespigniani (2001a, 2001b, 2002) have made an extensive study of this model using mean-field methods. Their nonrigorous computations suggest the following conjectures:

- If $\alpha \leq 3$, then $\lambda_c = 0$.
- If $3 < \alpha \leq 4$, then $\lambda_c > 0$ but the critical exponent β, which controls the rate at which the equilibrium density of infected sites goes to 0, satisfies $\beta > 1$.
- If $\alpha > 4$, then $\lambda_c > 0$ and the equilibrium density $\sim C(\lambda - \lambda_c)$ as $\lambda \downarrow \lambda_c$, i.e. the critical exponent $\beta = 1$.

Notice that the conjectured behavior of λ_c for the SIS model parallels the results for p_c in the SIR model quoted above.

Gómez-Gardeñes et al. (2008) have recently extended this calculation to the bipartite case, which they think of as a social network of sexual contacts between men and women. They define the polynomial decay rates for degrees in the two sexes to be γ_M and γ_F, and argue that the epidemic is supercritical when the transmission rates for the two sexes satisfy

$$\sqrt{\lambda_M \lambda_F} > \lambda_c = \sqrt{\frac{< k >_M < k >_F}{< k^2 >_F < k^2 >_M}}$$

where the pointy brackets indicate expected value and k is shorthand for the degree distribution. Here λ_c is positive when $\gamma_M, \gamma_F > 3$.

Berger, Borgs, Chayes, and Saberi (2004) have considered the contact process on the Barabási-Albert preferential attachment graph, which has a power law degree distribution with $\alpha = 3$. They have shown that $\lambda_c = 0$. Their proof starts with the following observation. Here, we follow the formulation in Lemma 4.8.2 of Durrett (2007).

Lemma 1.1. *Suppose G is a star graph with center 0 and leaves $1, 2, \ldots, k$. Let A_t be the set of vertices infected in the contact process at time t when $A_0 = \{0\}$. If $k\lambda^2 \to \infty$, then $P(A_{\exp(k\lambda^2/10)} \neq \emptyset) \to 1$.*

Using this Lemma and the fact that when there are n vertices, the vertex of largest degree in the preferential attachment graph has $O(n^{1/2})$ neighbors, they show that starting from all sites occupied the infection survives for time $\geq \exp(cn^{1/2})$ for some constant c. Based on results for the contact process on ($\mathbf{Z}$ mod n) by Durrett and Liu (1988) and Durrett and Schonmann (1988), and on ($\mathbf{Z}$ mod $n)^d$ by Mountford (1993), it is natural to conjecture that in the contact process on G_n, the infection survives for time $\geq \exp(cn)$ for some constant c. It certainly cannot last longer, because the total number of edges is $O(n)$, and so even if all sites are occupied at time 0, there is a probability $\geq \exp(-cn)$ that all sites will be vacant at time 1.

The next result is the main result of this paper. It shows that the physicists' mean field computations are incorrect, i.e., $\lambda_c = 0$ for all $\alpha > 3$, and it gives almost the right lower bound on the survival time.

Theorem 1. *Consider a Newman, Strogatz and Watts random graphs G_n on the vertex set $\{1, 2, \ldots, n\}$, where the degrees d_i satisfy $P(d_i = k) \sim Ck^{-\alpha}$ as $k \to \infty$ for some constant C and some $\alpha > 3$, and $P(d_i \leq 2) = 0$. Let $\{\xi_t^1 : t \geq 0\}$ denote the contact process on the random graph G_n starting from all sites occupied, i.e., $\xi_0^1 = \{1, 2, \ldots, n\}$. Then for any value of the infection rate $\lambda > 0$, there is a positive constant $p(\lambda)$ so that for any $\delta > 0$*

$$\inf_{t \leq \exp\left(n^{1-\delta}\right)} P\left(\frac{|\xi_t^1|}{n} \geq p(\lambda)\right) \to 1 \quad \text{as } n \to \infty.$$

One could assume that $\nu > 1$ and look at the process on the giant component, but we would rather avoid this complication. The assumption $P(d_i \leq 2) = 0$ is convenient, because it implies the following.

Lemma 1.2. *Consider a Newman, Strogatz and Watts graphs, G_n, on n vertices, where the degrees of the vertices, d_i, satisfy $P(d_i \leq 2) = 0$, and the mean of the size biased degree distribution $\nu < \infty$. Then $P(G_n$ is connected $) \to 1$ as $n \to \infty$, and if D_n is the diameter of G_n, $P(D_n > (1+\epsilon)\log n/\log \nu) \to 0$ for any $\epsilon > 0$.*

The size of the giant component in the graph is given by the nonextinction probability of the two-phase branching process, so $P(d_i \leq 2) = 0$ is needed to have the size $\sim n$. Intuitively, Lemma 1.2 is obvious because the worst case is the random 3-regular graph, and in this case, the graph is not only connected and has diameter $\sim (\log n)/(\log 2)$, see Sections 7.6 and 10.3 of Bollobás (2001), but the probability of a Hamiltonian cycle tends to 1, see Section 9.3 of Janson, Luczak, and Ruciński (2000). We have not been able to find a proof of Lemma 1.2 in the literature, so we give one in Section 5. By comparing the growth of the cluster with a branching process it is easy to show $P(D_n < (1-\epsilon)\log n/\log \nu) \to 0$ for any $\epsilon > 0$.

Theorem 1 shows that the fraction of infected sites in the graph G_n is bounded away from zero for a time longer than $\exp(n^{1/2})$. So using self-duality we can now define a quasi-stationary measure ξ_∞^1 on the subsets of $\{1, 2, \ldots, n\}$ as follows. For any subset of vertices A, $P(\xi_\infty^1 \cap A \neq \emptyset) \equiv P(\xi^A_{\exp(n^{1/2})} \neq \emptyset)$. Let X_n be uniformly distributed on $\{1, 2, \ldots, n\}$ and let $\rho_n(\lambda) = P(X_n \in \xi_\infty^1)$. Berger, Borgs, Chayes and Saberi (2004) show that for the contact process on the Bŕabasi-Albert preferential attachment graph, there are positive, finite constants so that

$$b\lambda^C \leq \rho_n(\lambda) \leq B\lambda^c.$$

In contrast, we get reasonably good numerical bounds on the critical exponent.

Theorem 2. *Suppose $\alpha > 3$. If $0 < \lambda < \lambda_0$ and $0 < \delta < 1$, then there exists two constants $c(\alpha, \delta)$ and $C(\alpha, \delta)$ so that as $n \to \infty$*

$$P(c\lambda^{1+(\alpha-2)(2+\delta)} \leq \rho_n(\lambda) \leq C\lambda^{1+(\alpha-2)(1-\delta)}) \to 1.$$

When α is close to 3 and δ is small, the powers in the lower and upper bounds are close to 3 and 2. As $\alpha \to \infty$ and $\delta \to 0$ the ratio of the two bounds converges to 2.

The intuition behind the lower bound is that if the infection starts from a vertex of degree $d(x) \geq (10/\lambda)^{2+\delta}$, then it survives for a long time with a probability bounded away from 0. The density of such points is $\sim C\lambda^{(2+\delta)(\alpha-1)}$, but we can improve the bound to the one given by looking at neighbors of these vertices, which have density $\sim C\lambda^{(2+\delta)(\alpha-2)}$ and will infect their large degree neighbor with probability $\geq c\lambda$.

For the upper bound we show that if $m(\alpha, \delta)$ is large enough and the infection starts from a vertex x such that there is no vertex of degree $\geq \lambda^{-(1-\delta)}$ within distance m from x, then its survival is very unlikely. To get the extra factor of λ we note that the first event must a birth. Based on the proof of Lemma 1.1, we expect that survival is unlikely if there is no nearby vertex of degree $\geq \lambda^{-2}$ and hence the lower bound gives the critical exponent.

Having seen $\rho_n(\lambda)$ in Theorem 2, it is natural to speculate that $\rho_n(\lambda) \to \rho(\lambda)$ as $n \to \infty$. By the heuristics for the computation of λ_c in the SIR model, it is natural to guess that, when $\alpha > 2$, $\rho(\lambda)$ is the expected probability of weak survival for the contact process on a tree generated by the two-phase branching process, starting with the origin occupied.

Here the phrase 'weak survival' refers to set of infected sites being not empty for all times, in contrast to 'strong survival' where the origin is reinfected infinitely often. As in the case of the contact process on the Bollobás-Chung small world studied by Durrett and Jung (2007), it is the weak survival critical value that is the threshold for prolonged persistence on the finite graph.

Sketch of the proof of Theorem 1.

The remainder of the paper is devoted to proofs. Let V_n^ϵ be the set of vertices in the graph G_n with degree at least n^ϵ. We call the points in V_n^ϵ *stars*. We say that a star of degree k is *hot* if at least $\lambda k/4$ of its neighbors are infected and is *lit* if at least $\lambda k/10$ of its neighbors are infected. Our first step, taken in Lemma 2.2, is to improve the proof of Lemma 1.1 to show that a hot star will remain lit for time $\exp(cn^\epsilon)$ with high probability.

To keep the system going for a long time, we cannot rely on just one star. There are $O(n^{1-\epsilon(\alpha-1)})$ stars in this graph which has diameter $O(\log n)$. If one star goes out, presence of a lit star can make it hot again within a time $2n^{\epsilon/3}$ with probability at least n^{-b}. See Lemmas 2.3 and 2.4 for this. Lemma 2.6 shows that a lit star gets hot within $2\exp(n^{\epsilon/3})$ units of time with probability

$$\geq 1 - 5\exp(-\lambda^2 n^{\epsilon/3}/16),$$

and Lemma 2.5 shows that a hot star eventually succeeds to make a non-lit star hot within $\exp(n^{\epsilon/2})$ units of time with probability

$$\geq 1 - 8e^{-\lambda^2 n^\epsilon/80}.$$

Using these estimates, we can show that the number of lit stars dominates a random walk with a strong positive drift, and hence more than 3/4's of the collection will stay lit for a time $O(\exp(n^{1-\alpha\epsilon}))$. See Proposition 1 at the end of Section 2 for the argument.

To get a lower bound on the density of infected sites, first we bound the probability of the event that the dual process, starting from a vertex of degree $(10/\lambda)^{2+\delta}$, reaches more than 3/4's of the stars. We do this in two steps. In the first step (see Lemma 3.2) we get a lower bound for the probability of the dual process reaching one of the stars. To do this, we consider a chain of events in which we reach vertices with degree $(10/\lambda)^{k+\delta}$ for $k \geq 2$ sequentially. In the second step (see Lemma 3.3) we again use a comparison with random walk to show that, with probability tending to 1, the dual process, starting from any lit star, will light up more than 3/4's of the stars. Then we show that the above events are asymptotically uncorrelated, and use a second moment argument to complete the proof of Theorem 1 and the lower bound for the density in Theorem 2.

Open Problem. *Improve the bounds in Theorem 2 and extend the result to $\alpha > 1$.*

When $2 < \alpha < 3$ the size biased distribution has infinite mean. Chung and Lu (2002, 2003) obtained bounds on the diameter in this case, and later van der Hofstadt, Hooghiemstra, and Zamenski (2007) showed

$$H_n \sim \frac{2 \log \log n}{-\log(\alpha - 2)}$$

When $1 < \alpha < 2$ the size-biased distribution has infinite mass. van der Hofstadt, Hooghiemstra, and Zamenski (2006) have shown in this case if H_n is the distance between 1 and 2 then

$$\lim_{n\to\infty} P(H_n = 2) = \lim_{n\to\infty} 1 - P(H_n = 3) = p \in (0, 1)$$

so the graph is very small.

All of the results about the persistence of infection at stars in Section 2 are valid for any α, since they only rely on properties of the contact process on a star graph and an upper bound on the diameter. The results in Section 3, rely on the existence of the size biased distribution and hence are restricted to $\alpha > 2$. The proof of the lower bound should be extendible to that case, but the proof of the upper bound given in Section 4 relies heavily on the size-biased distribution having finite mean. When $1 < \alpha < 2$, the size-biased distribution does not exist and the situation changes drastically. We guess that in this case $\rho_n(\lambda) = O(\lambda)$.

2 Persistence of infection at stars

Let $\epsilon > 0$ and let V_n^ϵ be the set of vertices in our graph G_n with degree at least n^ϵ. We call these vertices *stars*. We say that a vertex of degree k is *hot* if it has at least $L = \lambda k/4$ infected neighbors and we call it *lit* if it has at least $0.4L = \lambda k/10$ infected neighbors. We will show that if ϵ is small, then in the contact process starting from all vertices occupied, most of the stars in V_n^ϵ will remain lit for time $O(\exp(n^{1-\alpha\epsilon})$.

We begin with a slight improvement of Lemma 1.1 which gives a numerical estimate of the failure probability, but before that we need two simple estimates.

Lemma 2.1. *If $0 \leq x \leq a \leq 1$ then $e^x \leq 1 + (1 + a)x$ and $e^{-x} \leq 1 - (1 - 2a/3)x$.*

Proof. Using the series expansion for e^x

$$
\begin{aligned}
e^x &\leq 1 + x + \frac{ax}{2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right) \\
e^{-x} &\leq 1 - x + \frac{ax}{2}\left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots\right)
\end{aligned}
$$

and summing the geometric series gives the result. □

Lemma 2.2. *Let G be a star graph with center 0 and leaves $1, 2, \ldots, k$. Let A_t be the set of vertices infected in the contact process at time t. Suppose $\lambda \leq 1$ and $\lambda^2 k \geq 50$. Let $L = \lambda k/4$ and let $T = \exp(k\lambda^2/80)/4L$. Let $P_{L,i}$ denote the probability when at time 0 the center is at state i and L leaves are infected. Then*

$$
P_{L,i}\left(\inf_{t \leq T} |A_t| \leq 0.4L\right) \leq 7e^{-\lambda^2 k/80} \qquad \textit{for } i = 0, 1.
$$

Proof. Write the state of the system as (m, n) where m is the number of infected leaves and $n = 1$ if the center is infected and 0 otherwise. To reduce to a one dimensional chain, we will concentrate on the first coordinate. When the state is $(m, 0)$ with $m > 0$, the next event will occur after exponential time with mean $1/(m\lambda + m)$, and the probability that it will be the reinfection of the center is $\lambda/(\lambda + 1)$. So the number of leaf infections N that will die while the center is 0 has a shifted geometric distribution with success probability $\lambda/(\lambda + 1)$, i.e.,

$$
P(N = j) = \left(\frac{1}{\lambda + 1}\right)^j \cdot \frac{\lambda}{\lambda + 1} \quad \text{for } j \geq 0.
$$

Let N_L be the realization of N when the state of the system is $(L, 0)$. Then N_L will be more than $0.1L$ with probability

$$
P_{L,0}(N_L > 0.1L) \leq (1 + \lambda)^{-0.1L} \leq e^{-\lambda L/20} = e^{-\lambda^2 k/80}. \tag{2.1}
$$

Here we use the inequality $1 + \lambda \geq e^{\lambda/2}$. If $N_L \leq 0.1L$, then there will be at least $0.9L$ infected leaves when the center is infected.

The next step is to modify the chain so that the infection rate is 0 when the number of infected leaves is $L = \lambda k/4$ or greater. In this case the number of infected leaves $\geq Y_t$ where

$$
\begin{array}{rll}
 & \text{at rate} & \\
Y_t \to Y_t - 1 & \lambda k/4 & \\
Y_t \to Y_t + 1 & 3\lambda k/4 & \text{for } Y_t < L\ . \\
Y_t \to Y_t - N & 1 &
\end{array}
$$

To bound the survival time of this chain, we will estimate the probability that starting from $0.8L$ it will return to $0.4L$ before hitting L. During this time Y_t is a random walk that jumps at rate $\lambda k + 1$. Let X be the change in the random walk in one step. Then

$$X = \begin{cases} -1 & \text{with probability } (\lambda k/4)/(\lambda k+1) \\ +1 & \text{with probability } (3\lambda k/4)/(\lambda k+1) \\ -N & \text{with probability } 1/(\lambda k+1), \end{cases}$$

and so

$$\begin{aligned} Ee^{\theta X} &= e^{\theta} \cdot \frac{3}{4} \cdot \frac{\lambda k}{\lambda k+1} + e^{-\theta} \cdot \frac{1}{4} \cdot \frac{\lambda k}{\lambda k+1} \\ &\quad + \frac{1}{\lambda k+1} \sum_{j=0}^{\infty} e^{-\theta j} \left(\frac{1}{\lambda+1} \right)^j \cdot \frac{\lambda}{\lambda+1}. \end{aligned}$$

If $e^{-\theta}/(\lambda+1) < 1$, the third term on the right is

$$\frac{\lambda}{\lambda k+1} \cdot \frac{1}{1+\lambda-e^{-\theta}}.$$

If we pick $\theta < 0$ so that $e^{-\theta} = 1 + \lambda/2$, then

$$Ee^{\theta X} = \frac{\lambda k}{\lambda k+1} \left(\frac{1}{1+\lambda/2} \cdot \frac{3}{4} + (1+\lambda/2) \cdot \frac{1}{4} + \frac{2}{\lambda k} \right).$$

Since $1/(1+x) < 1 - x + x^2$ for $0 < x < 1$,

$$\begin{aligned} &\frac{1}{1+\lambda/2} \cdot \frac{3}{4} + (1+\lambda/2) \cdot \frac{1}{4} + \frac{2}{\lambda k} - 1 \\ &< \left(-\frac{\lambda}{2} + \frac{\lambda^2}{4} \right) \frac{3}{4} + \frac{\lambda}{8} + \frac{2}{\lambda k} \\ &< -\frac{3\lambda}{16} + \frac{\lambda}{8} + \frac{2}{\lambda k}, \end{aligned}$$

where in the last inequality, we have used $\lambda < 1$. Since we have assumed $\lambda^2 k \geq 50$, the right-hand side is < 0.

To estimate the hitting probability we note that if $\phi(x) = \exp(\theta x)$ and $Y_0 \geq 0.6L$, then $\phi(Y_t)$ is a supermartingale until it hits L. Let q be the probability that Y_t hits the interval $(-\infty, 0.4L]$ before returning to L. Since $\theta < 0$, we have $\phi(x) \geq \phi(0.4L)$ for $x \leq 0.4L$. So using the optional stopping theorem we have

$$q\phi(0.4L) + (1-q)\phi(L) \leq \phi(0.8L),$$

which implies that

$$q \leq \phi(0.8L)/\phi(0.4L) = \exp(0.4\theta L) \leq e^{-\lambda^2 k/40},$$

as $e^{-\theta} = 1 + \lambda/2 \geq e^{\lambda/4}$ when $\lambda/4 < 1/2$ (sum the series for e^x).

At this point we have estimated the probability that the chain started at a point $\geq 0.8L$ will go to L before going below $0.4L$. When the chain is at L, the time until the next jump is exponential with mean $1/(L+1) \geq 1/2L$. The probability that the jump takes us below $0.8L$ is (since $1 + \lambda \geq e^{\lambda/2}$)

$$\leq (1+\lambda)^{-0.2L} \leq e^{-\lambda L/10} = e^{-\lambda^2 k/40}.$$

Thus the probability that the chain fails to return to L, $M = e^{\lambda^2 k/80}$ times before going below $0.4L$ is

$$\leq 2e^{-\lambda^2 k/80}.$$

Using Chebyshev's inequality on the sum, S_M of M exponentials with mean 1 (and hence variance 1),

$$P(S_M < M/2) \leq 4/M.$$

Multiplying by $1/2L$ we see that the total time, T_M of the first M excursions satisfies

$$P(T_M < M/4L) \leq 4e^{-\lambda^2 k/80}.$$

Combining this with the previous estimate on the probability of having fewer than M returns and the error probability in (2.1) proves the desired result. □

Thus Lemma 2.2 shows that a hot star will remain lit for a long time with probability very close to 1. Our next step is to investigate the process of transferring the infection from one star to another. The first step in doing that is to estimate what happens when only the center of the star infected.

Lemma 2.3. *Let G be a star graph with center 0 and leaves $1, 2, \ldots, k$. Let $0 < \lambda < 1$, $\delta > 0$ and suppose $\lambda^{2+\delta} k \geq 10$. Again let $P_{l,i}$ denote the probability when at time 0 the center is in state i and l leaves are infected. Let τ_0 be the first time 0 becomes healthy, and let T_j be the first time the number of infected leaves equals j. If $L = \lambda k/4$, $\gamma = \delta/(4+2\delta)$, and $K = \lambda k^{1-\gamma}/4$, then for $k \geq k_0(\delta)$*

$$\begin{aligned} P_{0,1}(T_K > \tau_0) &\leq 2/k^\gamma, \\ P_{K,1}(T_0 < T_L) &\leq \exp(-\lambda^2 k^{1-\gamma}/16) \leq 1/k^\gamma, \\ E_{0,1}(T_L | T_L < \infty) &\leq 2. \end{aligned}$$

Combining the first two inequalities $P_{0,1}(T_L < \infty) \geq 1 - 2/k^\gamma$, and using Markov's inequality, if we can infect a vertex of degree at least k such that $k \geq k_0(\delta)$ and $\lambda^{2+\delta} k > 10$, then with probability $\geq 1 - 5/k^\gamma$ the vertex gets hot within the next k^γ units of time.

Proof. Note that $\tau_0 \sim \exp(1)$, and for any $t \leq \tau_0$, the leaves independently becomes healthy at rate 1 and infected at rate λ. Let $p_0(t)$ is the probability that leaf j is infected at time t when the central vertex of the star has remained infected for all times $s \leq t$. $p_0(0) = 0$ and

$$\frac{dp_0(t)}{dt} = -p_0(t) + (1 - p_0(t))\lambda = \lambda - (\lambda+1)p_0(t).$$

So solving gives $p_0(t) = \int_0^t \lambda e^{-(\lambda+1)(t-s)}\, ds = \frac{\lambda}{\lambda+1}\left(1 - e^{-(\lambda+1)t}\right)$. From this it follows that

$$P_{0,1}(T_K < \tau_0) \geq P(\text{Binomial}(k, p_0(k^{-\gamma})) > K) P(\tau_0 > k^{-\gamma}). \tag{2.2}$$

Now if $k^\gamma > 8/3$, $(\lambda+1)k^{-\gamma} \leq 3/4$ and it follows from Lemma 2.1 that

$$p_0(k^{-\gamma}) \geq \lambda k^{-\gamma}/2.$$

Writing $p = p_0(k^{-\gamma})$ to simplify formulas, if $\theta > 0$

$$P(\text{Binomial}(k,p) \leq K) \leq e^{\theta K}\left(1 - p + pe^{-\theta}\right)^k.$$

Since $\log(1+x) \leq x$ the right-hand side is

$$\leq \exp\left(\frac{\theta \lambda k^{1-\gamma}}{4} + (e^{-\theta} - 1)\frac{\lambda k^{1-\gamma}}{2}\right).$$

Taking $\theta = 1/2$ and using Lemma 2.1 to conclude $e^{-1/2} - 1 \leq -1/3$, the above is

$$\leq \exp(-\lambda k^{1-\gamma}/24) \leq \exp(-k^{1/2-\gamma}/8),$$

since $\lambda^2 k \geq 9$. Using this in (2.2), the right-hand side is

$$\geq (1 - \exp(-k^{1/2-\gamma}/8))(1 - k^{-\gamma}) \geq 1 - 2/k^\gamma,$$

if $k^{1/2-\gamma} \geq 8\gamma \log k$.

Using the supermartingale from the proof of Lemma 2.2, if $q = P_{K,1}(T_0 < T_L)$, then we have

$$q \cdot 1 + (1-q)e^{\theta L} \leq e^{\theta K},$$

and so $q \leq e^{\theta K} \leq e^{-\lambda K/4}$. In the last step we have used $e^\theta = 1/(1+\lambda/2) \leq e^{-\lambda/4}$, which comes from Lemma 2.1. Filling in the value of K, $e^{-\lambda K/4} = e^{-\lambda^2 k^{1-\gamma}/16}$. Now

$$\lambda^2 k^{1-\gamma} = (\lambda^{2+\delta} k)^{2/(2+\delta)} k^{1-\gamma-2/(2+\delta)} \geq 10^{2/(2+\delta)} k^{\delta/(4+2\delta)}.$$

So if $k^{\delta/(4+2\delta)} > 16 \cdot 10^{-2/(2+\delta)} \gamma \log k$, then $e^{-\lambda K/4} \leq 1/k^\gamma$.

To bound the time we use the lower bound random walk Y_t from Lemma 2.2. $EN = 1/\lambda$, so

$$EY_t = \left(\frac{\lambda k}{2} - \frac{1}{\lambda}\right) t = \left(\frac{\lambda^2 k - 2}{2\lambda}\right) t.$$

Let T_L^Y be the hitting time of L for the random walk Y_t. Using the optional stopping theorem for the martingale $Y_t - (\lambda^2 k - 2)t/2\lambda$ and the bounded stopping time $T_L^Y \wedge t$ we get

$$EY_{T_L^Y \wedge t} - \left(\frac{\lambda^2 k - 2}{2\lambda}\right) E\left(T_L^Y \wedge t\right) = EY_0 = 0.$$

Since $EY_{T_L^Y \wedge t} \le L = \lambda k/4$, it follows that

$$E(T_L^Y \wedge t) \le \left(\frac{2\lambda}{\lambda^2 k - 2} \right) L = \frac{\lambda^2 k/2}{\lambda^2 k - 2} = \frac{1}{2 - 4/\lambda^2 k} \le 1,$$

as by our assumption $\lambda^2 k \ge 4$. Letting $t \to \infty$ we have $ET_L^Y \le 1$. Since Y_t is a lower bound for the number of infected leaves, $T_L \mathbf{1}_{[T_L<\infty]} \le T_L^Y$. Hence

$$E_{0,1}(T_L | T_L < \infty) = \frac{E_{0,1}\left(T_L \mathbf{1}_{[T_L<\infty]}\right)}{P_{0,1}(T_L < \infty)} \le \frac{E_{0,1} T_L^Y}{P_{0,1}(T_K < \tau_0) P_{K,1}(T_L < T_0)} \le 1/(1/2) = 2$$

for large k. □

To transfer infection from one vertex to another we use the following Lemma.

Lemma 2.4. *Let $v_0, v_1, \ldots v_m$ be a path in the graph and suppose that v_0 is infected at time 0. Then the probability that v_m will become infected by time m is $\ge (e^{-1}(1 - e^{-\lambda})e^{-1})^m$.*

Proof. The first factor is the probability that the infection at v_0 lasts for time 1, the second the probability that v_0 infects v_1 by time 1, and the third the probability that the infection at v_1 remains until time 1. Iterating this m times brings the infection from 0 to m. □

When the diameter of the graph is $\le 2 \log n$, the probability in Lemma 2.4 is $\ge n^{-b}$ for some $b \in (1/2, \infty)$, and the time required is $\le 2 \log n$. Combining this with Lemma 2.3 (with $k = n^\epsilon$ and $\gamma = 1/3$) shows that if n is large, then with probability $\ge Cn^{-b}$ we can use one hot star to make another star hot within time $2n^{\epsilon/3}$. Using Lemma 2.2 and trying repeatedly gives the following Lemma.

Lemma 2.5. *Let s_1 and s_2 be two stars in V_n^ϵ and suppose that s_1 is hot at time 0. Then, for large n, s_2 will be hot by time $T = \exp(n^{\epsilon/2})$ with probability*

$$\ge 1 - 8e^{-\lambda^2 n^\epsilon/80}.$$

Proof. If n is large, Lemma 2.2 shows that s_1 remains lit for T units of time with probability $\ge 1 - 7e^{-\lambda^2 n^\epsilon/80}$. Let $t_n = 2n^{\epsilon/3}$ and consider the discrete time points $t_n, 2t_n, \ldots$. At all of these time points we can think of a path starting from an infected neighbor of s_1 up to s_2. Using one such path the infection gets transmitted to s_2 and it gets hot in $2n^{\epsilon/3}$ units of time with probability $\ge Cn^{-b}$ for some constant C. So s_1 fails to make s_2 hot by time T with probability

$$\le (1 - Cn^{-b})^{T/t_n} \le \exp(-Cn^{-b} T/t_n) \le \exp(-\lambda^2 n^\epsilon/80)$$

for large n. For the first inequality we use $1 - x \le e^{-x}$. Combining with the first error probability in this proof, we get the result. □

Next we show that a lit star becomes hot with a high probability, and then helps to make other non-lit stars lit.

Lemma 2.6. *Let s be a star of V_n^ϵ and suppose that s is lit at time 0. Then s will be hot by time $2\exp(n^{\epsilon/3})$ with probability*

$$\geq 1 - 5\exp(-\lambda^2 n^{\epsilon/3}/16),$$

if n is large.

Proof. Since s is lit, it has at least $\lambda n^\epsilon/10$ infected neighbors at time 0. If s itself is not infected at time 0, let N be the number of leaf infections that die out before s gets infected. Using similar argument as in the beginning of the proof of Lemma 2.2,

$$P(N = j) = \left(\frac{1}{\lambda+1}\right)^j \cdot \frac{\lambda}{\lambda+1} \quad \text{for } j \geq 0,$$

which implies

$$P(N > \lambda n^\epsilon/20) \leq (1+\lambda)^{-\lambda n^\epsilon/20} \leq e^{-\lambda^2 n^\epsilon/40},$$

as $1+\lambda > e^{\lambda/2}$ by Lemma 2.1. Also the time T_M taken for $M = \lambda n^\epsilon/20$ leaf infections to die out is a sum of M exponentials with mean at most $1/(\lambda+1)M \leq 1/M$. Now if $n^{2\epsilon/3} > 40/16$, the above error probability is $\leq e^{-\lambda^2 n^{\epsilon/3}/16}$.

Using Chebyshev's inequality on the sum, S_M of M exponentials with mean 1 (and hence variance 1), we see that if $\exp(n^{\epsilon/3}) \geq 2$, i.e., $n^{\epsilon/3} > \log 2$

$$P(S_M > M\exp(n^{\epsilon/3})) \leq \frac{1}{M(\exp(n^{\epsilon/3})-1)^2} \leq \frac{4}{M\exp(2n^{\epsilon/3})} \leq \exp(-\lambda^2 n^{\epsilon/3}/16).$$

where in the final inequality we have used $M > 4$, i.e., $n^\epsilon > 80/\lambda$, and $\lambda^2/16 < 2$.

Multiplying by $1/M$ we see that the total time, T_M, satisfies

$$P(T_M > \exp(n^{\epsilon/3})) \leq \exp(-\lambda^2 n^{\epsilon/3}/16).$$

Combining these two error probabilities gives that s will be infected along with at least $\lambda n^\epsilon/20$ infected neighbors within $\exp(n^{\epsilon/3})$ units of time with error probability

$$\leq 2\exp(-\lambda^2 n^{\epsilon/3}/16). \tag{2.3}$$

Now $\lambda n^\epsilon/20 \geq \lambda n^{\epsilon/3}/4$, when $n^{2\epsilon/3} > 5$. So if s is infected and has at least $\lambda n^\epsilon/20$ infected neighbors, then using the second inequality of Lemma 2.3 (with $\gamma = 2/3$ and $k = n^\epsilon$), s becomes hot with error probability

$$\leq \exp(-\lambda^2 n^{\epsilon/3}/16).$$

Finally using Markov's inequality and the third inequality of Lemma 2.3, the time T_s taken by s to get hot, after it became infected, is more than $T = \exp(n^{\epsilon/3})$ with probability

$$\leq 2\exp(-n^{\epsilon/3}) \leq 2\exp(-\lambda^2 n^{\epsilon/3}/16),$$

as $\lambda < 1$. Combining all these error probabilities proves the Lemma. □

We now use Lemmas 2.5, 2.6 and 2.2 to prove that if the contact process starts from all sites infected, then for a long time at least 3/4's of the stars will be lit.

Proposition 1. *Let* $I^{\epsilon}_{n,t}$ *be the set of stars in* V^{ϵ}_{n} *which are lit at time* t *in the contact process* $\{\xi^1_t : t \geq 0\}$ *on* G_n. *Let* $t_n = 2\exp(n^{\epsilon/2})$ *and* $M_n = \exp(n^{1-\alpha\epsilon})$. *Then there is a stopping time* T_n *such that* $T_n \geq M_n \cdot t_n$ *and*

$$P\left(\left|I^{\epsilon}_{n,T_n}\right| \leq (3/4)\left|V^{\epsilon}_{n}\right|\right) \leq \exp(-Cn^{\epsilon}).$$

Proof. Let $\alpha_n = |V^{\epsilon}_n|$. Clearly $|I^{\epsilon}_{n,0}| = \alpha_n$. We will estimate the probability that starting from $(7/8)\alpha_n$ lit stars, the number goes below $(3/4)\alpha_n$ before reaching α_n. Define the stopping times τ_is' and σ_is' as follows. Let $\tau_0 = \sigma_0 = 0$ and for $i \geq 0$ let

$$\begin{aligned} \tau_{i+1} &\equiv \inf\left\{t > \tau_i + \sigma_i t_n : \left|I^{\epsilon}_{n,t}\right| = (7/8)\alpha_n\right\}, \\ \sigma_{i+1} &\equiv \min\left\{s \in \mathbb{N} : \left|I^{\epsilon}_{n,\tau_{i+1}+s\cdot t_n}\right| \notin ((3/4)\alpha_n, \alpha_n)\right\}. \end{aligned}$$

We need to look at time lags that are multiples of t_n in the definition of σ_i because in our worst nightmare (which is undoubtedly a paranoid delusion) all the lit stars of degree $k \geq n^{\epsilon}$ at time τ_{i+1} have exactly $0.1k$ infected neighbors .

Lemma 2.6 implies that a lit star of V^{ϵ}_n gets hot within time $2\exp(n^{\epsilon/3}) \leq \exp(n^{\epsilon/2})$ (for large n) with probability $\geq 1 - 5\exp(-\lambda^2 n^{\epsilon/3}/16)$. Combining with Lemma 2.2 gives that a lit star at time 0 gets hot by time $t_n/2$ and remains lit at time t_n with probability $\geq 1 - 6\exp(-\lambda^2 n^{\epsilon/3}/16)$ for large n. Now if $|I^{\epsilon}_{n,t}| < \alpha_n$ for some t, then the number of lit stars will increase at time $t + t_n$ with probability $\geq P(A \cap B)$, where

A : All the lit stars will get hot by $t_n/2$ units of time, and be lit for time t_n.

B : A non-lit star will become hot by time $t_n/2$ in presence of another hot star, and remain lit for another $t_n/2$ units of time.

Now using the above argument $P(A) \geq 1 - 6n\exp(-\lambda^2 n^{\epsilon/3}/16)$, as there are at most n stars. Combining Lemma 2.5 and 2.2 gives $P(B) \geq 1 - 9\exp(-\lambda^2 n^{\epsilon}/80)$. So $P(A \cap B) \geq 1 - \exp(-n^{\epsilon/4})$ for large n. Using the stopping times $\left|I^{\epsilon}_{n,\tau_i + r\cdot t_n}\right| \geq W_r$ for $r \leq \sigma_i$, where $\{W_r : r \geq 0\}$ is a discrete time random walk satisfying

$$\begin{aligned} W_r &\to W_r - 1 \text{ with probability } \exp\left(-n^{\epsilon/4}\right), \\ W_r &\to W_r + 1 \text{ with probability } 1 - \exp\left(-n^{\epsilon/4}\right), \end{aligned} \tag{2.4}$$

and $W_0 = (7/8)\alpha_n$. Now θ^{W_r} is a martingale where

$$\theta = \frac{\exp(-n^{\epsilon/4})}{1 - \exp(-n^{\epsilon/4})} < \exp(-n^{\epsilon/4}/2). \tag{2.5}$$

If q is the probability that W_r goes below $(3/4)\alpha_n$ before hitting α_n, then applying the optional stopping theorem

$$q \cdot \theta^{(3/4)\alpha_n} + (1-q) \cdot \theta^{\alpha_n} \le \theta^{(7/8)\alpha_n},$$

which implies

$$q \le \theta^{(\alpha_n/8)} \le \exp\left(-Cn^{1-(\alpha-1)\epsilon}\right),$$

as $\alpha_n \sim Cn^{1-(\alpha-1)\epsilon}$ for some constant C. So the probability that the random walk fails to return to α_n at least $M_n = \exp(n^{1-\alpha\epsilon})$ times before going below $(3/4)\alpha_n$ is $\le \exp(-Cn^{\epsilon})$. Now if

$$K = \min\left\{i \ge 1 : \left|I^{\epsilon}_{n,\tau_i+\sigma_i\cdot t_n}\right| \le (3/4)\alpha_n\right\},$$

the coupling with the random walk will imply $P(K \le M_n) \le \exp(-Cn^{\epsilon})$, and hence for $T_n \equiv \tau_{M_n} + \sigma_{M_n} \cdot t_n$

$$P\left(\left|I^{\epsilon}_{n,T_n}\right| \le (3/4)\left|V^{\epsilon}_n\right|\right) \le \exp(-Cn^{\epsilon}).$$

As $\sigma_i \ge 1$ for all i, by our construction $T_n \ge M_n \cdot t_n$, and we get the result. □

So the infection persists for time longer than $\exp\left(n^{1-\alpha\epsilon}\right)$ in the stars of V^{ϵ}_n.

3 Density of infected stars

Proposition 1 implies that if the contact process starts with all vertices infected, most of the stars remain lit even after $\exp(n^{1-\alpha\epsilon})$ units of time. In this section we will show that the density of infected stars is bounded away from 0 and we will find a lower bound for the density. We start with the following Lemma about the growth of clusters in the random graph G_n, when we expose the neighbors of a vertex one at a time. For more details on this procedure see Section 3.2 of Durrett (2007).

Lemma 3.1. *Suppose* $0 < \delta \le 1/8$. *Let* A *be the event that the two clusters, starting from 1 and 2 respectively, intersect before their sizes grow to* n^{δ}. *Then*

$$P(A) \le Cn^{-(\frac{1}{4}-\delta)}.$$

Proof. If $d_1, \ldots, d_n$ are the degrees of the vertices, then

$$P\left(\max_{1\le i\le n} d_i > n^{3/(2\alpha-2)}\right) \le n \cdot P(d_1 > n^{3/(2\alpha-2)}) \le c/\sqrt{n} \tag{3.1}$$

for some constant c. Suppose all the degrees are at most $n^{3/(2\alpha-2)}$. Suppose R_1 and R_2 are the clusters starting from 1 and 2 up to size n^{δ}. Let B be the event that R_1 contains a vertex of degree $\ge n^{1/(2\alpha-2)}$. Let e_n be the sum of degrees of all those vertices with degree $\ge n^{1/(2\alpha-2)}$. While growing R_1 the probability that a vertex of degree $\ge n^{1/(2\alpha-2)}$ will be included on any step is

$$\le \frac{e_n}{\sum_{i=1}^{n} d_i - n^{\delta+3/(2\alpha-2)}} \equiv \beta_n.$$

Since the size biased distribution is $q_k \sim Ck^{-(\alpha-1)}$ as $k \to \infty$, $\sum_{s\geq k} q_s \sim Ck^{-(\alpha-2)}$ as $k \to \infty$, and we have $e_n \sim Cn^{1-(\alpha-2)/(2\alpha-2)}$ and hence $\beta_n \sim Cn^{-(\alpha-2)/(2\alpha-2)}$ as $n \to \infty$. So for large n $\beta_n \leq c_1 n^{-1/4}$ for some constant c_1, when $\alpha > 3$. Thus

$$P(B^c) \geq 1 - c_1/n^{1/4-\delta}.$$

If B^c occurs, all the degrees of the vertices of R_1 are at most $n^{1/(2\alpha-2)}$. In that case, while growing R_2 the probability of choosing one vertex from R_1 is

$$\leq \frac{n^{\delta+1/(2\alpha-2)}}{\sum_{i=1}^n d_i - n^{\delta+3/(2\alpha-2)}} \leq c_2/n^{1-\delta-1/(2\alpha-2)}.$$

So the conditional probability

$$P(A^c|B^c) \geq \left(1 - c_2 n^{-(1-\delta-1/(2\alpha-2))}\right)^{n^\delta} \geq 1 - c_2/n^{1-2\delta-1/(2\alpha-2)}.$$

Hence combining these two

$$P(A^c) \geq (1 - c_1/n^{1/4-\delta})(1 - c_2/n^{1-2\delta-1/(2\alpha-2)}) \geq 1 - C_1/n^{1/4-\delta},$$

and that completes the proof. □

Lemma 3.1 will help us to show that in the dual contact process, staring from any vertex of degree $\geq (10/\lambda)^{2+\delta}$ for some $\delta > 0$, the infection reaches a star of V_n^ϵ, with probability bounded away from 0.

Lemma 3.2. *Let ξ_t^A be the contact process on G_n starting from $\xi_0^A = A$. Suppose $0 < \epsilon < 1/20(\alpha-1)$. Then there are constants $\lambda_0 > 0$, $n_0 < \infty$, $c_0 = c_0(\lambda, \epsilon)$ and $p_i > 0$ independent of $\lambda < \lambda_0$, $n \geq n_0$ and ϵ such that if $T = n^{c_0}$, v_2 is a vertex with degree $d(v_2) \geq (10/\lambda)^{2+\delta}$ for some $0 < \delta < 1$ and v_1 is a neighbor of v_2,*

$$P\left(\xi_T^{\{v_2\}} \cap V_n^\epsilon\right) \geq p_2, \qquad P\left(\xi_{T+1}^{\{v_1\}} \cap V_n^\epsilon\right) \geq p_1\lambda.$$

Proof. The second conclusion follows immediately from the first, since the probability that v_1 will infect v_2 before time 1, and that v_2 will stay infected until time 1 is

$$\geq \frac{\lambda}{\lambda+1}(1 - e^{-(\lambda+1)})e^{-1} \geq c\lambda.$$

Let Λ_m be the set of vertices in G_n of degree $\geq (10/\lambda)^{m+\delta}$ for $m \geq 2$. Define $\gamma = \frac{\delta}{2(2+\delta)}$ and

$$\begin{aligned} B &= 2(\alpha-1)\log(10/\lambda), & u &= \left(e^{-1}(1-e^{-\lambda})e^{-1}\right)^{-(B+1)}, \\ w_n &\equiv \log(n^\epsilon)/\log(10/\lambda) - \delta & T_m &= T_m^1 + T_m^2 \quad \text{where} \quad T_m^1 = (10/\lambda)^{(m+\delta)\gamma} \quad T_m^2 = u^m. \end{aligned}$$

and let $n^{c_0} = \sum_{m=2}^{w_n} T_m$.

Define the chain of events E_m inductively as follows. Let $E_2 = \left\{\xi_{T_2}^{\{v_2\}} \cap \Lambda_3 \neq \emptyset\right\}$ and for $m \geq 3$, having defined $E_2, \ldots, E_{m-1}$, we let

$$E_m = \left\{\xi_{T_m}^{\{v_m\}} \cap \Lambda_{m+1} \neq \emptyset\right\}, \quad \text{and} \quad v_m \in \xi_{T_{m-1}}^{\{v_{m-1}\}} \cap \Lambda_m.$$

Let A_m be the event that the clusters of size $(10/\lambda)^{(m+\delta+1)(\alpha-2)}$ starting from two neighbors of v_m do not intersect and

$$F = \cap_{m=2}^{w_n} A_m.$$

Since $\epsilon < 1/20(\alpha - 1)$, the cluster size $(10/\lambda)^{(m+\delta+1)(\alpha-2)}$ is at most $n^{1/10}$ for $m \leq w_n$. So using Lemma 3.1 and the fact $\binom{k}{2} < k^2$,

$$P(F^c) \leq \left(\sum_{m=2}^{w_n} (10/\lambda)^{2m+2\delta}\right) cn^{-(1/4-1/10)} \leq n^{2\epsilon} cn^{-(1/4-1/10)} < cn^{-(1/4-3/20)} < 1/6$$

for large n.

Since each vertex has degree at least 3, if F occurs then by the choice of B the neighborhood of radius Bm around v_m will contain more than $(10/\lambda)^{(m+\delta+1)(\alpha-2)+m}$ vertices. Let G_m be the event that the neighborhood of radius Bm around v_m intersects Λ_{m+1}. In the neighborhood of v_m probability of having a vertex of Λ_{m+1} is at least $c(\lambda/10)^{(m+\delta+1)(\alpha-2)}$. Hence

$$P(G_m^c F) \leq \left(1 - c(\lambda/10)^{(m+\delta+1)(\alpha-2)}\right)^{(10/\lambda)^{m+(m+\delta+1)(\alpha-2)}} \leq \exp(-(10/\lambda)^m).$$

If λ is small, $\sum_{m=2}^{\infty} \exp(-(10/\lambda)^m) \leq 1/6$.

On the intersection of F and G_m we have a vertex of Λ_{m+1} within radius Bm of v_m. Using Lemma 2.2 and Lemma 2.3, in the contact process $\left\{\xi_t^{\{v_m\}} : t \geq 0\right\}$, v_m gets hot at time T_m^1 and remains lit till time T_m with error probability $\leq c\lambda^{(m+\delta)\gamma}$ for small λ. If v_m is lit, then Lemma 2.4 shows that v_m fails to transfer the infection to some vertex in Λ_{m+1} within time T_m^2 with probability

$$\leq \left[1 - (e^{-1}(1 - e^{-\lambda})e^{-1})^{Bm}\right]^{T_m^2/(Bm)} \leq \exp\left[-(e^{-1}(1 - e^{-\lambda})e^{-1})^{-m}/(Bm)\right] \equiv \eta_m.$$

where $\equiv$ indicates we are making a definition, and hence $P(E_m^c G_m F) \leq c\lambda^{(m+\delta)\gamma} + \eta_m$. If λ is small $\sum_{m=2}^{w_m}[c\lambda^{(m+\delta)\gamma} + \eta_m] \leq 1/6$, we can take $p_2 = 1/2$ and the proof is complete. □

Lemma 3.2 gives a lower bound on the probability that an infection starting from a neighbor of a vertex of degree $\geq (10/\lambda)^{2+\delta}$ reaches a star. Lemma 2.3 shows that if the infection reaches a star, then with probability tending to 1 the star gets hot within $n^{\epsilon/3}$ units of time. Combining these two we get the following.

Proposition 2. *Suppose $0 < \epsilon < 1/20(\alpha - 1)$. There are constants $\lambda_0 > 0$, $n_0 < \infty$ $c_1 = c_1(\lambda, \epsilon)$ and $p_1 > 0$, which does not depend on $\lambda < \lambda_0$, $n \geq n_0$ and ϵ, such that for any vertex v_1 with a neighbor v_2 of degree $d(v_2) \geq (10/\lambda)^{2+\delta}$ for some $\delta \in (0, 1)$, and $T = n^{c_1}$ the probability that $\xi_T^{\{v_1\}}$ contains a hot star is bounded below by $p_1\lambda$.*

Next we will show that if we start with one lit star, then after time $\exp(n^{\epsilon/2})$ at least 3/4's of the stars will be lit.

Lemma 3.3. *Let $I^{\epsilon}_{n,t}$ be the set of stars which are lit at time t in the contact process on G_n such that $|I^{\epsilon}_{n,0}| = 1$. Then for $T' = \exp(n^{\epsilon/2})$*

$$P(|I^{\epsilon}_{n,T'}| < (3/4)|V^{\epsilon}_n|) \leq 7\exp(-\lambda^2 n^{\epsilon/3}/16).$$

Proof. Let s_1 be the lit star at time 0. As seen in Proposition 1, s_1 remains lit at time $T' = \exp(n^{\epsilon/2})$ with probability $\geq 1 - 6\exp(-\lambda^2 n^{\epsilon/3}/16)$ for large n. With probability $\geq Cn^{-b}$ another star gets hot within time $t_n = 2n^{\epsilon/3}$ and remains lit at time T'. Using similar argument as in Lemma 2.5, the process fails to make $(3/4)|V^{\epsilon}_n|$ many stars lit by time T' with probability

$$\leq (3/4)|V^{\epsilon}_n|(1 - Cn^{-b})^{T'/t_n} \leq (3/4)|V^{\epsilon}_n|\exp(-Cn^{-b}T'/t_n) \leq \exp(-\lambda^2 n^{\epsilon/3}/16),$$

as $|V^{\epsilon}_n| = Cn^{1-(\alpha-1)\epsilon}$ and $1 - x \leq e^{-x}$. So combining with the earlier error probability we get the result. □

Now we are almost ready to prove our main result. However, we need one more Lemma that we will use in the proof of the theorem.

Lemma 3.4. *Let F and G be two events which involve exposing n^{δ} many vertices starting at 1 and 2 respectively for some $0 < \delta \leq 1/8$. Then*

$$|P(F \cap G) - P(F)P(G)| \leq Cn^{-(1/4-\delta)}.$$

Proof. Let R_1 and R_2 be the clusters for exposing n^{δ} many vertices starting from 1 and 2 respectively, and let A be the event that they intersect. Clearly

$$\begin{aligned} P(F \cap G) &\leq P(A) + P(F \cap G \cap A^c) \\ &= P(A) + P(F \cap A^c)\, P(G \cap A^c) \\ &\leq P(A) + P(F)P(G). \end{aligned}$$

Using similar argument for F^c and G we get

$$|P(F \cap G) - P(F)P(G)| \leq P(A).$$

We estimate $P(A)$ using Lemma 3.1. □

Lemma 3.4 shows that two events which involve exposing at most $n^{1/8}$ vertices starting from two different vertices are asymptotically uncorrelated. Now we give the proof of the main theorem.

Proof of Theorem 1. Given $\delta > 0$, choose $\epsilon = \min\{\delta/\alpha, 1/20(\alpha-1)\}$. Let A_n be the set of vertices in G_n with a neighbor of degree at least $(10/\lambda)^{2+\delta}$. Clearly $|A_n|/n \to c_0(\lambda/10)^{(2+\delta)(\alpha-2)}$ as $n \to \infty$ for some constant c_0. Define the random variables $Y_x, x \in A_n$ as $Y_x = 1$ if the dual contact process starting from x can light up a star of V_n^ϵ and 0 otherwise. By Proposition 2, $EY_x \geq p_1\lambda$ for some constant $p_1 > 0$ and for any $x \in A_n$.

If we grow the cluster starting from $x \in A_n$ and exposing one vertex at a time, we can find a star on any step with probability at least $cn^{-(\alpha-2)\epsilon}$. So with probability $1 - \exp(-cn^\epsilon)$, we can find a star of V_n^ϵ within the exposure of at most $n^{\alpha\epsilon}$ vertices. So, with high probability, lighting a star up is an event involving at most $n^{(\alpha+1)\epsilon}$ many vertices. As $(\alpha+1)\epsilon < 1/8$, using Lemma 3.4, we can say

$$\begin{aligned} &P(Y_x = 1, Y_z = 1) - P(Y_x = 1)\, P(Y_z = 1) \\ &\leq (1 - \exp(-cn^\epsilon))\, Cn^{-(1/4-(\alpha+1)\epsilon)} + \exp(-cn^\epsilon) \equiv \theta_n. \end{aligned}$$

Using our bound on the covariances,

$$\operatorname{var}\left(\sum_{x\in A_n} Y_x\right) \leq n + \binom{n}{2}\theta_n,$$

and Chebyshev's inequality gives

$$P\left(\left|\sum_{x\in A_n}(Y_x - EY_x)\right| \geq n\gamma\right) \leq \frac{n + \binom{n}{2}\theta_n}{n^2\gamma^2} \to 0 \quad \text{as } n \to \infty,$$

for any $\gamma > 0$, since $\theta_n \to 0$ as $n \to \infty$. Since $EY_x \geq p_1\lambda$ and $|A_n|/n \to c_0(\lambda/10)^{(2+\delta)(\alpha-2)}$, if we take $p_l \equiv p_1\lambda \cdot c_0(\lambda/10)^{(2+\delta)(\alpha-2)}/2$ then

$$\lim_{n\to\infty} P\left(\sum_{x\in A_n} Y_x \geq np_l\right) = 1. \tag{3.2}$$

Now if $Y_x = 1$, Proposition 2 says that the dual process starting from x makes a star hot after $T_1 = n^{c_1}$ units of time. Then by Lemma 3.3 within next $T_2 = \exp(n^{\epsilon/2})$ units of time the dual process lights up 75% of all the stars with probability $1 - 7\exp(-\lambda^2 n^{\epsilon/3}/16)$.

Let $I_{n,t}^\epsilon$ be the set of stars which are lit at time t in the contact process $\{\xi_t^1 : t \geq 0\}$ and

$$T_3 = \inf\left\{t > \exp(n^{1-\alpha\epsilon}) : \left|I_{n,t}^\epsilon\right| \geq (3/4)\left|V_n^\epsilon\right|\right\}.$$

By Proposition 1, $P(T_3 < \infty) \geq 1 - \exp(-cn^\epsilon)$. Let

$$\mathcal{S} = \left\{S \subset \{1, 2, \ldots, n\} : \xi_t^1 = S \Rightarrow |I_{n,t}^\epsilon| \geq (3/4)|V_n^\epsilon|\right\}.$$

Using the Markov property and self-duality of the contact process we get the following inequality. For any subset B of the vertex set, and for the event $F_n \equiv [T_3 < \infty]$ we have

$$
\begin{aligned}
&P\left[\left(\xi^{\mathbf{1}}_{T_1+T_2+T_3} \supset B\right) \cap F_n\right] \\
&= \sum_{S \in \mathcal{S}} P\left(\xi^S_{T_1+T_2} \supset B\right) P\left(\xi^{1}_{T_3} = S | F_n\right) P(F_n) \\
&= \sum_{S \in \mathcal{S}} P\left(\xi^{\{x\}}_{T_1+T_2} \cap S \neq \emptyset \forall x \in B\right) P\left(\xi^{1}_{T_3} = S | F_n\right) P(F_n) \\
&\geq \sum_{S \in \mathcal{S}} P\left(|\xi^{\{x\}}_{T_1+T_2} \cap I^\epsilon_{n,T_3}| > (3/4)|V^\epsilon_n| \forall x \in B\right) P\left(\xi^{1}_{T_3} = S | F_n\right) P(F_n) \\
&\geq P(Y_x = 1 \forall x \in B)\left(1 - 7|B| \exp\left(-\lambda^2 n^{\epsilon/3}/16\right)\right) P(F_n) \\
&\geq P(Y_x = 1 \forall x \in B)(1 - 2\exp\left(-cn^{\epsilon/4}\right)),
\end{aligned}
$$

as $|B| \leq n$ and $P(F_n) \geq 1 - \exp(-cn^\epsilon)$. Hence for $T = T_1 + T_2 + T_3$, combining with (3.2) and using the attractiveness property of the contact process we conclude that as $n \to \infty$

$$
\inf_{t \leq T} P\left(\frac{|\xi^{\mathbf{1}}_t|}{n} > p_l\right) = P\left(\frac{|\xi^{\mathbf{1}}_T|}{n} > p_l\right) \geq P\left(\xi^{\mathbf{1}}_T \supseteq \{x : Y_x = 1\}, \sum_{x \in A_n} Y_x \geq np_l\right) \to 1, \quad (3.3)
$$

which completes the proof of Theorem 1, and proves the lower bound in Theorem 2.

4 Upper bound in Theorem 2

For the upper bound, we will show that if the infection starts from a vertex x with no vertex of degree $> 1/\lambda^{1-\delta}$ nearby, it has a very small chance to survive. To get the 1 in upper bound we need to use the fact that first event in the contact process starting at x has to be a birth so we begin with that calculation.

Let Λ_δ be the set of vertices of degree $> \lambda^{\delta-1}$. Define $Z_x, x \in \{1, 2, \ldots, n\}$ as $Z_x = 1$ if the dual contact process $\{\xi^{\{x\}}_t : t \geq 0\}$ starting from x survives for $T' = 1/\lambda^{\alpha-1}$ units of time, and 0 otherwise. We will show $EZ_x \leq C\lambda^{1+(\alpha-2)(1-\delta)}$ for some constant C. If T_1 is the time for the first event in the dual process, then $ET_1 \leq 1$ and using Markov's inequality $P(T_1 > 1/\lambda^{\alpha-1}) < \lambda^{\alpha-1}$. So if $T_1 < 1/\lambda^{\alpha-1}$, the first event must be a birth for Z_x to be 1. So for $x \in \Lambda_\delta$,

$$
\begin{aligned}
P(Z_x = 1) &\leq P(T_1 > 1/\lambda^{\alpha-1}) + \sum_{i > \lambda^{\delta-1}} p_i \frac{\lambda i}{\lambda i + 1} \\
&\leq \lambda^{\alpha-1} + C\lambda \sum_{i > \lambda^{\delta-1}} i^{-(\alpha-1)} \\
&\leq \lambda^{\alpha-1} + C\lambda \cdot \lambda^{(\alpha-2)(1-\delta)}.
\end{aligned}
$$

For $x \in \Lambda_\delta^c$, let $w(\lambda) \le C\lambda^{(\alpha-2)(1-\delta)}$ be the size-biased probability of having a vertex of Λ_δ in its neighborhood. If $d(x) = i$, the expected number of vertices in a radius m around x is at most $i \cdot EZ_m$, where Z_m is the total progeny up to m^{th} generation of the branching process with offspring distribution $q_k = (k+1)p_{k+1}/\mu \sim ck^{\alpha-1}$. So the expected number of vertices, which are within a distance $m = \lceil(\alpha-1)/\delta\rceil$, the smallest integer larger than $(\alpha-1)/\delta$, from x and belong to Λ_δ, is

$$\le \sum_{i=2}^{(1/\lambda)^{1-\delta}} p_i \cdot i \cdot EZ_m \cdot C\lambda^{(\alpha-2)(1-\delta)} \le C\lambda^{(\alpha-2)(1-\delta)}.$$

Using Markov's inequality the probability of having at least one vertex of Λ_δ within a distance m from x has the same upper bound as above.

Until we reach Λ_δ, $|\xi_t^{\{x\}}| \le Y_t$ where

$$\begin{aligned} Y_t &\to Y_1 - 1 \quad \text{at rate} \quad Y_t \\ Y_t &\to Y_t + 1 \quad \text{at rate} \quad Y_t\lambda \cdot (1/\lambda)^{1-\delta} = Y_t\lambda^\delta \end{aligned}$$

So Y_t jumps at rate $Y_t(1+\lambda^\delta)$ and it jumps to $Y_t + 1$ with probability $\lambda^\delta/(1+\lambda^\delta) < \lambda^\delta$. If $T_1 < 1/\lambda^{\alpha-1}$, the first event in the dual process $\xi_t^{\{x\}}$ must be a birth for Z_x to be 1. Let T_{2m} is the time of the $2m^{th}$ event after the first event. Then $ET_{2m} \le 2m/(1+\lambda^\delta)$ and using Markov's inequality

$$P(T_{2m} > 1/\lambda^{\alpha-1}) \le C\lambda^{\alpha-1}.$$

Now if $T_{2m} < 1/\lambda^{\alpha-1}$ and there is no vertex of Λ_δ within a distance m of x, the infection starting at x survives for time T' only if Y_t has at least m up jumps before hitting 0. If there are $\le m-1$ up jumps in the first $2m$ then Y_t will hit 0 by T_{2m}, as $Y_0 = 2$. The probability of this event is

$$\begin{aligned} &\le P(B \ge m) \quad \text{where} \quad B \sim \text{ Binomial } (2m, \lambda^\delta) \\ &\le 2^{2m}\lambda^{m\delta} \le 2^{2m}\lambda^{\alpha-1}. \end{aligned}$$

Combining all three error probabilities, for any $x \in \Lambda_\delta^c$,

$$\begin{aligned} P(Z_x = 1) &\le P(T_1 > 1/\lambda^{\alpha-1}) + P(T_{2m} > 1/\lambda^{\alpha-1}) + \sum_{i \le \lambda^{\delta-1}} p_i \frac{\lambda i}{\lambda i + 1} \cdot C\lambda^{(\alpha-2)(1-\delta)} \\ &\le C\lambda^{1+(\alpha-2)(1-\delta)}. \end{aligned}$$

Using an argument similar to the one at the end of the proof of Theorem 1

$$P\left(\left|\sum_x (Z_x - EZ_x)\right| > n\gamma\right) \to 0 \quad \text{as } n \to \infty$$

for any $\gamma > 0$. Since $EZ_x \le C\lambda^{1+(\alpha-2)(1-\delta)}$ for all $x \in \{1, 2, \dots, n\}$, if we take $p_u = 3C\lambda^{1+(\alpha-2)(1-\delta)}$, then

$$P\left(\sum_x Z_x \ge np_u\right) \to 0 \quad \text{as } n \to \infty.$$

So by making C larger in the definition of p_u and using the attractiveness of the contact process

$$\inf_{t \ge T'} P(|\xi_t^1| \le p_u n) \to 1$$

as $n \to \infty$. □

5 Proof of connectivity and diameter

We conclude the paper with the proof of Lemma 1.2. We begin with a large deviations result. The fact is well-known, but the proof is short so we give it for completeness.

Lemma 5.1. *Let $X_1, X_2, \dots$ be i.i.d., nonnegative with mean μ. If $\rho < \mu$, then there is a constant $\gamma > 0$ so that*

$$P(X_1 + \cdots + X_k \le \rho k) \le e^{-\gamma k}$$

Proof. Let $\phi(\theta) = Ee^{-\theta X}$. If $\theta > 0$ then

$$e^{-\theta \rho k} P(X_1 + \cdots + X_k \le \rho k) \le \phi(\theta)^k.$$

So we have

$$P(X_1 + \cdots + X_k \le \rho k) \le \exp(k\{\theta\rho + \log\phi(\theta)\}).$$

$\log(\phi(0)) = 0$ and as $\theta \to 0$

$$\frac{d}{d\theta}\log(\phi(\theta)) = \frac{\phi'(\theta)}{\phi(\theta)} \to -\mu.$$

So $\log\phi(\theta) \sim -\mu\theta$ as $\theta \to 0$, and the result follows by taking θ small. □

Proof of Lemma 1.2. We will prove the result in the following steps.

Step 1: Let $k_n = (\log n)^2$. The size of the cluster C_x, starting from $x \in \{1, 2, \dots n\}$, reaches size k_n with probability $1 - o(n^{-1})$.

Step 2: There is a $B < \infty$ so that if the size of C_x reaches size $B \log n$, it will reach $n^{2/3}$ with probability $1 - O(n^{-2})$.

Step 3: Let $\zeta > 0$. Two clusters C_x and C_y, starting from x and y respectively, of size $n^{(1/2)+\zeta}$ will intersect with probability $1 - o(n^{-2})$.

Steps 2 and 3 follow from the proof of Theorem 3.2.2 of Durrett (2007), so it is enough to do Step 1. Before doing this, note that if $d_1, \dots, d_n$ are the degrees of the vertices, and $\eta > 0$ then as $n \to \infty$,

$$P\left(\max_{1\le i\le n} d_i > n^{(1+\eta)/(\alpha-1)}\right) \le n \cdot P(d_1 > n^{(1+\eta)/(\alpha-1)}) \sim Cn^{-\eta}.$$

Given $\alpha > 3$, we choose $\eta > 0$ small enough so that $(1+\eta)/(\alpha-1) < 1/2$.

To prove step 1, we will expose one vertex at a time. Following the notation of Durrett (2007), suppose A_t, U_t and R_t are the sets of active, unexplored and removed sites respectively at time t in the process of growing the cluster starting from 1, with $R_0 = \{1\}$, $A_0 = \{z : 1 \sim z\}$ and $U_0 = \{1, 2, \ldots, n\} - A_0 \cup R_0$. At time $\tau = \inf\{t : A_t = \emptyset\}$ the process stops. If $A_t \neq \emptyset$, pick i_t from A_t in some way measurable with respect to the process up to that time and let

$$\begin{aligned} R_{t+1} &= R_t \cup \{i_t\} \\ A_{t+1} &= A_t \cup \{z \in U_t : i_t \sim z\} - \{i_t\} \\ U_{t+1} &= U_t - \{z \in U_t : i_t \sim z\}. \end{aligned}$$

Here $|R_t| = t+1$ for $t \leq \tau$ and so $C_1 = \tau + 1$. If there were no collisions, then $|A_{t+1}| = |A_t| - 1 + Z$ where Z has the size biased degree distribution q. Let q^η be the distribution of $(Z|Z \leq n^{(1+\eta)/(\alpha-1)})$. Then on the event $\{\max_i d_i \leq n^{(1+\eta)/(\alpha-1)}\}$, $|A_t|$ is dominated by a random walk $S_t = S_0 + Z_1 + \cdots + Z_t$, where $S_0 = A_0$ and $Z_i \sim q^\eta$. Since $q_{k-1} = kp_k/\mu$, we have $q_0 = q_1 = 0$ and hence $q_0^\eta = q_1^\eta = 0$. Then S_t increases monotonically.

If we let $T = \inf\{m : S_m \geq k_n\}$ then

$$P(|C_1| \leq k_n) \leq P(S_t - |A_t| \geq 4 \text{ for some } t \leq T). \tag{5.1}$$

As observed above, if n is large, all of the vertices have degree $\leq n^\beta$ where $\beta = (1+\eta)/(\alpha - 1) < 1/2$. As long as $S_t \leq 2k_n$, each time we add a new vertex and the probability that it is in the active set is at most

$$\gamma_n = \frac{2k_n n^\beta}{\sum_{i=1}^n d_i - 2k_n n^\beta} \leq Ck_n n^{\beta-1}$$

for large n. Thus the probability of two or more collisions while $S_t \leq 2k_n$ is $\leq (2k_n)^2\gamma_n^2 = o(n^{-1})$.

If $S_T - S_{T-1} \leq k_n$, then the previous argument suffices, but $S_T - S_{T-1}$ might be as large as n^β. Letting $m > 1/(1-2\beta)$, we see that the probability of m or more collisions is at most

$$(n^\beta)^m (Cn^{\beta-1})^m = o(n^{-1}).$$

To grow the cluster we will use a breadth first search: we will expose all the vertices at distance 1 from the starting point, then those at distance 2, etc. When a collision occurs, we do not add a vertex, and we delete the one with which a collision has occurred, so two are lost. There is at most one collision while $S_t \leq 2k_n$. Since $S_0 \geq 3$, it is easy to see that the worst thing that can happen in terms of the growth of the cluster is for the collision to occur on the first step, reducing S_0 to 1. After this the number of vertices doubles at each step so size k_n is reached before we have gone a distance $\log_2 k_n$ from the starting point.

In the final step we might have a jump $S_\tau - S_{\tau-1} \geq k_n$ and m collisions, but as long as $k_n = (\log n)^2 > 2m$ we do not lose any ground. In the growth before time T, each vertex, except for possibly one collision, has added two new vertices to the active set. From this it is easy to see that the number of vertices in the active set is at least $k_n/2 - 2m$.

To grow the graph now, we will expose all of the vertices in the current active set, then expose all of the neighbors of these vertices, etc. Let $\epsilon > 0$. The proof of Theorem 3.2.2 in Durrett (2007) shows (see page 78) that if δ is small then until $n\delta$ vertices have been exposed, the cluster growth dominates a random walk with mean $\nu - \epsilon$. Let $J_1, J_2, \ldots$ be the successive sizes of the active set when these phases are complete. The large deviations result, Lemma 5.1, implies that there is a $\gamma > 0$ so that

$$P(J_{i+1} \le (\nu - 2\epsilon)J_i | J_i = j_i) \le \exp(-\gamma j_i)$$

Since $J_1 \ge (\log n)^2/2 - 8$, it follows from this result that with probability $\ge 1 - o(n^{-1})$, in at most

$$\left(\frac{1}{2} + \zeta\right) \frac{\log n}{\log(\nu - \epsilon)}$$

steps, the active set will grow to size $n^{(1/2)+\zeta}$. Using the result from Step 3 and noting that the initial phase of the growth has diameter $\le \log_2 k_n = O(\log\log n)$ the desired result follows.

□

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